Multidimensional color image storage, retrieval, and compression based on quantum amplitudes and phases

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ABSTRACT
In this study, we propose a new representation method for multidimensional color images, called an n-qubit normal arbitrary superposition state (NASS), where n qubits represent the colors and coordinates of 2^n pixels (e.g., a three-dimensional color image of 1024 x 1024 x 1024 using only 30 qubits). Based on NASS, we present an (n + 1)-qubit normal arbitrary superposition state with relative phases (NASSRP) and an (n + 2)-qubit normal arbitrary superposition state with three components (NASSTC) for lossless and lossy quantum compression, respectively. We also design three general quantum circuits to generate NASS, NASSRP, and NASSTC states, where we retrieve an image from a quantum system using different projection measurement operators. Finally, we define the quantum compression ratio and analyze lossless and lossy quantum compression algorithms of multidimensional quantum images. For the first time, we implemented the compression of multidimensional color images on a quantum computer. Thus, we address the theoretical and practical aspects of image processing on a quantum computer.

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1. Introduction

Quantum computing [7] exploits the unique computing performance of quantum coherence, entanglement, superposition of quantum states and other inherent characteristics, and it has become an international research focus. Indeed, by utilizing these unique properties, Shor’s discrete logarithms and integer factoring algorithms in polynomial time [18], Deutsh’s parallel computing algorithm with quantum parallelism and coherence [4], and Grover’s quadratic speedup for unordered database search algorithms [9] deliver better performance than any known classical algorithms. Quantum algorithms such as the quantum search algorithm [17,19], quantum watermarking [11], quantum encryption and decryption [22,26], quantum-behaved particle swarm optimization [20,21], quantum Fourier transform (QFT) [15,18], and quantum wavelet transform (QWT) [8] are also more efficient than their classical counterparts.

In a quantum system, the frequency of the physical nature of color could represent a color instead of the RGB model or the HSI model, thus a color may be represented by only a 1-qubit quantum state and an image can be stored in a quantum array [23,24]. A flexible representation of a quantum image can store the colors and coordinates of a two-dimensional grayscale image of 2^n pixels using n + 1 qubits [12]. A set of quantum states for M colors and a set of quantum states for N coordinates
were proposed to represent $M$ the colors and coordinates of $N$ pixels in an image, respectively [13]. A previous study [13] also discussed the retrieval of images stored in a quantum system. Phase-space distribution functions (Wigner and Husimi functions) have been used to store an image in a quantum system [16]. Information storage and retrieval were achieved based on the quantum amplitude in previous studies, as well as the quantum phase [1,25].

Quantum computing can be implemented using quantum gate operations. A finite set of basic gate operations can be used to construct any quantum gate operation [5]. Universal quantum gates are expressed as combinations of one-bit and two-bit gates [2,6,15]. An efficient scheme has been proposed for initializing a quantum register with an arbitrary superposed state and the application of this scheme to three specific cases was discussed [14].

In this study, we propose an $n$-qubit normal arbitrary superposition state (NASS) that represents a multidimensional color image with $2^n$ pixels. Supposing that 1 qubit is equivalent to 1 bit, the classic compression ratio (i.e., memory) of NASS is $2^n \times 24/n$ (the classic compression ratio of a $1024 \times 1024$ RGB color image is 1258291.2). Based on NASS, we present an $(n+1)$-qubit normal arbitrary superposition state with relative phases (NASSRP) to represent a multidimensional color image with $2^n$ pixels and some additional information. Thus, an $(n+1)$-qubit NASSRP can represent $2^n$ colors, $2^n$ coordinates, and $2^{n+1}$ integers. In order to reduce computational resources required by a quantum computer, we employ NASSRP and an algorithm (Algorithm 2) to implement lossless quantum compression. The simulation results showed that NASSRP facilitates the lossless quantum compression of binary images. We also apply QFT and QWT based on NASS to enable the lossy quantum compression of grayscale images. Our simulation results showed that NASS facilitates the lossy quantum compression of grayscale images. We also use an $(n+2)$-qubit normal arbitrary superposition state with three components (NASSTC) for the lossy quantum compression of color images. Our simulation results demonstrated that NASSTC allowed the lossy quantum compression of color images.

The paper is organized as follows: some basic quantum gate operations and representations of multidimensional color images are described in Section 2. Multidimensional color image storage is explained in Section 3 and multidimensional color image retrieval is discussed in Section 4. Multidimensional image compression is presented in Section 5. Our conclusions are provided in Section 6.

2. Basic quantum gates and representations of multidimensional color images

2.1. Basic quantum gates

A state of a quantum system is described as a vector in a Hilbert space, which is called a ket by Dirac. $|\psi\rangle$ and $\langle\psi|$ are the symbols used to represent the right ket and left ket, respectively. $|\psi\rangle$ and $\langle\psi|$ are a pair of Hermite conjugate states in a quantum system, which are defined as

$$|\psi\rangle = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix}$$

and

$$\langle\psi| = |\psi\rangle^\dagger = \begin{bmatrix} v_0^* & v_1^* & \cdots & v_{n-1}^* \end{bmatrix}$$

where $v_i$ $(i = 0, 1, \ldots, n-1)$ is a complex number.

The symbol $\otimes$ represents the tensor product of matrix, which is defined as

$$|u\rangle \otimes |\psi\rangle = \begin{bmatrix} u_0|\psi\rangle \\ \vdots \\ u_{n-1}|\psi\rangle \end{bmatrix} = \begin{bmatrix} u_0v_0 \\ \vdots \\ u_{n-1}v_{n-1} \end{bmatrix}$$

where $|u\rangle = [u_0 \ u_1 \ \cdots \ u_{n-1}]^T$ and $u_i$ $(i = 0, 1, \ldots, n-1)$ is a complex number. $|u\rangle \otimes |\psi\rangle$ can also be represented as $|u\rangle|\psi\rangle$ or $|u\psi\rangle$.

The notations of some basic quantum gates and their corresponding matrices are shown in Fig. 1. The identity ($I$), Hadamard ($H$) and Pauli-$X$ ($X$) gates were defined in Ref. [15]. Let $U$, 1-Controlled $U$ ($1CU$), 0-Controlled $U$ ($0CU$), and $n$ qubit Controlled $U$ ($nCU$) be the fourth, fifth, sixth, and seventh gates in Fig. 1, where the explanations of these gates are as follows.
$U_{gate}$:

$U|0\rangle = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = u_{00}|0\rangle + u_{10}|1\rangle$

and

$U|1\rangle = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = u_{01}|0\rangle + u_{11}|1\rangle$

$1CU$ and $0CU$ gates:

$1CU|0\rangle|i\rangle = |0\rangle|i\rangle, \quad 1CU|1\rangle|i\rangle = |1\rangle(U|i\rangle)$

and

$0CU|0\rangle|i\rangle = |0\rangle(U|i\rangle), \quad 0CU|1\rangle|i\rangle = |1\rangle|i\rangle$

where $i = 0$ or $i = 1$.

$nCU$ gate:

$nCU|i_1i_2\cdots i_{n-1}x\rangle = |i_1i_2\cdots i_{n-1}\rangle(U|x\rangle)$

where $i_1, i_2, \ldots, i_{n-1}$ are numbers in $nCU$ (see Fig. 1) and $i_1, i_2, \ldots, i_{n-1}, x \in \{0, 1\}$.

When $|j_1\cdots j_{n-1}\rangle \neq |i_1\cdots i_{n-1}\rangle$, the quantum state is not altered:

$nCU|j_1j_2\cdots j_{n-1}x\rangle = |j_1j_2\cdots j_{n-1}x\rangle$

2.2. Representations of multidimensional color images

Based on a previously reported treatment strategy for colors and coordinates [13,12], representations of multidimensional color images are proposed in this sub-section.

Briefly, we describe an angle (i.e., a coefficient) that represents a color as follows [13]. A bijective function $F_1$ specifies a one-to-one relationship between a color and an angle

$F_1 : Color \leftrightarrow \phi$ \hspace{1cm} (1)

Fig. 1. Notations of some basic quantum gates and their corresponding matrices.
where \( \text{Color} = \{ \text{color}_1, \text{color}_2, \ldots, \text{color}_M \} \), \( \text{color} \) corresponds to the \( i \)-th color in the ordered \( M \) colors, 
\( \phi = \{ \phi_0, \phi_1, \ldots, \phi_{M - 1} \} \). \( \phi_i = \frac{\pi}{2M - 1} \). \( F_i(\text{color}_{i+1}) = \phi_i \) and \( i \in \{0, 1, \ldots, M - 1\} \).

For binary images, \( M = 2 \),
\[ \phi_0 = 0, \quad \phi_1 = \frac{\pi}{2} \] (2)
where \( \phi_0 \) and \( \phi_1 \) correspond to white and black, respectively.

For grayscale images, \( M = 256 \), colors are sorted in ascending order based on the grayscale values. Thus,
\[ \phi_i = \frac{(i)\pi}{2(256 - 1)} \] (3)
where \( i \) is a grayscale value. For example, \( \phi_0 \) and \( \phi_{255} \) correspond to grayscale values 0 and 255, respectively.

For color images, \( M = 2^{24} \),
\[ \phi_i = \frac{(i)\pi}{2(2^{24} - 1)} \] (4)
where \( i = x \times 256 \times 256 + y \times 256 + z \). \( x, y, z \) are gray scale values of three components of a \( RGB \) color. For example, \( \phi_0 \) and \( \phi_{16,777,255} \) correspond to \( RGB \) values \((0, 0, 0)\) and \((255, 255, 255)\), respectively.

If \( V \) is a \( k \)-dimensional Euclidean space spanned by the orthogonal basis vectors \( b_1, b_2, \ldots, b_k \), a \( k \)-dimensional digital image is denoted as the function \( f : V \rightarrow R \) where \( V \) represents the position information of an image and \( f(V) \) is a color set of pixels that corresponds to the position \( V \). If we let \( f(V) \subset \text{Color} \), we can obtain the angle set \( \phi \) that represents the colors of a \( k \)-dimensional digital image using (1).

To store some additional information related to an image in a quantum system, we create a bijective function \( F_2 \) that specifies a one-to-one relationship between an angle and an integer
\[ F_2 : \text{Number} \rightarrow \text{Angle} \] (5)
where \( \text{Number} = \{1, 2, \ldots, m\} \), \( \text{Angle} = \{\beta_0, \beta_2, \ldots, \beta_{m-1}\} \). \( F_2(i + 1) = \beta_i, F_2(\text{Number}_i + 1) = \beta_i \) and \( i \in \{0, 1, \ldots, m - 1\} \). If \( m = 1 \), let \( \beta_0 = 0 \).

A quantum superposition state in \( 2^n \) dimensional Hilbert space may be expressed as \( |\psi_0\rangle = \sum_{i=0}^{2^{n-1}} a_i |i\rangle \) where \( \{|i\rangle, \ldots, |2^{n-1}\rangle\} \) is a set of orthogonal basis and \( a_i \) is an arbitrary real.

To represent a \( k \)-dimensional color digital image, \( |\psi_2\rangle = \sum_{i=0}^{2^{n-1}} a_i |i\rangle \) is modified as follows
\[ |\psi_2\rangle = \sum_{i=0}^{2^n-1} a_i |v_1\rangle |v_2\rangle \cdots |v_k\rangle \] (6)
where \( i = i_1 \cdots i_{b_1} \cdots i_{b_m} \cdots i_{b_k} \) and \( v_1 = i_1 \cdots i_j, v_2 = i_{j+1} \cdots i_k \) are the binary expansions for \( i \). \( v_1, v_2 \) and \( v_k \), respectively. \(|i\rangle = |v_1\rangle |v_2\rangle \cdots |v_k\rangle \) is a coordinate \( (v_1, v_2, \ldots, v_k) \) in the \( k \)-dimensional space; \( a_i \in \phi \) (see (1)) represents the color of the pixel that corresponds to the coordinate \(|i\rangle\).

To normalize the state \(|\psi_2\rangle\) in (6), we set
\[ \theta_i = \frac{a_i}{\sqrt{\sum_{j=0}^{2^n-1} \theta_j^2}} \] (7)
Substituting (7) for \( a_i \) in (6), we obtain a NASS state \(|\psi_A\rangle\) that represents a multidimensional color image with \( 2^n \) pixels
\[ |\psi_A\rangle = \sum_{j=0}^{2^{n-1}} \theta_j |v_1\rangle |v_2\rangle \cdots |v_k\rangle \] (8)
where \( (\sum_{j=0}^{2^{n-1}} \theta_j^2) = 1 \) and \(|j\rangle = |v_1\rangle |v_2\rangle \cdots |v_k\rangle\).

To represent a multidimensional color image and its additional information, (8) is changed as follows
\[ |\psi_{AP}\rangle = \sum_{j=0}^{2^{n-1}} \theta_j |v_1\rangle |v_2\rangle \cdots |v_k\rangle |\chi_j\rangle \] (9)
where \(|\chi_j\rangle = \cos \gamma_j |0\rangle + e^{i\alpha} \sin \gamma_j |1\rangle \) represents some additional information related to the pixel that corresponds to the coordinate \(|j\rangle\). \( e^{i\alpha} \) is a relative phase in \(|\chi_j\rangle\). \( \gamma_j, \alpha \in \text{Angle} \) correspond to two integers from (5). \(|j\rangle = |v_1\rangle |v_2\rangle \cdots |v_k\rangle \) and \( \theta_j \) are the same in (8) and (9).

Since \(|\psi_{AP}\rangle| = \sqrt{\sum_{i=0}^{2^n-1} \theta_i^2 (\cos^2 \gamma_j + \sin^2 \gamma_j)} = 1 \), \(|\psi_{AP}\rangle\) is called as NASSRP. From (9), we can see that an \( n + 1 \)-qubit NASSRP state can represent \( 2^n \) colors, \( 2^n \) coordinates, and \( 2^{n+1} \) integers.

Supposing that \( y_1, y_2 \) and \( y_3 \) are the grayscale values of three components of the \( RGB \) color of the pixel at the coordinate \(|j\rangle\), we calculate three angles \( r_j, g_j \) and \( b_j \), which correspond to \( y_1, y_2 \) and \( y_3 \), respectively, using (3).
\[ i \) and \( 1i \cdots i_1 \) are the binary expansions for integers \( x \) and \( y \), respectively, then \( \theta_{0i1} = \theta_x, \theta_{1i1} = \theta_y, \theta_0 \) and \( \theta_1 \) are the coefficients of items \( |x\rangle \) and \( |y\rangle \) in (8). \( i_1i_2 \cdots i_{j-1} \) is the binary expansion of an integer \( i \in \{0, 1, \ldots, (2^n-1)\} \) and \( j = 2, 3, \ldots, n \).

For example, when \( j = 2, (18) \) is rewritten
We obtain the NASSRP state successively on the initial state 0.

The controlled-\(R_{sji}\) operation is defined

\[
R_{sji} = \left( \sum_{k=0}^{s-1} |k\rangle \langle k| \right) \otimes I + |i\rangle \langle i| \otimes R_s(x_{ji})
\]  

(21)

The controlled-\(R_{sji}\) operation is unitary, because \(R_{sji}R_{sji}^\dagger = I^j\) where \(I^j = I \otimes I \otimes \cdots \otimes I\) is the tensor product of matrix \(I\) \(j\) times.

A unitary matrix \(R_s\) is defined for \(j = 1\) and \(j \geq 2\)

\[
R_s = R_s(x_1) \otimes I^{(n-1)}
\]  

(22)

and

\[
R_{sji} = \prod_{i=0}^{2^{j-1}-1} \left( R_{sji} \otimes I^{(n-j)} \right), \quad j \geq 2
\]  

(23)

By applying the unitary matrix \(R_{sji}\) successively on the initial state \(|0\rangle^\otimes n\), we obtain

\[
|\psi_{\alpha}\rangle = \left( \prod_{j=1}^{n} R_{sji} \right) |0\rangle^\otimes n = \sum_{i=0}^{2^n-1} \theta_i |v_1\rangle |v_2\rangle \cdots |v_k\rangle
\]  

(24)

\(|\psi_{\alpha}\rangle\) in (24) is implemented by the quantum circuit shown in Fig. 2.

\(|\psi_{\gamma}\rangle\) in (13) is implemented by the quantum circuit shown in Fig. 3. By applying Dashed box 1 in Fig. 3 to the initial state \(|0\rangle^\otimes n\), we obtain a quantum state

\[
|\psi_{\gamma}\rangle_1 = \frac{1}{\sqrt{3}} \left( |0\rangle^\otimes n |01\rangle + |10\rangle + |11\rangle \right)
\]  

(25)

The implementations of Dashed box \(i (i = 2, 3, 4)\) in Fig. 3 are shown in Fig. 4. For example, Dashed box 2 is implemented in Fig. 4 when \(i_1 = 0\) and \(i_2 = 1\). By applying Dashed box \(i (i = 2, 3, 4)\) successively to \(|\psi_{\gamma}\rangle_1\) in (25), we reach the state \(|\psi_{\gamma}\rangle\) in (13).

A controlled-\(R_j\) operation is defined

\[
R_j = \left( \sum_{l=0}^{2^{j-1}-1} |l\rangle \langle l| \right) \otimes I + |j\rangle \langle j| \otimes R(\gamma_j, \lambda_j)
\]  

(26)

where \(\gamma_j, \lambda_j \in \text{Angle}\) in (5). By applying the unitary matrix \(R_j\) successively to the state \(|\psi_{\alpha}\rangle \otimes |0\rangle\), we obtain the NASSRP state \(|\psi_{\gamma}\rangle\)

![Fig. 2. Implementation of a NASS state. The circuit implementation of a NASS state on the left of the figure is named as RNASS (Realization of NASS) shown on the right. \(|0\rangle, |0\rangle, \ldots, |0\rangle\) in (24) is an input of \(n\) qubits, which is also notated as \(|0\rangle^\otimes n\) in (24). Dashed box 1 and box \(j (j = 2, 3, \ldots, n)\) correspond to the implementations of \(R_{sji}\) (see (22)) and \(R_s\) (see (23)), respectively. The ith \((i = 0.1.2.\ldots,2^{j-1}-1)\) gate of box \(j\) is the implementation of \(R_{sji}\) (see (21)). The output of the circuit is the state \(|\psi_{\alpha}\rangle\) in (24).](image-url)
\[ w_{AP} \frac{\alpha_{ji}}{C_{0}^{1}} = R_{j}(\arctan(\sqrt{2})) \] (27) is implemented by the quantum circuit shown in Fig. 5.

A nCU gate can be constructed using one-bit and two-bit gates with a total number of \( O(\log^{2}N) \) where \( N = 2^{n} \). The circuit in Fig. 2 is built with \( 2^{n} - 1 \) quantum gates, so it can be constructed using \( O(N\log^{2}N) \) one-bit and two-bit gates. Similarly, we know that the circuits in Figs. 3 and 5 are also constructed using \( O(N\log^{2}N) \) one-bit and two-bit gates, respectively.

**Fig. 5.** Implementation of a NASSRP state. The RNASS circuit is shown in Fig. 2. \( |\psi_{AP}\rangle \) is an input of \( n+1 \) qubits. The \( j \)-th (\( j = 0, 1, \ldots, 2^{n-1} - 1 \)) gate of Dashed box \( n+1 \) is gate nCU shown in Fig. 1 and the implementation of \( R_{j} \) (see Eq. (26)). Dashed box \( n+1 \) corresponds to \( H^{\otimes n+1} R_{j} \).
4. Multidimensional color image retrieval

4.1. Image retrieval for NASSRP

For simplicity, we substitute $|n_1\cdots n_k\rangle$ for $|j\rangle$ in (9) in this sub-section, i.e., the NASSRP state is expressed as

$$\psi_{AP}(r) = \sum_{j=0}^{2^n-1} \theta_j |j\rangle$$  \hspace{1cm} (28)

where $|j\rangle = \cos \gamma_j |0\rangle + e^{i \delta_j} \sin \gamma_j |1\rangle$. $\gamma_j, \delta_j \in \text{Angle}$ correspond to two integers from (5).

The image retrieval process is divided into three steps.

Step 1, we detect the first $n$ qubits in $|\psi_{AP}(r)\rangle$ using the following method. We define the observable operator $M_1$

$$M_1 = \sum_{j=0}^{2^n-1} m_j P_j, \quad P_j = (|j\rangle\langle j|) \otimes I$$  \hspace{1cm} (29)

Apply $M_1$ to measure the quantum state $|\psi_{AP}(r)\rangle$, which yields $m_j$ with probability $p(m_j) = \langle \psi_{AP}(r) | P_j | \psi_{AP}(r) \rangle = \theta_j^2$, i.e.,

$$\theta_j = \sqrt{p(m_j)}$$  \hspace{1cm} (30)

The state after measurement is

$$P_j |\psi_{AP}(r)\rangle = |j\rangle |X_j\rangle, \quad j = 0, 1, \ldots, 2^n - 1$$  \hspace{1cm} (31)

Step 2, we define the observable operator $M_2$

$$M_2 = \mu_0 |0\rangle\langle 0| + \mu_1 |1\rangle\langle 1|$$  \hspace{1cm} (32)

Apply $M_2$ to measure the quantum state $|X_j\rangle$ in (31), which yields $\mu_1$ with probability $p(\mu_1) = \sin^2 \gamma_j$, i.e.,

$$\gamma_j = \arcsin \left( \sqrt{p(\mu_1)} \right)$$  \hspace{1cm} (33)

Step 3, the observable operator $M_3$ is defined as

$$M_3 = \sum_{i=0}^{1} \eta_i H_i$$  \hspace{1cm} (34)

where $H_0 = \left( \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) (|0\rangle + |1\rangle) \right)$ and $H_1 = \left( \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) (|0\rangle - |1\rangle) \right)$.

To obtain a correct angle $\theta_j$, we must employ many identical quantum states $|\psi_{AP}(r)\rangle$, so there many identical quantum states exist after Step 1, $|j\rangle |X_j\rangle$ in (31). Therefore, applying $M_3$ to measure $|X_j\rangle$ in (31) yields $\eta_0$ with probability $p(\eta_0) = 1/2 \pm \cos \gamma_j \sin \delta_j \cos \lambda_j$, i.e.,

$$\lambda_j = \arccos \left( \frac{p(\eta_0) - 1/2}{\cos \gamma_j \sin \delta_j} \right)$$  \hspace{1cm} (35)

Thus, we obtain $\theta_j, \gamma_j$, and $\lambda_j$, which correspond to the coordinate $|j\rangle$ ($j = 0, 1, \ldots, 2^n - 1$), using the three steps. Thus, we have retrieved a multidimensional color image represented by a NASSRP state from a quantum system.

Suppose that we must employ $n_0, n_j, n$ identical quantum states to acquire correct angles $\theta_j, \gamma_j, \lambda_j$ with the probability of $1 - \alpha$. Next, we describe how to solve the values of $n_0, n_j, n$.

4.1.1. Calculating $n_0$

When $\theta_j = \sqrt{p(m_j)}$ (see (30)), we define

$$Z = \begin{cases} 0, & \text{result of measurement is not } m_j \\ 1, & \text{result of measurement is } m_j \end{cases}$$  \hspace{1cm} (36)

where the result $m_j$ is obtained by using $M_1$ to measure the quantum state $|\psi_{AP}(r)\rangle$ in (28). Since $Z$ is either 1 or 0, $Z$ is a Bernoulli random variable. The probability mass function of the random variable $Z$ is given by

$$\begin{cases} p(\tilde{m}_j) = p(Z = 0) = 1 - p \\ p(m_j) = p(Z = 1) = p \end{cases}$$  \hspace{1cm} (37)

where $\tilde{m}_j$ indicates that the measurement result is not $m_j$. 

The expectation $\mu$ and variance $\sigma$ of $Z$ are $\mu = p$ and $\sigma^2 = p(1-p)$, respectively. Suppose that $Z_1, Z_2, \ldots, Z_n$ are $n$ samples of $Z$ and that $n$ is sufficiently large, then $\left(\sum_{t=1}^{n} z_t - np\right)/\sqrt{n} p(1-p) = (nZ - np)/\sqrt{n} p(1-p)$ has an approximately standard normal distribution according to the Central Limit Theorem. Thus, $P\left((nZ - np)/\sqrt{n} p(1-p) < z_{n/2}\right) = 1 - \alpha$, where $1 - \alpha$ is the confidence level (the value of $z_{n/2}$ can be found in standard normal distribution look-up tables, e.g., when $\alpha = 0.05$, $z_{0.025} = 1.96$). By solving the inequality $\left[\sum_{t=1}^{n} z_t - np(1-p)\right]/\sqrt{n} p(1-p) < z_{n/2}$, we find that the confidence interval of $p$ is $[p_{\min}, p_{\max}]$ with an approximate confidence level of $1 - \alpha$. $p_{\min}$ and $p_{\max}$ are expressed as follows

$$p_{\min} = \frac{2nZ + \lambda - \sqrt{(2nZ + \lambda)^2 - 4(n + \lambda)nZ^2}}{2n_z + 2\lambda}$$

(38)

and

$$p_{\max} = \frac{2nZ + \lambda + \sqrt{(2nZ + \lambda)^2 - 4(n + \lambda)nZ^2}}{2n_z + 2\lambda}$$

(39)

where $\lambda = Z^2_{n/2}$, $Z = \sum_{t=1}^{n} z_t/n$.

The size of the confidence interval $[p_{\min}, p_{\max}]$ is

$$\Delta p = p_{\max} - p_{\min} = \frac{\sqrt{4nZ - 4nZ^2 + \lambda^2}}{n_z + \lambda}$$

(40)

Since $\theta_j = \sqrt{p(m_j)}$, we define

$$\theta_{\max} = \sqrt{p_{\max}}, \quad \theta_{\min} = \sqrt{p_{\min}}$$

(41)

From (40) and (41), we infer

$$\Delta \theta = \sqrt{p_{\max}} - \sqrt{p_{\min}} \leq \sqrt{p_{\max} - p_{\min}} = \sqrt{\Delta p}$$

(42)

where $\Delta \theta = \theta_{\max} - \theta_{\min}$.

By defining $\hat{\theta}_j = \sqrt{p(m_j)} = \sqrt{Z}_j$, we know that $\hat{\theta}_j \approx \sqrt{p}$ and $Z \approx p$ based on the Law of Large Numbers. The confidence interval of $p$ is $[p_{\min}, p_{\max}]$, therefore $\hat{\theta}_j = Z \in [p_{\min}, p_{\max}]$ and $\theta^2_j = p(m_j) = p \in [p_{\min}, p_{\max}]$. From (41), we conclude that

$$\theta_j \in [\theta_{\min}, \theta_{\max}], \quad \hat{\theta}_j \in [\theta_{\min}, \theta_{\max}]$$

(43)

and

$$|\hat{\theta}_j - \theta_j| \leq \Delta \theta = \theta_{\max} - \theta_{\min}$$

(44)

Since $a_j = \theta_j \sqrt{\sum_{y=0}^{z_j-1} a_y^2}$ (see (7)), from (43) and (44), we yield

$$\frac{|\hat{a}_j - a_j|}{G_{\phi}} \leq \Delta \theta$$

(45)

and

$$a_j, \quad \hat{a}_j \in [G_{\phi} \theta_{\min}, G_{\phi} \theta_{\max}]$$

(46)

where $G_{\phi} = \sqrt{\sum_{y=0}^{z_j-1} a_y^2}$ and $a_j = \theta_j \sqrt{\sum_{y=0}^{z_j-1} a_y^2}$.

According to (1), we calculate

$$\Delta \phi = |\phi_{s,i} - \phi_i| = \frac{\pi}{2(M - 1)}$$

(47)

where $i \in \{1, 2, \ldots, (M - 1)\}$ and $M \in \{2, 256, 2^{24}\}$.

Suppose

$$\sqrt{\Delta p} < \frac{\Delta \phi}{G_{\phi}}$$

(48)

Setting $\Delta \phi = |a_j - \hat{a}_j|$, from (45), (42) and (48), we can see that

$$\Delta \phi \leq G_{\phi} \Delta \theta \leq \sqrt{\Delta p} G_{\phi} \leq \Delta \phi$$

(49)

$$\Delta \phi, \Delta \theta, \quad a_j, \quad \hat{a}_j, \quad G_{\phi} \theta_{\min}, \quad G_{\phi} \theta_{\max}, \quad \text{and} \quad G_{\phi} \Delta \theta \text{ are shown in Fig. 6.}$$

Formula (48) is equivalent to

$$y^4 n_i^2 + [2\lambda y^4 - 4\lambda (Z - Z^2)] n_i + y^4 \lambda^2 - \lambda^2 > 0$$

(50)
where $y = \frac{\sqrt{Z}}{C_0}$.

By solving (50), the two solutions are

$$n_+ = \frac{\sqrt{2(Z - Z^2)}}{y^2} - 1 - \sqrt{\left(\frac{2(Z - Z^2)}{y^2} - 1\right)^2 + (1 - y^2)}$$

$$n_- = \frac{\sqrt{2(Z - Z^2)}}{y^2} - 1 + \sqrt{\left(\frac{2(Z - Z^2)}{y^2} - 1\right)^2 + (1 - y^2)}$$

where $\lambda = \frac{Z^2}{C_0^2}, \ y = \frac{\sqrt{Z}}{C_0}, \ \Delta \phi = \frac{\sqrt{\sum_{i=0}^{n-1} \alpha_i^2}}{C_0}$ and $Z = \sum_{i=0}^{n-1} Z_i/n_i$.

$n_+ \leq 0$ in (51), so $n_-$ is not selected. We take

$$n_0 = [n_-]$$

where $[\cdot]$ indicates rounding up. Thus, we can obtain the correct angle $\theta_j$ after at most $n_0$ measurements with an approximate probability of $1 - x$.

4.1.3. Calculating $n_f$

When $\gamma_j = \arcsin \left(\sqrt{p(\mu_i)}\right)$ (see (33)), we define

$$X = \begin{cases} 0, & \text{result of measurement is not } \mu_1 \\ 1, & \text{result of measurement is } \mu_1 \end{cases}$$

where the result $\mu_1$ is obtained by using $M_2$ to measure the quantum state $|\chi_i\rangle$ in (31). Since $X$ is either 1 or 0, $X$ is also a Bernoulli random variable.

The second step is essentially measuring a single quantum state, so we use the method described in Section 2.2 of a previous study [13] to calculate $n_f$ (suppose that we have taken $n_x$ measures on $|\chi_i\rangle$ and $X_1, X_2, \ldots, X_{n_f}$ are $n_f$ samples of $X$)

$$n_f = \frac{\lambda}{\sin^2(\Delta \beta)} - 1 + \sqrt{\left(\frac{2(X - X^2)}{\sin^2(\Delta \beta)} - 1\right)^2 + (1 - \sin^2(\Delta \beta))}$$

where $\lambda = \frac{Z^2}{C_0^2}, \ X = \sum_{i=1}^{\frac{\pi}{2}} \frac{\lambda}{\sin^2(\Delta \beta)}$, and $[\cdot]$ indicates rounding up.

4.1.1. Calculating $n_s$

We define

$$\xi_j = \frac{\pi}{2} - \theta_j, \ \sqrt{p(\xi)} = \frac{2p(\eta_0) - 1}{\sin 2\gamma_j}$$

where $\xi$ is assumed to be a measurement result.

By substituting $\theta_j = \arcsin \left(\frac{p(\eta_0)}{\sqrt{\gamma_j^2 + \gamma_{ij}^2}}\right)$ (see (35)) into (56), we infer

$$\xi_j = \arcsin \left(\frac{2p(\eta_0) - 1}{\sin 2\gamma_j}\right) = \arcsin \left(\sqrt{p(\xi)}\right)$$

Since $\sqrt{p(\xi)} = \frac{2p(\eta_0) - 1}{\sin 2\gamma_j}$, we can state that the result $\eta_0$ with probability $p(\eta_0)$ is equivalent to the result $\xi$ with $p(\xi)$.

Set

$$Y = \begin{cases} 0, & \text{result of measurement is not } \xi \\ 1, & \text{result of measurement is } \xi \end{cases}$$

Fig. 6. Relationships between $\Delta \phi, \ \Delta \phi$, $a_i$, $\hat{a}_i$, $G_a \theta_{\max}$, $G_\phi \theta_{\min}$, and $G_\phi \Delta \theta$. When $\hat{a}_j \in \Delta \theta$ (i.e., $\hat{a}_j \in G_\phi \Delta \theta$), we derive $a_i = \phi_i$ from the figure.
and

\[ W = \begin{cases} 0, & \text{result of measurement is not } \eta_0 \\ 1, & \text{result of measurement is } \eta_0 \end{cases} \]

(59)

where the result \( \eta_0 \) is obtained by using \( M_1 \) to measure the quantum state \( |X_j \rangle \) in (31). Suppose that we have made \( n_k \) measurements on \( |X_j \rangle, Y_1, Y_2, \ldots, Y_n \), and \( W_1, W_2, \ldots, W_n \), for \( n_k \) samples of \( Y \) and \( W \).

Define

\[ Y = \sum_{i=1}^{n_k} Y_i, \quad W = \sum_{i=1}^{n_k} W_i \]

(60)

\[ \xi_j = \arcsin \left( \sqrt{p(\xi)} \right) \text{ compares with } \gamma_j = \arcsin \left( \sqrt{p(\mu_i)} \right), \text{ so substituting } Y \text{ for } X \text{ in (55), we obtain} \]

\[ n_c = \left\lfloor 2 \left( \frac{2(Y - Y_j^2)}{\sin^4(\Delta t)} - 1 \right) + \left( \frac{2(Y - Y_j^2)}{\sin^4(\Delta t)} - 1 \right)^2 + (1 - \sin^4(\Delta t)) \right\rfloor \]

(61)

where \( n_c \) is the number of identical quantum states, and we obtain the correct angle \( \xi_j = \frac{Z}{2}, Y = \sum_{i=1}^{n_k} Y_i \approx p(\xi) = \left( \frac{2p(\mu_i)}{\sin^2 Z} \right)^2 \approx \left( \frac{2p(\mu_i)}{\sin^2 Z} \right)^2 \), and \( \Delta t = |\beta_i - \beta_{i-1}| = \frac{\pi}{2(m-1)} \) (see (5)).

Since \( \xi_j = \frac{Z}{2} - \lambda_j \), we have

\[ n_k = n_c \]

(62)

4.2. Image retrieval for NASS

If a NASS state in (8) is employed to store an image, we simply apply the observable operator \( \tilde{M}_1 \)

\[ \tilde{M}_1 = \sum_{j=0}^{n_k} m_j P_j, \quad P_j = |j><j| \]

(63)

on \( |\psi_A \rangle \), which yields \( m_j \) with probability \( p(m_j) = \langle \psi_A | P_j | \psi_A \rangle = \theta_j^2 \), i.e.,

\[ \theta_j = \sqrt{p(m_j)} \]

(64)

(63) and (64) are comparable with (29) and (30), thus we conclude that the number of identical quantum states needed to obtain the correct angle \( \theta_j \) of a NASS state is also \( n_y = \lfloor n_c \rfloor \), where \( n_c \) is calculated using (52).

4.3. Image retrieval for NASSTC

We define the observable operator \( M_c \)

\[ M_c = \sum_{j=0}^{n_k-1} \left( m_j^c P_j^c + m_j^r P_j^r + m_j^b P_j^b \right) \]

(65)

where \( P_j^c = |j0 \rangle \langle j1|, \quad P_j^r = |j10 \rangle \langle j11|, \quad P_j^b = |j11 \rangle \langle j11| \).

If a NASSTC state \( |\psi_C \rangle \) in (13) is used to represent an image, we simply apply the observable operator \( M_c \) on \( |\psi_C \rangle \), which yields \( m_j^c \) with probability \( p(m_j^c) = \langle \psi_C | P_j^c | \psi_C \rangle = \theta_j^2 \), i.e.,

\[ \theta_j = \sqrt{p(m_j^c)} \]

(66)

Similarly,

\[ \theta_{ji} = \sqrt{p(m_j^r)} \]

(67)

\[ \theta_{ij} = \sqrt{p(m_j^b)} \]

(68)

Suppose that \( n_y, n_{yj} \), and \( n_{yb} \) are the numbers of identical quantum states required to obtain the correct angles \( \theta_j, \theta_{ji}, \text{ and } \theta_{ij} \), respectively. (66)–(68) are comparable with (30), so we can calculate \( n_y, n_{yj} \), and \( n_{yb} \) using (52) and (53).
5. Multidimensional image compression

Image compression solves the problem of reducing the amount of computational resources required to store or reconstruct digital images. Classical image compression (such as Ordered Dither Block Truncation Coding [10]) reduces the redundant data to economize the memory resources. For multidimensional quantum images based on NASS state, the resources required are mainly the number of simple quantum gates used for image storage and the number of identical quantum states used to retrieve an image from a quantum system. These two values are related to the number of items in superposition states NASS or NASSRP. Thus, quantum image compression reduces the number of items in superposition states. Therefore, we define quantum compression ratio (QCR)

$$r = \frac{n_1 \times m_1}{n_2 \times m_2}$$

where $m_1$ and $m_2$ are the numbers of items where the coefficient is not zero in the superposition states that represent the original image and the compressed image, respectively. $n_1$ and $n_2$ are the qubit numbers of the original image and the compressed image, respectively.

5.1. Lossless compression

An image may contain many pixels with the same color, which means that an image has redundant color data. The coordinates in an image are continuous and related to each other, which means that redundant coordinate data exist in an image. We designed Algorithm 1 to reduce these two forms of data redundancy.

To reduce a $k$-dimensional color image to a one-dimensional image, the simplest and most natural approach is to let $v_{x_{ji}} v_{y_{ji}} /C_1 /C_1 /C_1 \ldots /C_1 /C_1 /C_1$ (a one-dimensional image is increased to a $k$-dimensional image in the opposite manner in (8)), but the effect is not good for some images, such as the image in Fig. 8. Thus, we use Algorithm 1 to reduce the dimensions of an image.

Algorithm 1. Dimension reduction and sorting algorithm for a k-dimensional color image (DRS).

1. Analyze a k-dimensional color image and select a plane where the same colors of adjacent pixels are maximized, assuming that the selected plane is spanned by $v_{x_{ji}} v_{y_{ji}} /C_1 /C_1 /C_1 /C_1$.
2. Select an axis of the plane $v_{x_{ji}} v_{y_{ji}} /C_1 /C_1 /C_1 /C_1$ where the same colors of adjacent pixels are maximized, assuming that the selected axis is $v_{x_{ji}}$ or $v_{y_{ji}}$, and transform the plane into a line in the manner shown in (a) or (b) of Fig. 7.
3. Reduce the k-dimensional image to a one-dimensional image by $j v_{y_{ji}} v_{x_{ji}} /C_1 /C_1 /C_1 /C_1 \ldots$ where $j v_{y_{ji}} v_{x_{ji}}$ indicates that the image is sorted initially by $v_{x_{ji}} \ldots v_{x_{j-1}} v_{y_{j-1}} v_{y_{j-1}} v_{x_{j-1}} \ldots$ and then in the manner described in (b) of Fig. 7.

To make the above algorithm explicit, let us consider the $4 \times 2 \times 2$ image shown in (a) of Fig. 8 as an example. We select the plane $v_{x_{ji}} v_{y_{ji}} /C_1 /C_1 /C_1 /C_1$ (i.e., the plane $X \times Z$) shown in (b) of Fig. 8 after Step 1 and the axis $v_{x_{ji}}$ (i.e., X-axis) shown in (c) of Fig. 8 after Step 2. After Step 3, we obtain the state $|\psi_{si} \rangle$ with the sorted items

$$|\psi_{si} \rangle = \sum_{x_0 y_0 z_0} \theta_j |x_0 x_1 y_0 \rangle + \sum_{x_0 y_0 z_0} \theta_j |x_0 x_1 y_0 \rangle$$

where $j = x_0 x_1 y_0$ or $j = x_0 x_1 y_0$. From (c) of Fig. 8, we find that the $\sum_{x_0 y_0 z_0}$ order is

$|000 \rangle \rightarrow |010 \rangle \rightarrow |100 \rangle \rightarrow |110 \rangle \rightarrow |111 \rangle \rightarrow |101 \rangle \rightarrow |011 \rangle \rightarrow |001 \rangle$

Fig. 7. Scanning by $|v_{x_{ji}} \rangle$ axis in (a) and scanning by $|v_{y_{ji}} \rangle$ axis in (b). $|v_{x_{ji}} \rangle$ and $|v_{y_{ji}} \rangle$ axes are all expanded by three binary bits. The pixels are sorted in the direction of the arrow in (a) or (b).
are represents a four-dimensional color image, the axes \( p, q, r, \) and come to a suspension after encountering a different color and counting the number of colors that \( \in (70) \) into \( \{C_0\} \).

Thus, the sorted items of \( |\psi_A\rangle \) are

\[
\begin{align*}
|0000\rangle & \rightarrow |0100\rangle \rightarrow |1000\rangle \rightarrow |1100\rangle \rightarrow |1101\rangle \rightarrow |1001\rangle \rightarrow |0101\rangle \rightarrow |0001\rangle \\
& \rightarrow |0010\rangle \rightarrow |0110\rangle \rightarrow |1110\rangle \rightarrow |1111\rangle \rightarrow |1011\rangle \rightarrow |0111\rangle \rightarrow |0110\rangle \rightarrow |0011\rangle
\end{align*}
\]

If we rewrite \( |\psi_A\rangle = \sum_{j=0}^{15} \theta_j |v_1\rangle |v_2\rangle |v_3\rangle |v_4\rangle \), where \( \theta_j \) \( j = 0, 1, \ldots, 15 \) still represents 16 colors in (a) of Fig. 8, but \( |\psi_A\rangle \) represents a four-dimensional color image, the axes \( |v_1\rangle = |j_1\rangle, |v_2\rangle = |j_2\rangle, |v_3\rangle = |j_3\rangle, |v_4\rangle = |j_4\rangle \), where \( j_1, j_2, j_3, j_4 \in \{0, 1\} \). We select the plane \( |v_1\rangle \times |v_2\rangle \) after Step 1 and the axis \( |v_1\rangle \) shown in (d) of Fig. 8 after Step 2. After Step 3, \( |\psi_A\rangle = \sum_{x=0}^{x_{\max}} \sum_{j_1} |j_1| |j_2| |j_3| |j_4| \), where \( x = j_3 j_4 \) and \( j = j_1 j_2 j_3 j_4 \). The \( \sum_{j_1} \) order is \( |00\rangle \rightarrow |10\rangle \rightarrow |11\rangle \rightarrow |01\rangle \).

**Algorithm 2.** Lossless compression algorithm for quantum images (LCQI).

1. Apply the DRS algorithm (Algorithm 1) to a \( k \)-dimensional color image

\[
|\psi_A\rangle = \sum_{j=0}^{2^k-1} \theta_j |v_1\rangle \cdots |v_k\rangle
\]

to obtain a one-dimensional color image

\[
|\psi_A(r)\rangle = \sum_{j=0}^{2^k-1} \theta_j |j\rangle
\]  

(70)

where \( |j_0\rangle, |j_1\rangle, \ldots, |j_{2^k-1}\rangle \) are the coordinates of the one-dimensional image and \( \theta_j \) denotes the corresponding color.

2. Start from \( |j_0\rangle \) and come to a suspension after encountering a different color and counting the number of colors that are the same (suppose that the number is \( m_0 \)). Convert items \( \sum_{x=0}^{x_{\max}-1} \theta_x |j_x\rangle \) of \( |\psi_A(r)\rangle \) in (70) into

\[
\sum_{x=0}^{x_{\max}-1} \theta_x |j_x\rangle = \sum_{x=0}^{m_0-1} \sqrt{m_0} \theta_x |j_x\rangle = \sqrt{m_0} \theta_0 |j_0\rangle
\]  

(71)

where \( m_0 = 0 (x = 1, 2, \ldots, m_0 - 1) \), the effect of \( \sqrt{m_0} \) makes the converted state remain normal.

3. In order to retrieve the correct items in (71) from a quantum system, by substituting \( j_0 \), \( j_1 \), \( j_2 \), \( j_3 \), \( j_4 \) in \( |j\rangle \) of Eq. (9), we use \( |\lambda_{j_0}\rangle = \cos \frac{\pi}{4} |0\rangle + e^{i\pi} \sin \frac{\pi}{4} |1\rangle \) to indicate the integer \( m_0 \). Thus, we transform (71) into

\[
\sqrt{m_0} \theta_0 |j_0\rangle |0\rangle + e^{i\pi} |1\rangle
\]  

(72)

where the angle \( \lambda_{j_0} \) represents the integer \( m_0 \).

4. \( |j_{m_0}\rangle \) is used as the new starting point and we repeat Step 2 to Step 3.

5. Repeat Step 4 until it encounters \( |j_{2^k-1}\rangle \).
By applying the LCQI algorithm (Algorithm 2) to a \( k \)-dimensional color image represented by \( |\psi_A\rangle = \sum_{j=0}^{2^k-1} \theta_j |v_1\rangle |v_2\rangle \cdots |v_k\rangle \), we achieve a NASSRP state

\[
|\psi_{AP}(c)\rangle = \sum_{h \in A} \tilde{\theta}_h |j_h\rangle |X_h\rangle
\]

(73)

where \( |X_h\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{j \lambda_h} |1\rangle) \), angle \( \lambda_h \) represents the number \( m_h \) of adjacent pixels with the same color. \( A = \{ j_h \mid \tilde{\theta}_h = 0, h = 0, 1, \ldots, 2^k - 1 \} \), \( \tilde{\theta}_h = \sqrt{m_h} \theta_h \) when \( j_h \in A \).

For example, a three-dimensional color image in (a) of Fig. 8 is compressed by the LCQI algorithm (Algorithm 2), where the compressed image is expressed as

\[
|\psi_{AP}(c)\rangle = 2\sqrt{2} (\theta_0 |X_0\rangle + \theta_2 |X_2\rangle + \theta_4 |X_4\rangle + \theta_6 |X_6\rangle) \quad (74)
\]

where \( |X_n\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{j \lambda_n} |1\rangle) \), \( j_n = 0, 2, 4, 6 \). In this case, \( Number = \{ 1, 2, \cdots, 8 \} \) and \( Angle = \{ 0, \frac{\pi}{16}, \cdots, \frac{7\pi}{16} \} \) in (5), so we derive \( \lambda_n = \frac{\pi}{8} \) from (a) of Fig. 8 and \( |X_0\rangle = |X_2\rangle = |X_4\rangle = |X_6\rangle = \frac{\sqrt{2}}{2} (|0\rangle + e^{j \frac{\pi}{8}} |1\rangle) \). Thus, (74) can be rewritten

\[
|\psi_{AP}(c)\rangle = 2 (\theta_0 |0\rangle + \theta_2 |2\rangle + \theta_4 |4\rangle + \theta_6 |6\rangle) (|0\rangle + e^{j \frac{\pi}{8}} |1\rangle)
\]

(75)

To retrieve \( |\psi_{AP}(c)\rangle \) in (73), we simply use Step 1 and Step 3 in Section 4.1 to obtain \( \tilde{\theta}_h \) and \( \lambda_h \).

We calculate that the QCR of the LCQI algorithm (Algorithm 2) for the image in (a) of Fig. 8 from 69, 70, and 75 is \( r = \frac{54.55}{39.23} = 6.7 \).

5.2. Lossless compression simulation experiments

The 256 \( \times \) 256 Lena image shown in (a) of Fig. 9 is represented as

\[
|\psi_A\rangle = \sum_{j=0}^{256-1} \theta_j |v_1\rangle |v_2\rangle = \sum_{j=0}^{256-1} \theta_j |x_0 x_1 \cdots x_7\rangle |y_0 y_1 \cdots y_7\rangle
\]

(76)

where \( j = x_0 x_1 \cdots x_7 y_0 y_1 \cdots y_7, |v_1\rangle = |x_0 \cdots x_7\rangle, |v_2\rangle = |y_0 \cdots y_7\rangle \). We applied the DRS algorithm (Algorithm 1) to transform the image Lena into a one-dimensional image (Step 1, selecting a plane \( |v_1\rangle \times |v_2\rangle \); Step 2, scanning along the axis \( |v_1\rangle \) or \( |v_2\rangle \)), before compressing the one-dimensional image using the LCQI algorithm (Algorithm 2). The distribution of the adjacent pixels with the same color is shown in Fig. 10.

Similarly, we performed tests using the set of eight standard Consultative Committee of International Telegraph and Telephone (CCITT) facsimile binary images [3] shown in Fig. 11 and the images (grayscale and color) shown in Fig. 9. Table 1 shows the lossless compression results obtained using the LCQI algorithm (Algorithm 2). In Table 1, the number of bits is

![Fig. 9. Original images of 256 \( \times \) 256 pixels. From (a) to (e) (grayscale images): Lena, airplane, baboon, couple, and peppers. From (f) to (j) (color images): Lena, airplane, baboon, couple, and peppers. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)](https://example.com/fig9.png)
the size of an image, i.e., the sizes of binary images, grayscale images, and color images are 1728 × 2376 = 4,105,728 × 256 × 24 = 1,572,864, respectively. The number of qubits before coding is the size of the NASS state used to represent an image, e.g., we can employ a 23-qubit NASS state to represent a two-dimensional binary image CCITT8 (1728 × 2376 < 2^{11} × 2^{12}). The number of qubits after coding is the size of the NASSRP state used to represent a compressed image by the LCQI algorithm (Algorithm 2). The DRS algorithm (Algorithm 1) translates a multidimensional image into a one-dimensional image, e.g., CCITT8 (1728 × 2376 = 4,105,728 < 2^{23}), so the size of the NASSRP is 23 qubits, where 22 qubits store the pixels of the compressed image and other 1-qubit stores the number of adjacent pixels with the same colors. For lossless compression, we select the maximum value of QCR after scanning by row and by column, and calculate the average value (i.e., Mean in Table 1).

From the perspective of QCR (see (69)), we conclude that NASSRP and the LCQI algorithm (Algorithm 2) have good effects on binary images, and poor or no effects on grayscale images and color images according to Table 1. Supposing that 1 qubit is equivalent to 1 bit, the classic compression ratios of NASSRP are approximately 2^{23}/(n + 1), 2^{23} × 8/(n + 1), and 2^{23} × 24/(n + 1), which correspond to binary, grayscale, and color images, where 2^{23} is the number of pixels of an image. Thus, the classic compression ratio of NASSRP is most efficient for color images. For example, in Table 1, the classic compression ratios of NASSRP for binary, grayscale, and color images are 4,105,728/23 = 178,510,524, 288/17 = 16,840, and 1,572,864/17 = 92,521, respectively.

We know that the QCR of CCITT8 is 72.6 by scanning along the row, according to Table 1. The distribution of the adjacent pixels with the same colors is shown in Fig. 12. We use the CCITT8 image as an example to describe the quantum compression process for binary images.

Suppose that CCITT8 is stored in a NASS state \(|\psi_{\text{ccitt}}\rangle\) as an example of \(|\psi_{\text{A}}\rangle\) in (8)

\[
|\psi_{\text{ccitt}}\rangle = \sum_{j=0}^{2^{23} - 1} \theta_j |j_1 j_2 \cdots j_{11} j_{12} j_{13} \cdots j_{23}\rangle
\]

where \(j = j_1 j_2 \cdots j_{23}\). The image CCITT8 includes 1,766,467 black and 2,339,261 white pixels, so we conclude that \(\theta_0 = 0\) and \(\theta_j = \frac{1}{\sqrt{1,766,467}}\) correspond to white and black by (2), (6) and (7). Thus, (77) is rewritten as

\[
|\psi_{\text{ccitt}}\rangle = \frac{1}{\sqrt{1,766,467}} \sum_{j=0}^{2^{23} - 1} |j_1 j_2 \cdots j_{11} j_{12} j_{13} \cdots j_{23}\rangle
\]

where \(B = \{j|\theta_j \neq 0, j = 0, 1, \ldots, 2^{23} - 1\}\), the size of set \(B\) is 1,766,467 and \(j = j_1 j_2 \cdots j_{23}\).

By applying the LCQI algorithm (Algorithm 2), we obtain the NASSRP (see (73))

\[
|\psi_{\text{np}}(\text{ccitt8})\rangle = \sum_{j_h \in A} \bar{\theta}_{j_h} |j_h\rangle |\chi_{j_h}\rangle
\]

where \(|\chi_{j_h}\rangle = \sqrt{\frac{1}{2}} (|0\rangle + e^{i\phi_h} |1\rangle)\). angle \(\phi_h\) represents the number \(m_h\) of adjacent pixels with the same color. \(A = \{j_h|\bar{\theta}_{j_h} \neq 0, h = 0, 1, \ldots, 2^{23} - 1\}\), \(\bar{\theta}_{j_h} = \sqrt{m_h} \theta_{j_h}\) when \(j_h \in A\). The maximum of \(m_h\) (i.e., \(m = 3360\)) and the size of set \(A\) (i.e., \(np = 24,329\)) are shown in Fig. 12.

![Fig. 10. Distribution of the adjacent pixels with the same color in the Lena image. (a) Scanning along the axis \(|\psi_2\rangle\), i.e., scanning by row, the maximum number of adjacent pixels with the same color \(m = 6\), the number of pixels in the compressed image \(np = 53,589\), and the compression ratio \(r = 1.1510\). (b) Scanning along the axis \(|\psi_1\rangle\), i.e., scanning by column, \(m = 10\), \(np = 49,175\), and the compression ratio \(r = 1.2543\).](image-url)
We calculate QCR by using (69)

\[ r = \frac{n_1 \times m_1}{n_2 \times m_2} = \frac{23 \times 1,766,467}{23 \times 24,329} \approx 72.6 \]  

(80)

where \( n_1 \) and \( n_2 \) are the sizes of NASS and NASSRP, respectively, and \( m_1 \) and \( m_2 \) correspond to the sizes of \( B \) in (78) and \( A \) in (79).

5.3. Lossy compression

In classic image compression, an image is converted into a new image using a Fourier transform or wavelet transform. Coefficients less than a threshold are forced to zero in the new image. We can also use QFT or QWT to implement lossy compression for images based on NASS states. QFT is the key ingredient of many quantum algorithms (such as quantum factoring and quantum discrete logarithm), so we illustrate how to compress a quantum image by QFT. QFT is defined as

\[ QFT|j\rangle = \frac{1}{\sqrt{N}} \sum_{h=0}^{N-1} e^{2\pi i jh/N} |h\rangle \]  

(81)

where \( |0\rangle, |1\rangle, \ldots, |N-1\rangle \) is an orthonormal basis and \( N = 2^n \). A circuit based on QFT can be constructed using one-bit and two-bit gates where the total number is \( O(\log^2 N) \) (i.e., the complexity of the circuit is \( O(\log^2 N) \)). However, the complexity of the best classical algorithms, which use discrete Fourier transform such as the fast Fourier transform, is \( O(N \log N) \) (see Section 5.1 in a previous study [15]).

We apply QFT to a NASS state \( |\psi\rangle \)

\[ QFT|\psi\rangle = \sum_{j=0}^{N-1} a_j |j\rangle = \sum_{h=0}^{N-1} \psi_h |h\rangle \]  

(82)

Table 1

<table>
<thead>
<tr>
<th>Name of image</th>
<th>Number of bits</th>
<th>Number of qubits</th>
<th>Quantum compression ratio (QCR)</th>
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<td>After coding</td>
<td>Scanning by row</td>
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<tr>
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<td>16</td>
<td>1.1510</td>
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<td>1.1865</td>
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<td>peppers</td>
<td>1,572,864</td>
<td>16</td>
<td>0.9475</td>
</tr>
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Fig. 11. Eight standard CCITT facsimile images of 1728 × 2376. (a), (b), . . . , (h) correspond to CCITT PBM(i = 1, 2, . . . , 8).
We detect the transformed image $\sum_{h=0}^{N-1} y_h |h\rangle$ using the observable operator $\bar{M}_1$ in (63). Suppose that the results of $\zeta$ measurements are $\zeta_h$ times of $|h\rangle$, we make $\zeta_h = 0$ if $\zeta_h/\zeta < \zeta$ where $\zeta$ is a threshold. To ensure that the normal state is retrieved from a quantum system, we define

$$\hat{y}_h = \sqrt{\frac{\zeta_h}{\sum_{j \in D} \zeta_j}}$$

(83)

where $D = \{j | \zeta_j \neq 0, j = 0, 1, \ldots, N-1\}$.

Substituting $\hat{y}_h$ for $y_h$ in (82), we obtain the compressed image $|\psi_{\text{L}}\rangle = \sum_{h \in D} \hat{y}_h |h\rangle$ (84)

By applying quantum the Fourier inverse transform to $|\psi_{\text{L}}\rangle$, we can calculate $\hat{\theta}_j (j = 0, 1, \ldots, N-1)$, which is the estimate of $\theta_j$.

Since we only need to obtain $\hat{y}_h (h \in D)$ by detecting the state $\sum_{h=0}^{N-1} y_h |h\rangle$, the compression rate is

$$r = \frac{N}{\text{dim}(D)}$$

(85)

where $\text{dim}(D)$ is the size of the set $D$.

If the QFT in (82) is replaced with the quantum Haar wavelet transform, which was described previously [8], we can compress a quantum image using QWT.

5.4. Lossy compression simulation experiments

5.4.1. Grayscale images

To make the above description (5.3 Lossy compression) explicit, let us consider the $256 \times 256$ image shown in (a) of Fig. 9 as an example. The process applied to the example comprises the following steps.

Step 1: store the original image in (a) of Fig. 9 in quantum systems as a NASS state.

Step 2: apply QFT to the NASS state $|\psi_{\text{L}}\rangle$ in (88) to compress the image.

Step 3: In Step 1, we use the angles $\{\phi_i\}$ (see (3)) to represent the colors of the original image

$$\phi_i = \frac{f(i)\pi}{2(256 - 1)}$$

(86)

where $f(i)$ is the grayscale value of the pixel at the coordinate $|i\rangle$,

$$\theta_j = \frac{\phi_j}{G_\phi}$$

(87)

where $G_\phi = \sqrt{\sum_{i=0}^{2^8-1} \phi_i^2}$ (e.g., $G_\phi = 209.449$ for the grayscale image Lena.)

By substituting $\theta_j$ of (87) and $N = 2^{16} - 1$ into (8), we obtain the NASS state $|\psi_{\text{L}}\rangle$ that represents the Lena image

$$|\psi_{\text{L}}\rangle = \sum_{j=0}^{2^{16}-1} \theta_j |j\rangle |v_1\rangle |v_2\rangle = \sum_{j=0}^{2^{16}-1} \theta_j |j_1 j_2 \cdots j_8 j_9 j_{10} \cdots j_{16}\rangle |v_1\rangle |v_2\rangle$$

(88)
where $\ket{v_1} = \ket{j_1j_2 \cdots j_{16}}$ and $\ket{v_2} = \ket{j_9j_{10} \cdots j_{16}}$ are the x-axis and y-axis of the two-dimensional image, respectively. $j_1j_2 \cdots j_{16}$ is the binary expansion for the integer $j$. Next, we create the quantum state using the quantum circuit in Fig. 2.

In Step 2, we execute Eqs. (82)–(84) to complete the compression and decompression based on QFT.

Using the same procedure, we compressed the grayscale images in Fig. 9 by applying QFT or QWT. The simulation results and the compression performance are shown in Fig. 13 and Table 2. We can see that QFT and QWT allowed the NASS states to represent grayscale images.

### 5.4.2. Color images

Similarly, we substitute $\phi_i = \frac{i\pi}{\sqrt{2^{24} - 1}}$ (see (4)) for $\phi_i = \frac{f(i)\pi}{\sqrt{2^{256} - 1}}$ (see (86)) and obtain a NASS state that represents the color image Lena shown in (f) of Fig. 9.

$$
\ket{\psi_{CL}} = \sum_{j=0}^{2^{16} - 1} \theta_j \ket{v_1} \ket{v_2} = \sum_{j=0}^{2^{16} - 1} \theta_j \ket{j_1j_2 \cdots j_{16}} \ket{\overline{j_9j_{10} \cdots j_{16}}} \tag{89}
$$

where $\theta_j = \phi_j G_0$, $G_0 = \sqrt{\sum_{i=0}^{2^{16} - 1} \phi_i^2}$, $\phi_i = \frac{i\pi}{\sqrt{2^{24} - 1}}$.

The results shown in Fig. 14 were obtained after applying QTT and QWT to the NASS state $\ket{\psi_{CL}}$, which show that QFT and QWT are not efficient for allowing a NASS state to represent a color image. This shows that the sorting method based on RGB colors using (4) is not efficient for the lossy quantum compression of color images.

QFT and QWT had beneficial effects on grayscale images (see Table 2 and Fig. 13) and RGB color can be view as three grayscale, so we can use a NASSTC state $\ket{\psi_C}$ (see (13)) to represent a color image.

For clarity, we present the equation for NASSTC, as follows:

$$
\ket{\psi_C} = \frac{1}{\sqrt{3}} \ket{\psi_{Ar}} \ket{01} + \frac{1}{\sqrt{3}} \ket{\psi_{Ag}} \ket{10} + \frac{1}{\sqrt{3}} \ket{\psi_{Ab}} \ket{11} = \left( \sum_{j=0}^{2^n - 1} \theta_r \ket{j} \right) \ket{01} + \left( \sum_{j=0}^{2^n - 1} \theta_g \ket{j} \right) \ket{10} + \left( \sum_{j=0}^{2^n - 1} \theta_b \ket{j} \right) \ket{11}
$$

Fig. 13. Examples of 256 $\times$ 256 grayscale images after compression.
where
\[ \omega_{Ar} = \frac{P}{C_0^1/j_0} \left( \sqrt{3} \theta_j j \right); \]
\[ \omega_{Ag} = \frac{P}{C_1^2/C_1^2/C_1^1 j_0} \left( \sqrt{3} \theta_j j \right); \]
\[ \omega_{Ab} = \frac{P}{C_0^1/j_0} \left( \sqrt{3} \theta_j j \right) \text{ (see (12))}, \]
\[ h_{rj} = \frac{r_j G_{rgb}}{256}; \]
\[ h_{gj} = \frac{g_j G_{rgb}}{256}; \]
\[ h_{bj} = \frac{b_j G_{rgb}}{256}; \]
\[ G_{rgb} = \sqrt{\frac{2^{2s-1}}{\sum_{i=0}^{2s-1} (r_i^2 + g_i^2 + b_i^2)}} \text{ (see (11))}, \]
\[ r_j = \frac{y_1}{C_0^1}; \]
\[ g_j = \frac{y_2}{C_0^1}; \]
\[ b_j = \frac{y_3}{C_0^1} \text{ (see (10))}, \]
and \( y_1, y_2, \) and \( y_3 \) are the grayscale values of three components of the RGB color of the pixel at the coordinate \( j \).

By applying QT and QWT to the NASSTC state \( |\psi_{Ar}\rangle, \) we can compress the color images shown in Fig. 9. The simulation results and the compression performance are shown in Fig. 15 and Table 3, which demonstrate that QT and QWT allowed

Table 3

<table>
<thead>
<tr>
<th>Performance of QT and QWT with the color images in Fig. 9.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of bits</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>Lena</td>
</tr>
<tr>
<td>Airplane</td>
</tr>
<tr>
<td>Baboon</td>
</tr>
<tr>
<td>Couple</td>
</tr>
<tr>
<td>Peppers</td>
</tr>
<tr>
<td>Mean</td>
</tr>
<tr>
<td>Standard deviation</td>
</tr>
</tbody>
</table>
the NASSTC state to represent a color image. However, the size of the NASSTC state was 2 qubits more than the NASS state for the same image. For a RGB color image with $2^n$ pixels, the normalized color difference (NCD) is defined

$$NCD = \frac{\sum_{i=0}^{3^n-1} \sqrt{(O^1_{i0} - C^1_{i0})^2 + (O^2_{i0} - C^2_{i0})^2 + (O^3_{i0} - C^3_{i0})^2}}{\sum_{i=0}^{3^n-1} \sqrt{(O^1_{i0})^2 + (O^2_{i0})^2 + (O^3_{i0})^2}}$$

(90)

where $O^1_{i0}$, $O^2_{i0}$, and $O^3_{i0}$ are three components of the RGB color of the pixel at coordinate $|i\rangle$ in the original image. $C^1_{i0}$, $C^2_{i0}$, and $C^3_{i0}$ are the three corresponding components in the compressed image. The NCD criterion is added in Table 3.

6. Conclusion

This study shows that an n-qubit NASS state can represent a k-dimensional color image with $2^n$ pixels. Suppose that 1 qubit is equal to 1 bit, from the perspective of the classic compression ratio (i.e., memory), the maximum compression ratio of NASS is $2^n \times 24/n$. In a quantum computer, we applied NASSRP and the LCQI algorithm (Algorithm 2) to eight standard CCITT facsimile images to obtain quantum lossless compression. The simulation results showed that the mean QCR was 15.93. Unfortunately, NASSRP had little or no effect on grayscale and color images using lossless QCR. Thus, we applied QFT and QWT to NASS states that represented the grayscale images shown in Fig. 9. The simulation results showed that the mean PSNR was 28.7 with QFT and 31.25 with QWT, when the mean QCR was approximately 10. Unfortunately, NASS had no effects on color images with lossy QCR. However, we applied NASSTC to the color images in Fig. 9 to achieve quantum lossy compression. The simulation results showed that the mean PSNR was 30.45 with QFT and 31.25 with QWT, when the mean QCR was approximately 10. We also retrieved images from a quantum system using different projection measurement operators for NASS, NASSRP, and NASSTC. Thus, we conclude that NASS, NASSRP, and NASSTC are effective for image storage, retrieval, and compression.

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References


