On the orientable regular embeddings of complete multipartite graphs

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\begin{abstract}
Let $K_{m[n]}$ be the complete multipartite graph with $m$ parts, while each part contains $n$ vertices. The regular embeddings of complete graphs $K_{m[1]}$ have been determined by Biggs (1971) \cite{1}, James and Jones (1985) \cite{12} and Wilson (1989) \cite{23}. During the past twenty years, several papers such as Du et al. (2007, 2010) \cite{6,7}, Jones et al. (2007, 2008) \cite{14,15}, Kwak and Kwon (2005, 2008) \cite{16,17} and Nedela et al. (2002) \cite{20} contributed to the regular embeddings of complete bipartite graphs $K_{2[n]}$ and the final classification was given by Jones \cite{13} in 2010. Since then, the classification for general cases $m \geq 3$ and $n \geq 2$ has become an attractive topic in this area. In this paper, we deal with the orientable regular embeddings of $K_{m[n]}$ for $m \geq 3$. We in fact give a reduction theorem for the general classification, namely, we show that if $K_{m[n]}$ has an orientable regular embedding $\mathcal{M}$, then either $m = p$ and $n = p^k$ for some prime $p \geq 5$ or $m = 3$ and the normal subgroup $\text{Aut}_+^2(\mathcal{M})$ preserving each part setwise is a direct product of a $3$-subgroup $Q$ and an abelian $3'$-subgroup, where $Q$ may be trivial. Moreover, we classify all the embeddings when $m = 3$ and $\text{Aut}_+^2(\mathcal{M})$ is abelian. We hope that our reduction theorem might be the first necessary approach leading to the general classification.
\end{abstract}

\section{Introduction}

A map is a 2-cell embedding of a connected graph into a closed surface. The embedded graph is called the \textit{underlying graph} of the map. An \textit{automorphism} of a map is an automorphism of the underlying graph which can be extended to a self-homeomorphism of the supporting surface. It is
well known that the automorphism group of a map acts freely on the set of flags (that is, triples of mutually incident i-cells, \(0 \leq i \leq 2\)). If it acts regularly, then the map is called regular.

For an embedding \(\mathcal{M}\) on orientable surface, we use \(\text{Aut}^+ (\mathcal{M})\) to denote the group of all orientation-preserving automorphisms of \(\mathcal{M}\). If \(\text{Aut}^+ (\mathcal{M})\) acts regularly on the arcs, then we call \(\mathcal{M}\) an orientable regular map. Such maps fall into two classes: those that admit also orientation-reversing automorphisms, called reflexible, and those that do not, called chiral.

One of the central problems in topological graph theory is to classify all regular or orientable regular embeddings of a given class of graphs. In a general setting, the classification problem was treated by Gardiner et al. in [10]. However, for particular classes of graphs, it has been solved only in a few cases. Let \(K_{m|n}\) be the complete multipartite graph with \(m\) parts, while each part contains \(n\) vertices and two vertices are adjacent if and only if they belong to the different parts. All the regular embeddings of complete graphs \(K_{m|1}\) have been determined by Biggs, James and Jones [1,12] for orientable case and by Wilson [23] for nonorientable case. As for the complete bipartite graphs \(K_{2|n}\), the nonorientable regular embeddings of these graphs have recently been classified by Kwak and Kwon [18]; during the past twenty years, several papers [6,7,14–17,20] contributed to the orientable case, and the final classification was given by Jones [13] in 2010. Since then, the classification for general case \(m \geq 3\) and \(n \geq 2\) was started. The only known result is the determination of orientable regular embeddings of the graphs \(K_{m|p}\) (where \(p\) is a prime) given by Du et al. in [8].

In this paper, we shall focus on the orientable regular embeddings, which are simply called regular embeddings. In general, to classify the regular embeddings \(\mathcal{M}\) of a given graph, one has to first analyze the possible group structure of \(\text{Aut}^+ (\mathcal{M})\). We have noted that in the classification of regular embeddings of \(K_{2|n}\), the key point is a determination of the so called isobicyclic groups \(H = \langle x \rangle \langle y \rangle\), where \(|x| = |y| = n\), \((x) \cap \langle y \rangle = 1\) and \(x^\alpha = y\) for an involution \(\alpha \in \text{Aut}(H)\). Therefore, to classify the regular embeddings of \(K_{m|n}\) for \(m \geq 3\) and \(n \geq 2\), one should first analyze the structure of \(\text{Aut}^+ (\mathcal{M})\) and then obtain a reduction theorem. Basing on the reduction, one may eventually give the final classification. Our following main theorem might be the first necessary approach leading to the general classification.

**Theorem 1.1.** Let \(\mathcal{M}\) be an orientable regular embedding of \(K_{m|n}\) where \(m \geq 3\) and \(n \geq 2\), and let \(\text{Aut}^+_0 (\mathcal{M})\) be the normal subgroup of \(\text{Aut}^+ (\mathcal{M})\) consisting of automorphisms preserving each part setwise. Then \(\text{Aut}^+_0 (\mathcal{M})\) is an isobicyclic group. Moreover, we have

1. if \(m \geq 4\), then \(m = p\) and \(n = p^n\) for some prime \(p\); or
2. if \(m = 3\), then \(\text{Aut}^+_0 (\mathcal{M}) = Q \times K\), where \(Q\) is a 3-subgroup (may be trivial) and \(K\) is an abelian 3′-subgroup. In particular, when \(\text{Aut}^+_0 (\mathcal{M})\) is abelian, there is only one such map if \(3 \nmid n\) and there are three if \(3 \mid n\).

The paper is organized as follows. After this introduction section, some notations, terminologies and preliminary results will be given in Section 2; some group theoretical results used later will be proved in Section 3; the cases \(m \geq 4\) and \(m = 3\) will be discussed in Sections 4 and 5, separately. Finally, the proof of Theorem 1.1 can be summarized immediately from Sections 4 and 5.

### 2. Preliminaries

Throughout this paper, all graphs are finite, simple and undirected. For a graph \(\Gamma\), we use \(V(\Gamma)\) and \(E(\Gamma)\) to denote the vertex set and the edge set of \(\Gamma\) respectively. For any positive integer \(n\), let \([n] = \{1, \ldots, n\}\). For two integers \(s\) and \(t\), we use \(\gcd(s, t)\) to denote the greatest common divisor of them. For a finite group \(G\) and a positive integer \(s\), let \(G^s = \langle g^s | g \in G \rangle\). For a ring \(S\), we use \(S^s\) to denote the multiplicative group of \(S\). The center of a group \(G\) will be denoted by \(Z(G)\). The dihedral group of order \(n\) will be denoted by \(D_n\), and the cyclic group of order \(n\) as well as the integer residue ring modulo \(n\) will be denoted by \(\mathbb{Z}_n\). When we denote the quotient group \(G/N\) by \(\mathbb{Z}_n\), we use the standard ‘bar’ convention, in which the overbar denotes the canonical homomorphism from \(G\) onto \(\mathbb{Z}_n\) (thus \(\overline{g} = gn\) and \(\overline{H} = H/N\) for every element \(g \in G\) and every subgroup \(H\) with \(N \leq H \leq G\)).
Lemma 2.3. \( \text{gcd} \{ i, p \} = \text{gcd} \{ l, p \} = 1 \) and \( p^{e-d} \mid k \). Then by the choices of \( i, j, k \) and \( l \), we have \( |\text{Aut}(G)| = p^{3d+e-2}(p - 1)^2 \).

It is well known that the automorphism group \( G = \text{Aut}^+(\mathcal{M}) \) of a regular map is generated by a generator \( a \) of the stabilizer (which is necessarily cyclic) of a vertex, say \( \gamma \) and by an involution \( b \) inverting the direction of an edge incident with \( \gamma \), see [10]. Moreover, the embedding is determined by the group \( G \) and the choice of generators \( a \) and \( b \) [19,9]. A regular map given by \( G = \langle a, b \rangle \), with \( b^2 = 1 \), is called an algebraic map \( \mathcal{M}(G; a, b) \). Two algebraic maps \( \mathcal{M}(G; a, b) \) and \( \mathcal{M}(G; a', b') \) are isomorphic if and only if there is a group automorphism in \( \text{Aut}(G) \) taking \( a \mapsto a' \) and \( b \mapsto b' \). If the order of \( ab \) and \( a \) are \( s \) and \( t \) respectively, then \( \mathcal{M}(G; a, b) \) has type \( \{s, t\} \) in the notation of Coxeter and Moser [3], meaning that the faces are all \( s \)-gons and the vertices all have valency \( t \).

Now we introduce some Propositions and Lemmas which will be used later.

**Proposition 2.1** ([11, I.4.5]). Let \( G \) be a group and \( H \leq G \). Then \( N_G(H)/C_G(H) \) is isomorphic to a subgroup of \( \text{Aut}(H) \).

**Proposition 2.2** ([4]). Let \( p \) be an odd prime. Then

1. the maximal subgroups of the projective special linear group \( \text{PSL}(2, p) \) are: one class of subgroups isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_{p-1} \); one class isomorphic to \( \mathbb{D}_{p-1} \), when \( p \geq 13 \); one class isomorphic to \( \mathbb{D}_{p+1} \), when \( p \neq 7 \); two classes isomorphic to \( A_5 \) when \( p \equiv \pm 1 \pmod{10} \); two classes isomorphic to \( S_4 \), when \( p \equiv \pm 1 \pmod{8} \); and one class isomorphic to \( A_4 \), when \( p = 5 \) or \( p \equiv 3, 13, 27, 37 \pmod{40} \) and \( p \geq 5 \).

2. the maximal subgroups of the projective general linear group \( \text{PGL}(2, p) \) are: one class of subgroups isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_{p-1} \); one class isomorphic to \( \mathbb{D}_{2(p-1)} \), when \( p \geq 7 \); one class isomorphic to \( \mathbb{D}_{2(p+1)} \); one class isomorphic to \( S_4 \), when \( p = 5 \) or \( p \equiv 3, 13, 27, 37 \pmod{40} \) and \( p \geq 5 \); and one subgroup \( \text{PSL}(2, p) \).

**Lemma 2.3.** Let \( p \) be a prime. Then we have the following conclusions:

1. If \( \text{AGL}(1, m) \) is isomorphic to a subgroup of \( \text{GL}(2, p) \) for a prime power \( m \geq 3 \), then either \( m = 3 \) or \( m = p \);
2. If \( N \unlhd H \unlhd \text{GL}(2, p) \) and \( H/N \cong \text{AGL}(1, q) \) for some prime \( q \), then \( p = q \) and \( N = Z(H) \).

**Proof.** Set \( Z = Z(\text{GL}(2, p)) \). Then \( \text{GL}(2, p)/Z = \text{PGL}(2, p) \)

(1) Let \( K \) be a subgroup of \( \text{GL}(2, p) \) isomorphic to \( \text{AGL}(1, m) \). Then \( Z(K) = 1 \) and hence \( K \cap Z = 1 \). It follows that \( K \cong KZ/Z \leq G/Z \) and then \( \text{AGL}(1, m) \cong \text{PGL}(2, p) \). By checking Proposition 2.2, we have \( m = 3, 4, \) or \( p \). Now we only need to show that \( m \neq 4 \). Suppose to the contrary that \( m = 4 \), then \( K \cong A_4 \). Clearly, \( K \) contains three elements of order 2, one of which is contained in \( \text{SL}(2, p) \). However, \( \text{SL}(2, p) \) contains only one involution, which is the center involution, a contradiction.

(2) Clearly,

\[
\text{NZ}/Z \leq HZ/Z \leq \text{PGL}(2, p)
\]

and

\[
(HZ/Z)/(NZ/Z) \cong H/N \cong \text{AGL}(1, q).
\]

Then by checking Proposition 2.2 again, we get \( \text{AGL}(1, q) \cong HZ/Z \cong \text{AGL}(1, p) \). Therefore, \( q = p \) and \( Z(H) = N \). \( \Box \)

**Lemma 2.4.** Let \( G \cong \mathbb{Z}_{p^e} \times \mathbb{Z}_{q^d} \) where \( p \) is a prime and \( e > d \). Then,

1. \( \text{Aut}(G) \) is a 2-group when \( p = 2 \);
2. every \( p^e \)-subgroup of \( \text{Aut}(G) \) is abelian when \( p \) is odd.

**Proof.** Suppose that \( G = \langle a \rangle \times \langle b \rangle \) where \( |\langle a \rangle| = p^e \) and \( |\langle b \rangle| = p^d \). One can check that the mapping defined by \( a \mapsto a^i b^j \) and \( b \mapsto a^k b^l \) for some \( i, k \in \mathbb{Z}_{p^e} \) and \( j, l \in \mathbb{Z}_{q^d} \) is an automorphism of \( G \) if and only if \( \text{gcd}(i, p) = \text{gcd}(l, p) = 1 \) and \( p^{e-d} \mid k \). Then by the choices of \( i, j, k \) and \( l \), we have \( |\text{Aut}(G)| = p^{3d+e-2}(p - 1)^2 \).
Clearly, if $p = 2$, then $\text{Aut}(G)$ is a 2-group. If $p$ is odd, then $\text{Aut}(G)$ has an abelian Hall $p'$-subgroup $F$ which is contained in 
\[ \langle \alpha \in \text{Aut}(G) | \alpha(a) = a, \alpha(b) = a, \gcd(i, p) = \gcd(l, p) = 1 \rangle. \]
The theorem in [21, 9.1.10] tells us that every $p'$-subgroup of $\text{Aut}(G)$ is contained in a conjugate of $F$ and so it is abelian. □

3. Isobicycle groups

As mentioned before, if $H$ is a group with cyclic subgroups $X = \langle x \rangle$ and $Y = \langle y \rangle$ of order $n$ such that $H = XY, X \cap Y = 1$ and there is an automorphism $\alpha$ of $H$ transposing $x$ and $y$, then the group $H$ or the triple $(H, x, y)$ is said to be $n$-isobicyclic (or isobicyclic for brevity). In this section, by using some known results we shall deduce some properties of isobicyclic groups.

Lemma 3.1. Let $(H, x, y)$ be a $n$-isobicyclic triple. Then $H$ has a characteristic series
\[ 1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_l = H \]
of subgroups $H_i = H^i = \langle x^i \rangle \langle y^i \rangle$ with $H_i/H_{i-1} \cong \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$ for all $i \in [l]$, where $p_1 \geq \cdots \geq p_l$ are the prime divisors of $n$ and $s_i = n/(p_1 \cdots p_i)$.

Proof. We proceed the proof by induction on the order of $H$.

Suppose that $p$ is the maximal prime divisor of $n$ and $P$ is a Sylow $p$-subgroup of $H$. A result of Wielandt [22] on products of nilpotent groups shows that $P \triangleleft H$ and hence $P$ is a characteristic subgroup of $H$. Since $P$ is the unique Sylow $p$-subgroup of $H$, we have $P = H^{n/p^d}$ where $p^d$ is the highest power of $p$ dividing $n$. Noting that $\langle x^{n/p^d} \rangle \cap \langle y^{n/p^d} \rangle = 1$ and $|\langle x^{n/p^d} \rangle \langle y^{n/p^d} \rangle| = p^{2d} = |P|$, we have $P = H^{n/p^d} = \langle x^{n/p^d} \rangle \langle y^{n/p^d} \rangle^1$. Clearly, $(P, x^{n/p^d}, y^{n/p^d})$ is a $p^d$-isobicyclic triple. By Lemma 3 in [14], $P$ has a central series $1 = Z_0 \triangleleft Z_1 \triangleleft Z_{d-1} \triangleleft Z_d = P$ of subgroups $Z_i = \langle x^{n/p^d} \rangle \langle y^{n/p^d} \rangle^{d-i}$ with $Z_i/Z_{i-1} \cong \mathbb{Z}_n \times \mathbb{Z}_p$.

Now we consider the quotient group $\overline{H} = H/P$. By induction hypothesis, $\overline{H}$ has a characteristic series
\[ \overline{T} = \overline{N}_0 < \overline{N}_1 < \cdots < \overline{N}_j = \overline{H} \]
of subgroups $\overline{N}_i = \overline{H}^i = \langle \overline{x}^i \rangle \langle \overline{y}^i \rangle$ with $\overline{H}_i/\overline{H}_{i-1} \cong \mathbb{Z}_{q_i} \times \mathbb{Z}_{q_i}$ for all $i \in [j]$, where $q_1 \geq \cdots \geq q_j$ are the prime divisors of $n/p^d$ and $t_i = n/p^d(q_1 \cdots q_j)$. Set
\[ p_i = \begin{cases} p, & 0 \leq i \leq d; \\ q_i-d, & d < i \leq d+j \end{cases} \quad \text{and} \quad H_i = \begin{cases} Z_i, & 0 \leq i \leq d; \\ N_{i-d}, & d < i \leq d+j. \end{cases} \]

Then $p_1 \geq \cdots \geq p_{d+j}$ are the prime divisors of $n$. Write $s_i = n/(p_1 \cdots p_i)$. Then $H_i = H^{n/p^d} = \langle x^{n/p^d} \rangle \langle y^{n/p^d} \rangle$ and $1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_{d+j} = H$ is a characteristic series of $H$ with $H_i/H_{i-1} \cong \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$ for all $i \in [d+j]$. □

Lemma 3.2. Suppose that $(H, x, y)$ is a $n$-isobicyclic triple and $p$ is the maximal prime divisor of $n$. Let $L = H^{n/p^d}$. Then $H/C_H(L)$ is an isobicyclic group.

Proof. By Lemma 3.1, $L = \langle x^{n/p^d} \rangle \langle y^{n/p^d} \rangle = \langle x^{n/p} \rangle \times \langle y^{n/p} \rangle$. Let $t$ be the minimal positive integer such that $x^t y^{n/p} = y^{n/p} x^t$. Since there is an automorphism $\alpha$ of $H$ transposing $x$ and $y$, we also have $y^t x^{n/p} = x^{n/p} y^t$. Hence $\langle x^t, y^t \rangle \leq C_H(L)$. On the other hand, taking any $x^i y^j \in C_H(L)$, from $x^i y^j x^{n/p} = x^{n/p} x^i y^j$, we obtain $y^j x^{n/p} = x^{n/p} y^j$. Let $d = \gcd(t, j)$. Then there exist two integers $m$ and $k$ such that $d = mt + kj$. Therefore
\[ y^d x^{n/p} = y^{mt+kj} x^{n/p} = x^{n/p} y^{mt+kj} = x^{n/p} y^d. \]

By the minimality of $t$, we get $t = d$ and then $t | j$. Symmetrically, $t | i$ and hence $x^i y^j \in \langle x^t, y^t \rangle$. It follows that $C_H(L) = \langle x^t, y^t \rangle$. 

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Let\( \gamma \) be a non-abelian \( p^n \)-isobicyclic triple. Then \( H/H' \) is an inhomogeneous abelian group of rank 2.

4. Case \( m \geq 4 \)

The main result of this section is the following theorem.

**Theorem 4.1.** Let \( \mathcal{M} \) be a regular embedding of \( K_{m|n} \), where \( m \geq 4 \) and \( n \geq 2 \), and let \( \text{Aut}_0^+ (\mathcal{M}) \) be the kernel of \( \text{Aut}^+ (\mathcal{M}) \) on the set of \( m \) parts. Then \( m = p \) and \( n = p^e \) for some prime \( p \geq 5 \). Moreover, \( Z(\text{Aut}^+ (\mathcal{M})) = 1 \) and \( \text{Aut}_1^+ (\mathcal{M}) \) is a \( n \)-isobicyclic group.

**Proof.** To prove the theorem, set \( \Gamma = K_{m|n} \), with the vertex set

\[
V(\Gamma) = \bigcup_{i=1}^{m} \Delta_i, \quad \text{where} \quad \Delta_i = \{ \gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{im} \}
\]

and the edges are all pairs \( \{ \gamma_{ij}, \gamma_{ik} \} \) of vertices with \( i \neq k \). Then \( \text{Aut}(\Gamma) = S_n : S_m \), which has blocks \( \Delta_i \) for \( 1 \leq i \leq m \).

Set \( H = \text{Aut}_0^+ (\mathcal{M}) \) and \( G = \text{Aut}^+ (\mathcal{M}) = \langle a, b \rangle \), where \( \langle a \rangle = G_{\gamma_{11}} \) and \( b \) reverses the arc \( (\gamma_{11}, \gamma_{21}) \). Let \( x = a^{m-1} \) and \( y = x^b \). Then \( H = \langle x, y \rangle \). Write \( \overline{G} = G/H \) and we use \( \overline{T} \) to denote the quotient (block) graph of \( \Gamma \) induced by \( H \). Clearly, \( \overline{T} \cong K_m \).

Then we prove the theorem by the following four steps:

**Step 1.** Show that \( m \) is a prime power, \( \overline{G} \cong AGL(1, m) \) and \( H \) is a \( n \)-isobicyclic group.

By considering the order of \( G \), we know that \( |H| = n^2 \) and \( \overline{G} \) acts arc-regularly on \( \overline{T} \). From the classification of regular embeddings of \( K_m \), \( m \) is a prime power and \( \overline{G} \cong AGL(1, m) \) (see [1,12]). Since \( \langle x \rangle \leq H_{\gamma_{11}} \) and \( \langle y \rangle \leq H_{\gamma_{21}} \), we have \( \langle x \rangle \bigcap \langle y \rangle = H_{(\gamma_{11},\gamma_{21})} = 1 \). Noting that \( x^b = y, y^b = x \) and \( |\langle x \rangle||\langle y \rangle| = n^2 = |H| \), we have \( H = \langle x \rangle \langle y \rangle \) is a \( n \)-isobicyclic group.
Step 2. Show that $C_G(H_i) = C_H(H_i)$ and $C_{G/H_i}(H/H_i) = Z(H/H_i)$, where $H_i = H^{s_i}$ for $s_i = n/(p_1 \cdots p_i)$ and $n = p_1 \cdots p_i$ where $p_1 \geq \cdots \geq p_i$ are primes.

Taking any $g \in G \setminus H$, there exists $k \in [m]$ such that $A^g_k \neq A_k$. Write $H_{\gamma_k} = \langle z \rangle$. Clearly, $(H, z, z^g)$ is a $n$-isocyclic triple. By Lemma 3.1,

$$1 = H_0 < H_1 < \cdots < H_i = H$$

is a series of characteristic subgroups of $H$ and $H_i = \langle z^i \rangle \langle (z^g)^i \rangle$ with $H_i/H_{i-1} \cong \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_j}$ for all $i \in [l]$. Therefore $z^i \neq (z^g)^i$ and $zH_i \neq z^gH_i$ for all $i \in [l]$. It follows that $g \notin C_G(H_i)$ and $gH_i \notin C_{G/H_i}(H/H_i)$, from which we have $C_G(H_i) = C_H(H_i)$ and $C_{G/H_i}(H/H_i) = Z(H/H_i)$.

Step 3. Show that $m = p$ and $Z(G) = 1$ where $p$ is the minimal prime divisor of $n$.

Let $p$ be the minimal prime divisor of $n$. Set $N = H^p$. By Lemma 3.1, $N = \langle x^p \rangle \langle y^p \rangle$ is a characteristic subgroup of $H$ with $H/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and so $N \triangleleft G$. As shown in Step 2, $C_{G/N}(H/N) = Z(H/N) = H/N$.

Then from Proposition 2.1, we have

$$\text{AGL}(1, m) \cong \tilde{G} \cong (G/N)/(H/N) \triangleleft \text{Aut}(H/N) \cong \text{GL}(2, p).$$

Now by Lemma 2.3(1), we get $m = p$ and then $\tilde{G} \cong \text{AGL}(1, p)$.

Since $Z(G) \cong Z(\text{AGL}(1, p)) = 1$, we have $Z(G) \leq H$. Suppose that $x^iy^j \in Z(G)$ for some $i, j \in [n]$. From $x = a^{n-1}$, we have $[y^i, a] = [x^iy^j, a] = 1$ and hence $(y^a)^i = (y^i)^a = y^i$. Noting that $(y \bigcap (y^p)) = 1$, we get $j = n$ and then $x^i = x^iy^j \in Z(G)$. Since $x^{-i}y^j = [x^i, b] = 1$, we get $x = y^j \in \langle x \rangle (y^p) = 1$ and then $i = n$. Therefore $Z(G) = 1$.

Step 4. Show that $n = p^e$ for some $e$.

By Step 3, $m = p$ is the minimal prime divisor of $n$. Let $q$ be the maximal prime divisor of $n$ and set $J = H^{n/q}$. By Lemma 3.1, $J \cong \mathbb{Z}_q \times \mathbb{Z}_q$ is a characteristic subgroup of $H$ and then $J \triangleleft C_G(J)$. Set $L = C_G(J)$. Then the Step 2 implies that $L = C_H(J)$. By Proposition 2.1, $G/L \leq \text{Aut}(J) \cong \text{GL}(2, q)$. On the other hand, $(G/L)/(H/L) \cong G/H \cong \text{AGL}(1, p)$. By Lemma 2.3(2), we have $q = p$, that is, $n = p^e$ for some integer $e$. □

Remark 4.2. It is easy to see from the proof that the conclusions in Steps 1–3 of Theorem 4.1 hold for $m = 3$ as well.

Proposition 4.3. For each pair of admissible parameters $(p, p^e)$, there exists at least one regular embedding of $K_{p[p^e]}$.

Proof. We prove the proposition by constructing a family of regular embeddings of $K_{p[p^e]}$ as follows.

Suppose that $p \geq 5$ is a prime. We identify $\mathbb{Z}_{p^e+1}$ with the set

$$\{0, 1, 2, \ldots, p^e+1 - 1\}.$$

Let

$$A_i = \{jp + il | j = 0, 1, 2, \ldots, p^e - 1\}$$

for $i = 0, 1, \ldots, p - 1$.

Then we have $\mathbb{Z}_{p^e+1} = A_0 \cup A_1 \cup \cdots \cup A_{p-1}$. Now we identify the vertex set of $K_{p[p^e]}$ with $\mathbb{Z}_{p^e+1}$ and its edge set with $E = \{\{\alpha, \beta\} | \alpha, \beta \in \mathbb{Z}_{p^e+1}, \alpha \neq \beta\}$. Clearly, $A_0, A_1, \ldots, A_{p-1}$ are the $p$ parts of $K_{p[p^e]}$.

Let

$$G = \mathbb{Z}_{p^e+1} \times \mathbb{Z}_{p^e+1} = \{(\pi, \tau) | \pi \in \mathbb{Z}_{p^e+1}, \tau \in \mathbb{Z}_{p^e+1}\},$$

and $(\pi, \tau)(\mu, \nu) = (\pi \nu + \mu, \tau \nu)$, for all $(\pi, \tau), (\mu, \nu) \in G$. Then define an action of $G$ on $\mathbb{Z}_{p^e+1}$ by $\alpha(\pi, \tau) = \alpha \pi + \tau$ for all $\alpha \in \mathbb{Z}_{p^e+1}$ and $(\pi, \tau) \in G$. It is easy to verify that this is indeed an faithful action of $G$ on $\mathbb{Z}_{p^e+1}$. Noting that

$$p | \alpha - \beta \iff p | (\alpha \tau + \pi) - (\beta \tau + \pi) \iff p | \alpha^{(\pi, \tau)} - \beta^{(\pi, \tau)}$$

for all $(\pi, \tau) \in G$, we have

$$\{\alpha, \beta\} \in E \iff \{\alpha^{(\pi, \tau)}, \beta^{(\pi, \tau)}\} \in E,$$

and hence $G$ is a subgroup of $\text{Aut}(K_{p[p^e]})$.  

It is well known that $\mathbb{Z}_{p^e+1}^*$ is a cyclic group of order $p^e(p - 1)$. Set $\mathbb{Z}_{p^e+1}^* = \langle \theta \rangle$ for some $\theta \in \mathbb{Z}_{p^e+1}^*$. Let $a = (0, \theta), b = (1, -1)$. Clearly $0^a = 0, 0^b = 1$ and $1^b = 0$, moreover $\langle a, b \rangle$ cyclically permutes the elements of $\mathbb{Z}_{p^e+1}^*$. Noting that $\{0, 1\} \in E$ and $\mathbb{Z}_{p^e+1}^*$ is the neighborhood of 0, we have $\langle a, b \rangle$ is an arc-transitive subgroup of $\text{Aut}(\text{K}_p)$. Since the number of arcs of $\text{K}_p$ is $p^{2e+1}(p - 1)$, we have $|\langle a, b \rangle| \geq p^{2e+1}(p - 1)$. Clearly $|G| = p^{2e+1}(p - 1)$, and hence we have $G = \langle a, b \rangle$ is an arc-regular subgroup of $\text{Aut}(\text{K}_p)$ with cyclic vertex stabilizer $G_0 = \langle a \rangle$.

Set $\mathcal{M}_\theta = \mathcal{M}(G; a, b)$. Then $\mathcal{M}_\theta$ is a regular embedding of $\text{K}_p$. □

**Proposition 4.4.** The genus of $\mathcal{M}_\theta$ in Proposition 4.3 is

$$g(\mathcal{M}_\theta) = \begin{cases} 1 + \frac{p^e+1(p^e-1- p^e-4)}{4}, & p \equiv 1 \pmod{4}; \\ 1 + \frac{p^e+1(p^e-1- p^e-6)}{4}, & p \equiv 3 \pmod{4}. \end{cases}$$

**Proof.** Since $ab = (0, \theta)(1, -1) = (1, -\theta)$ and the identity of $G$ is $(0, 1)$, we get

$$(ab)^n = (1, -\theta)^n = (1 - \theta + \cdots + (-\theta)^{n-1}, (-\theta)^n) = \left(\frac{1 - (-\theta)^n}{1 + \theta}, (-\theta)^n\right).$$

Therefore, $(ab)^n = (0, 1) \iff (-\theta)^n = 1$, which implies that $ab$ and $-\theta$ have the same order. Noting that

$$(-\theta)^\frac{p^e(p-1)}{2} = (-1)^\frac{p^e(p-1)}{2} \theta^\frac{p^e(p-1)}{2} = (-1)^\frac{p^e(p-1)}{2},$$

the order of $ab$ is $p^e(p-1)$ for $p \equiv 1 \pmod{4}$; and $\frac{p^e(p-1)}{2}$ for $p \equiv 3 \pmod{4}$. It follows that the number of faces of $\mathcal{M}$ is $p^{2e+1}$ for $p \equiv 1 \pmod{4}$; and $2p^{e+1}$ for $p \equiv 3 \pmod{4}$. Thus we get the desired formula for $g(\mathcal{M}_\theta)$. □

5. Case $m = 3$

In this section we study the regular embeddings of $\text{K}_3[n]$. As before, set $\Gamma = \text{K}_3[n]$, with the vertex set $\mathcal{V}(\Gamma) = \Delta_1 \cup \Delta_2 \cup \Delta_3$ where $\Delta_i = \{\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}\}$ for $1 \leq i \leq 3$ and the edges are all pairs $\{\gamma_{ij}, \gamma_{jk}\}$ of vertices with $i \neq k$.

Suppose that $\mathcal{M}$ is a regular embedding of $\text{K}_3[n]$, where $n \geq 2$. Let $\text{Aut}^+_0(\mathcal{M})$ be the kernel of $\text{Aut}^+(\mathcal{M})$ on the set of three parts. As before, write $G = \text{Aut}^+(\mathcal{M})$ and $H = \text{Aut}^+_0(\mathcal{M})$. Let $G = \langle a, b \rangle$, where $\langle a \rangle = G_{\gamma_{11}}$ and $b$ reverses the arc $\langle \gamma_{11}, \gamma_{21} \rangle$. Set $x = a^2$ and $y = \theta^h$. Then by Remark 4.2, $H = \langle x, y \rangle$ is an $n$-isobicyclic group. Moreover, we have the following theorem.

**Theorem 5.1.** If $n = 3^ek$ with $e \geq 0$ and $3 \nmid k$, then $H = Q \times K$, where $Q$ is a $3^e$-isobicyclic group and $K$ is an abelian $k$-isobicyclic group.

**Proof.** We prove the theorem by following the following two steps:

**Step 1.** Show that $H$ is nilpotent.

Set $n = p_1 \cdots p_t$ where $p_1 \geq \cdots \geq p_t$ are the prime divisors of $n$. Let $H_i$ and $s_i$ be defined as in Lemma 3.1. Then $1 = H_0 < H_1 < \cdots < H_{i-1} < H_i = H$ is a series of characteristic subgroups of $H$ and hence $H_i \leq G$ for all $i \in [l]$. Consider the quotient graphs $\mathcal{M}_{H_i}$ as well as the quotient maps $\mathcal{M}_{H_i-1}$ induced by the normal subgroups $H_{i-1}$ for all $i \in [l]$. Then $\mathcal{K}_{3[n]}(H_{i-1} \cong K_{3[n-1]}$, here we set $s_0 = n$ and $\text{Aut}(\mathcal{M}_{H_{i-1}}) \cong G/H_{i-1}$. From Remark 4.2, we know

$$Z(G/H_{i-1}) = 1 \quad \text{and} \quad C_{G/H_{i-1}}(H_i/H_{i-1}) = C_{G/H_{i-1}}(H_i/H_{i-1}).$$

Then by Proposition 2.1, we have

$$(G/H_{i-1})/C_{H/H_{i-1}}(H_i/H_{i-1}) = (G/H_{i-1})/C_{G/H_{i-1}}(H_i/H_{i-1})$$

\leq \text{Aut}(H_i/H_{i-1}) \cong \text{GL}(2, p_i).$$
If $C_{H/H_i-1}(H_i/H_i-1) < H/H_i-1$, then by Lemma 3.2, $(H/H_i-1)/C_{H/H_i-1}(H_i/H_i-1)$ is a nontrivial isobicyclic group. Since $(G/H_i)/H_i/H_i-1) \cong G/H \cong S_3$, we have $(G/H_i-1)/C_{H/H_i-1}(H_i/H_i-1)$ is an extension of a nontrivial isobicyclic group by $S_3$, which contradicts to Lemma 3.3. Therefore, $C_{H/H_i-1}(H_i/H_i-1) = H/H_i-1$ and then we get a central series of $H$, namely the series

$$1 = H_0 < H_1 < \cdots < H_{i-1} < H_i = H.$$ 

It follows that $H$ is a nilpotent group.

Step 2. Show that the Hall $3'$-subgroup of $H$ is abelian.

Write $H = Q \times K$ where $Q$ and $K$ are the Sylow 3-subgroup and Hall $3'$-subgroup of $H$ respectively. Suppose to the contrary that $K$ is nonabelian. Then there exists a prime divisor $p$ of $n$ such that the Sylow $p$-subgroup $P$ of $H$ is nonabelian. Clearly, both $P$ and $P'$ are normal subgroups of $G$. Consider the quotient group $G = G/P'$. Since $H$ is a nilpotent group and $P$ is an abelian Sylow $p$-subgroup of $H$, we get $H \leq C_G(P)$. Taking any element $c \in G \setminus H$, there exists $1 \leq i \leq 3$ such that $\Delta_i^c \neq \Delta_i$. Set $H_i = \langle z \rangle$, we have $(H, z, z^c)$ is a $n$-isobicyclic triple. Let $n = sp^d$ where $\gcd(s, p) = 1$. Then $P = \langle z^c \rangle$ is a $p^d$-isobicyclic group. By Lemma 3.3, $\overline{P}$ is an inhomogeneous abelian group generated by two elements. Hence we have $\overline{c} \neq (\overline{c'})^3$, which implies that $\overline{c} \notin C_{\overline{c}}(\overline{P})$. It follows that $C_{\overline{c}}(\overline{P}) \leq \overline{H}$ and hence $\overline{H} = C_{\overline{c}}(\overline{P})$. By Proposition 2.1, we have

$$S_3 \cong G/H \cong (G/\langle H \rangle) \leq \text{Aut}(\overline{P}),$$

which contradicts to Lemma 2.4.

If $H$ is abelian, then we have the following lemma.

**Lemma 5.2.** Suppose that $H$ is abelian. Then $G$ has one of the following presentations

$$G = G(n, k) = \langle a, b \mid a^{2^n} = b^2 = 1, a^2 = x, x^b = y, [x, y] = 1, y^a = x^{-1}y^{-1}, (ab)^3 = x^{k_3}y^{-k_3} \rangle,$$

where $k = 0$ if $3 \mid n$ and $k = 0$ or $1$ if $3 \nmid n$, and $M$ is isomorphic to one of the maps

$$M(n, k, j) = M(G(n, k); a^j, b),$$

where $(k, j) = (0, 1)$ for $3 \mid n$ and $(k, j) = (0, 1), (1, 1)$ or $(1, -1)$ for $3 \nmid n$. Moreover, $M(n, k, j)$ has the type $\{3, 2n\}$ if $k = 0$ and $\{9, 2n\}$ if $k = 1$.

**Proof.** We prove the lemma by the following two steps:

*Step 1.* Determine the presentation of $G$.

Write $c = ab$. Since $H \leq G$ and $c^3 \in H$, we can set

$$y^a = x^ty^t \quad \text{and} \quad c^3 = x^uy^v$$

where $s$, $t$, $u$ and $v$ are integers to be determined. Since

$$x^c = x^{ab} = x^b = y \quad \text{and} \quad y^c = y^{ab} = (x^ty^t)^b = x^ty^t,$$

we have

$$x^3 = y^2 = (x^ty^t)^c = y^t(x^ty^t)^s = x^{st}y^{t+2}.$$ 

On the other hand, $c^3 = x^uy^v$ implies that $x^3 = x^{2uy^v} = x$. Therefore,

$$st = 1 \pmod n \quad \text{and} \quad t + s^2 \equiv 0 \pmod n.$$

Then from

$$x^ty^t = y^c = x^2 = x^{c-1}x^{2v} = x^{-1} = x^{ba^{-1}} = y^{a^{-1}} = y^{x^{-1}a} = y^a = x^ty^t,$$

we have $s \equiv t \equiv -1 \pmod n$ and hence $y^a = x^{-1}y^{-1}$.

Noting that $x^a y^b = (x^ay^b)^c = y^b(x^ay^{-1})^v = x^{-v}y^{a-v}$, we get

$$u \equiv -v \pmod n \quad \text{and} \quad v \equiv u - v \pmod n.$$
Then $3u \equiv -3v \equiv 0 \pmod{n}$, that is, $u \equiv v \equiv 0 \pmod{n}$ if $3 \nmid n$ and $u \equiv -v \equiv \frac{k\alpha}{3} \pmod{n}$ where $k = 0, 1, 2$ if $3 \mid n$.

Now we set

$$G(n, k) = \langle a, b \mid a^{2n} = b^2 = 1, a^2 = x, x^b = y, [x, y] = 1, y^a = x^{-1}y^{-1}, (ab)^3 = x^{\frac{b^2}{3}} y^{\frac{b^2}{3}} \rangle,$$

where $k = 0$ if $3 \nmid n$ and $k = 0, 1, 2$ if $3 \mid n$. Then $G(n, k) \leq G$. It is straightforward to check that $|G(n, k)| = 6n^2 = |G|$. Thus we have $G = G(n, k)$.

If $3 \mid n$ and $(ab)^3 = x^{\frac{b^2}{3}} y^{\frac{b^2}{3}}$, then

$$(a^{-1}b)^3 = b(ab)^{-3}b = b(x^{\frac{2n}{3}} y^{\frac{-2n}{3}})^{-1}b = x^{\frac{2n}{3}} y^{\frac{-2n}{3}} = (x^{-1})^{\frac{n}{3}} (y^{-1})^{-\frac{n}{3}}.$$

It follows that

$$G(n, 2) = \langle a^{-1}, b \mid (a^{-1})^{2n} = b^2 = 1, (a^{-1})^2 = x^{-1}, (x^{-1})^b = y^{-1}, [x^{-1}, y^{-1}] = 1, (y^{-1})^a = (x^{-1})^{-1}(y^{-1})^{-1}, (a^{-1}b)^3 = (x^{-1})^{\frac{n}{3}} (y^{-1})^{-\frac{n}{3}} \rangle,$$

from which we have $G(n, 2) \cong G(n, 1)$. Therefore, we get the desired presentation of $G$.

Step 2. Determine $\mathcal{M}$.

Recalling that $G_{\gamma_1} = \langle a \rangle$ and $b$ reverses the arc $(\gamma_1, \gamma_2)$, we know $\mathcal{M} = \mathcal{M}(G, a', b)$ for some $j \in [2n]$ with $\gcd(j, 2n) = 1$. Write $i = (j - 1)/2$. Then $a' = (a^2)^{(j-1)/2}a = a^i a$. It follows that

$$(y^i)^a = (y^a)^i = (y^i)^i = (y^i)^{-1} = (a^{-1})^{-i} = (a^{-1})^{-i} (y^{-i})^{-1}$$

and

$$(a'b)^3 = (x^i c)^3 = c(x^i)^x c(x^i)^c = c y^i x^i c y^i c = c^2 (y^i x^i c y^i c = c^2 x^{-i} y^{-i} y^i x c = c^3.$$

If $3 \mid n$, then the equality $\gcd(j, 2n) = 1$ implies that $j \equiv 1$ or $5$ (mod 6). It follows that

$$n/3 \equiv jn/3 \pmod{n} \quad \text{or} \quad n/3 \equiv -jn/3 \pmod{n}.$$

Therefore we have

$$\begin{cases} (a'b)^3 = 0, & k = 0; \\ (a'b)^3 = (x^i c)^{n/3} (y^i)^{-n/3}, & k = 1 \quad \text{and} \quad j \equiv 1 \pmod{6}; \\ (a'b)^3 = (x^i)^{n/3} (y^i)^{n/3}, & k = 1 \quad \text{and} \quad j \equiv 5 \pmod{6}. \end{cases}$$

Basing on the above paragraph, one may check that the following two arguments hold.

1. The mapping $a^i \mapsto a$, $b \mapsto b$ can be extended to an automorphism of $G$ when $G = G(n, 0)$ or $G = G(n, 1)$ and $j \equiv 1 \pmod{6}$;
2. The mapping $a^i \mapsto a^{-1}$, $b \mapsto b$ can be extended to an automorphism of $G$ when $G = G(n, 1)$ and $j \equiv 5 \pmod{6}$.

Therefore $\mathcal{M}$ is isomorphic to one of the maps

$$\mathcal{M}(n, k, j) = \mathcal{M}(G(n, k); a^i, b),$$

where $(k, j) = (0, 1)$ for $3 \nmid n$ and $(k, j) = (0, 1), (1, 1)$ or $(1, -1)$ for $3 \mid n$. Clearly, the type of $\mathcal{M}(n, k, j)$ is $\{3, 2n\}$ if $k = 0$ and $\{9, 2n\}$ if $k = 1$. Thus the map $\mathcal{M}(n, 0, 1)$ is different from $\mathcal{M}(n, 1, 1)$ and $\mathcal{M}(n, 1, -1)$ up to map isomorphism. Now we prove that $\mathcal{M}(n, 1, 1)$ is not isomorphic to $\mathcal{M}(n, 1, -1)$. Suppose to the contrary that $\mathcal{M}(n, 1, 1) \cong \mathcal{M}(n, 1, -1)$. Then there exists $\phi \in \text{Aut}(G(n, 1))$ such that $\phi(a) = a^{-1}$ and $\phi(b) = b$ and hence

$$\phi(c^3) = [\phi(c)^3] = [\phi(a)\phi(b)]^3 = (a^{-1}b)^3 = c^3 = x^{n/3} y^{-n/3}.$$

On the other hand, since

$$\phi(x) = \phi(a^2) = a^{-2} = x^{-1} \quad \text{and} \quad \phi(y) = \phi(x^i) = (x^{-1})^b = y^{-1},$$
we have
\[ \phi(c^3) = \phi(x^{n/3}y^{-n/3}) = x^{-n/3}y^{n/3}. \]

Therefore, \( x^{n/3}y^{-n/3} = x^{-n/3}y^{n/3} \). It follows that \( n/3 \equiv -n/3 \pmod{n} \), a contradiction. Thus we have \( M(n, 1, 1) \) is not isomorphic to \( M(n, 1, -1) \). □

Acknowledgments

The authors thank the referees for their helpful comments and suggestions. This work was supported by the National Natural Science Foundation of China and Natural Science Foundation of Beijing.

References