Electron beam dynamics and self-cooling up to PeV level due to betatron radiation in plasma-based accelerators

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In plasma-based accelerators, electrons are accelerated by ultrahigh gradient of 1–100 GV/m and undergo the focusing force with the same order as the accelerating force. Heated electrons are injected in a plasma wake and exhibit the betatron oscillation that generates synchrotron radiation. Intense betatron radiation from laser-plasma accelerators is attractive x-ray/gamma-ray sources, while it produces radiation loss and significant effects on energy spread and transverse emittance via the radiation reaction force. In this article, electron beam dynamics on transverse emittance and energy spread with considering radiation reaction effects are studied numerically. It is found that the emittance growth and the energy spread damping initially dominate and balance with radiative damping due to the betatron radiation. Afterward the emittance turns to decrease at a constant rate and leads to the equilibrium at a nanometer radian level with growth due to Coulomb scattering at PeV-level energies. A constant radiation loss rate $R_T = 2/3$ is found without regard to the electron beam and plasma conditions. Self-cooling of electron beams due to betatron radiation may guarantee TeV-range linear colliders and give hints on astrophysical ultrahigh-energy phenomena.

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I. INTRODUCTION

In this decade, vital research on beam- and laser-driven plasma-based acceleration [1,2] of charged particles has achieved great progress in tens of GeV energy doubling in a meter-scale plasma [3], high-energy, high-quality electron beams with energies of GeV-level in a cm-scale plasma [4–7], qualities of 1%-level energy spread [8], 1 $\pi$ mm $\text{mrad}$-level transverse emittance [9,10], and 1 fs-level bunch duration [11], ensuring that the stability of reproduction is as high as that of present high-power ultrashort-pulse lasers [7,12–14]. These high-energy high-quality particle beams make it possible to open the door for a wide range of applications in fundamental research, medical, and industrial uses. For many applications of laser wakefield accelerators, stability and controllability of the beam performance such as beam energy, energy spread, emittance, and charge are indispensable as well as compact and robust features of the system. In particular, there is great interest in applications for high-energy physics and astrophysics that explore unprecedented high-energy frontier phenomena much beyond 1 TeV, for which plasma-based accelerator concepts provide us with promising tools if beam-quality issues are figured out as well as an achievable highest energy.

To date, most of the experimental results have been obtained from interaction of ultrashort laser pulses, $\tau_L = 30–80$ fs with a short-scale plasma such as a few mm long gas jet and a few cm long plasma channel at the plasma density in the range of $n_p = 10^{18}–10^{19}$ cm$^{-3}$, where a large-amplitude plasma wave of the order of 100 GV/m is excited and traps energetic electrons to be efficiently accelerated inside a wake to high energies of the order of 1 GeV. The leading experiments that demonstrated the production of quasi monoenergetic electron beams [15–17] have been elucidated in terms of self-injection and successive acceleration of electrons in the nonlinear wakefield, referred to as a “bubble” that is a region where plasma electrons are blown out by radiation pressure of a laser pulse with the relativistic intensity given by its normalized vector potential $a_L = eA_L/mc^2 \gg 1$, where $A_L$ is the peak amplitude of the vector potential and $mc^2$ the rest energy of electron [18,19]. The self-injection is a robust method [20–22] relying on the adiabatically bubble evolving due to the self-focusing and self-compression that occur during the propagation of relativistic laser pulses. In this mechanism, initially heated electrons with large transverse momentum are injected into nonlinear wakefields that excite betatron oscillation of accelerated electrons due to a strong focusing field. Hence, suppressing the self-injection and the deterioration of beam qualities, high-quality electron beams required for most of applications have been produced with controlled injection.

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schemes [14] such as colliding optical injection [23,24], density-transition injection [25,26], and ionization-induced injection [27–29], in the quasilinear regime of wakefields driven by a laser pulse with a moderate intensity $a_L \sim 1$. These injection schemes provide us with high-quality electron beam injectors for a front end of the multi-stage laser-plasma accelerators, aiming at the high-energy frontier. Recently, two-stage laser-plasma acceleration has been successfully demonstrated in combination with ionization-induced injection [30,31]. Based on recent results on vital experiments and large-scale particle-in-cell simulations [32], the design considerations and the feasibility studies on applications for high-energy frontier collider with the TeV-class center-of-mass energy have been carried out [33,34]. In these considerations, the most critical issue is a choice of the operating plasma density that is an underlying parameter controlling the size, the performance, and the beam dynamics. Generally speaking, from the viewpoint of the beam qualities, the higher energy regime is in favor of the lower operating density, though such an option leads to a larger size and a higher laser peak power. Furthermore, initially heated electrons with large transverse momentum are injected and accelerated in plasma waves and exhibit the betatron oscillation that generates the emission of intense synchrotron radiation [35–38], which is an attractive x-ray/gamma-ray radiation source [39,40]. On the other hand, it produces a radiation loss of the beam energy [35] and a significant effect on the beam qualities such as the energy spread and the transverse emittance via the radiation reaction force [8,41].

We consider the feasibility of accelerating electron beams up to energies much beyond 1 TeV and evolution of the beam qualities such as emittance and energy spread at the final beam energy. First, we review the scaling formulas for designing laser-plasma accelerators in the quasilinear laser wakefield regime in Sec. II and derive the basic equations describing evolution of the normalized transverse emittance and the energy spread in Sec. III, taking into account radiation reaction due to the betatron oscillation of electrons that undergo strong acceleration and focusing forces simultaneously. In Sec. IV, the evolutions of transverse emittance and energy spread up to 1 PeV are presented by numerically solving the coupled equations describing the beam dynamics and the equilibrium at a nanometer radial level with growth due to Coulomb scattering at PeV-level energies is evaluated. In Sec. V, we mention that self-cooling of electron beams due to betatron radiation may guarantee TeV-range linear colliders [42] and give hints on astrophysical ultrahigh-energy phenomena [43].

II. DESIGN OF LASER-PLASMA ACCELERATORS

In underdense plasma, an ultraintense laser pulse excites a large-amplitude plasma wave with frequency $\omega_p = (4\pi e^2 n_p / m)^{1/2}$ and electric field of the order of

$$E_0 = \frac{mc\omega_p}{e} \approx 96 \text{ [GV/m]} \left( \frac{n_p}{10^{18} \text{[cm}^{-3}]^2} \right)^{1/2},$$

for the plasma density $n_p$ due to the ponderomotive force expelling plasma electrons out of the laser pulse and the space charge force of immovable plasma ions restoring expelled electrons on the back of the ion column remaining behind the laser pulse. Since the phase velocity of the plasma wave is approximately equal to the group velocity of the laser pulse $v_p/c = (1 - \omega_p^2 / \omega_0^2)^{1/2} = 1$ for the laser frequency $\omega_0$, and the accelerating field of $\sim 100$ GV/m for the plasma density $n_p \sim 10^{18}$ cm$^{-3}$, electrons trapped into the plasma wave are likely to be accelerated up to $\sim 1$ GeV energy in a 1 cm-length plasma.

In the quasilinear laser wakefield regime, the normalized laser intensity,

$$a_L = \left( \frac{2e^2 \lambda_L^2 I_L}{\pi m c^3} \right)^{1/2} \approx 0.855 \times 10^{-9} \lambda_L^{1/2} \text{[W/cm}^2] \lambda_L \text{[\mu m]}.$$  (2)

is set to be $a_L = 1$, where $I_L$ is the laser intensity and $\lambda_L = 2mc/\omega_L$ is the laser wavelength. In this regime, the wake potential $\Phi$ is obtained by [44,45]

$$\frac{\partial^2 \Phi}{\partial \xi^2} + k_p^2 \Phi = \frac{1}{2} k_p^2 mc^2 a^2(r, \xi),$$  (3)

where $\xi = z - v_p t$, $k_p = \omega_p / c$, and $a(r, \xi) = eA(r, \xi)/mc^2$ is the normalized vector potential of the laser pulse. The wake potential is calculated by

$$\Phi(r, \xi) = -\frac{mc^2 k_p}{2} \int_{-\infty}^{\infty} d\xi' \sin k_p (\xi - \xi') a^2(r, \xi'),$$  (4)

and the axial and radial electric fields are obtained by

$$eE_z = -\frac{\partial \Phi}{\partial z}, \quad \text{and} \quad eE_r = -\frac{\partial \Phi}{\partial r},$$  (5)

respectively. Considering a bi-Gaussian laser pulse with $1/e$ half-width $\sigma_L$ and $1/e^2$ spot radius $r_L$, the ponderomotive potential is given by

$$a^2(r, \xi) = \frac{a_1^2}{2} \exp\left( -\frac{2r^2}{r_L^2} - \frac{\xi^2}{\sigma_L^2} \right).$$  (6)

Behind the laser pulse at $\xi \ll -\sigma_L$, the axial and radial wakefields are

$$E_z(r, \xi) = -\sqrt{\frac{\pi}{4}} a_1^2 k_p \sigma_L E_0 \exp\left( -\frac{2r^2}{r_L^2} - \frac{k_p^2 \sigma_L^2}{4} \right) \cos k_p \xi,$$  (7)

and

$$E_r(r, \xi) = -\sqrt{\frac{\pi}{4}} a_1^2 \sigma_L r \frac{E_0}{r_L} \exp\left( -\frac{2r^2}{r_L^2} - \frac{k_p^2 \sigma_L^2}{4} \right) \sin k_p \xi,$$  (8)
respectively. Setting $k_p\sigma_L = 1$, i.e., the full width at the half maximum (FWHM) pulse length, $c\tau_L = 2(\ln 2)^{1/2}\sigma_L = 0.265\lambda_p$, the maximum accelerating field is

$$E_{z,\text{max}} = 0.35a_L^2 E_0 \approx 1.06 [\text{GV/m}] a_L^2 \left(\frac{n_p}{10^{15} \text{[cm}^{-3}]}\right)^{1/2}. \quad (9)$$

In this condition, the laser pulse length is shorter than a half plasma wavelength so that a transverse field at the tail of the laser pulse is negligible in the accelerating phase of the first wakefield. The net accelerating field $E_z$ that accelerates the bunch containing the charge $Q_b = eN_b$, where $N_b$ is the number of electrons in the bunch, is determined by the beam loading [46,47] that means the energy absorbed per unit length,

$$Q_b E_z = \frac{m c^2}{4e} k_p^2 \rho^2 \sigma_r^2 \frac{E_{z,\text{max}}^2}{E_0^2} \left(\frac{1 - E_z^2}{E_{z,\text{max}}^2}\right). \quad (10)$$

where $\rho = e^2/mc^2$ is the classical electron radius and $1 - E_z^2/E_{z,\text{max}}^2 = \eta_b$ is the beam loading efficiency that is the fraction of the plasma wave energy absorbed by particles of the bunch with the rms radius $\sigma_r$. In the beam-loaded field $E_z = (1 - \eta_b)^{1/2}E_{z,\text{max}}$, the charge is obtained as

$$Q_b = \frac{e}{4k_L \rho^2} \eta_b k_p^2 \sigma_r^2 \sigma_z \left(\frac{n_e}{n_p}\right)^{1/2} \approx 2.4 [\text{nC}] \eta_b k_p^2 \sigma_r^2 \sigma_z \left(\frac{n_e}{n_p}\right)^{1/2} \left(\frac{n_p}{10^{15} \text{[cm}^{-3}]}\right)^{-1/2}, \quad (11)$$

where $n_e = m \omega_p^2 / 4 \pi e^2 = \pi / (r_e \lambda_p) = 1.115 \times 10^{41} \text{[cm}^{-3}] / (\lambda_p \text{[\mu m]}^{-1})$ is the critical plasma density and for $k_p\sigma_L = 1$, $E_z/E_0 \approx 0.35a_L^2(1 - \eta_b)^{1/2}$. Since the loaded charge depends on the accelerating field and the bunch radius, it will be determined by considering the required accelerating gradient and the transverse beam dynamics.

First, we consider a design of multi-TeV linear accelerators composed of multistaging laser-plasma accelerators, each of which is driven by different laser pulse capable of providing a single-stage energy gain of 1 TeV in a relatively compact scale. Ideally, the stage length $L_{\text{stage}}$ is limited by the pump depletion length $L_{\text{dp}}$ for which the total field energy is equal to half the initial laser energy. For a Gaussian laser pulse with pulse length $k_p\sigma_L = 1$, the pump depletion length is given by [33,48,49]

$$k_p L_{\text{dp}} \approx \frac{8}{\sqrt{\pi a_L^2 k_p \sigma_L}} \frac{\omega_p^2}{\omega_p^2} \exp\left(\frac{k_p^2 \sigma_L^2}{2}\right) = 7.4 \frac{n_e}{a_L^2 n_p}. \quad (12)$$

In laser wakefield accelerators, accelerated electrons eventually overrun the acceleration phase to the deceleration phase, of which the velocity is approximately equal to the group velocity of the laser pulse. In the linear wakefield regime, the dephasing length $L_{\text{dp}}$, where the electrons undergo both focusing and acceleration is approximately given by [48]

$$k_p L_{\text{dp}} \approx \frac{\omega_p^2}{\omega_p^2} = \frac{\pi}{n_e}. \quad (13)$$

In the condition for the dephasing length less than the pump depletion length, $L_{\text{dp}} \leq L_{\text{pd}}$, the normalized vector potential should be $a_L \leq 1.5$. Setting $a_L = 1.5$, the maximum accelerating field is $E_{z,\text{max}} = 0.79E_0$ for $k_p\sigma_L = 1$. Assuming the beam loading efficiency $\eta_b = 0.5$, the net accelerating field becomes

$$E_z = \frac{E_{z,\text{max}}}{\sqrt{2}} \approx 0.56E_0 = 1.69 [\text{GV/m}] \left(\frac{n_p}{10^{15} \text{[cm}^{-3}]}\right)^{1/2}. \quad (14)$$

For the stage length approximately equal to the dephasing length,

$$L_{\text{stage}} = L_{\text{dp}} = \frac{\lambda_p}{2} \frac{n_e}{n_p} = \frac{\lambda_p}{2} \left(\frac{n_e}{n_p}\right)^{3/2} \approx 589 [\text{m}] \left(\frac{1 [\mu \text{m}]}{\lambda_L}\right)^2 \left(\frac{10^{15} \text{[cm}^{-3}]}{n_p}\right)^{3/2}, \quad (15)$$

and the accelerating field Eq. (14), the energy gain per stage is given by

$$W_{\text{stage}} = E_z L_{\text{stage}} = \pi mc^2 \frac{E_z}{E_0} \frac{n_e}{n_p} \approx 1 \text{[TeV]} \frac{10^{15} \text{[cm}^{-3}]}{n_p} \left(1 \text{[\mu m]}\right)^2. \quad (16)$$

Here we assume that the accelerating field keeps constant over the stage length. In fact, the accelerating field is set to be constant over the dephasing length defined by Eq. (15) with good approximation as a result of synergetic effects in conjunction with the pump depletion, self-focusing, and self-compression of the drive laser pulse as shown in the 1D nonlinear simulations [37] and the 2D PIC simulations [50]. The total number of stages becomes

$$N_{\text{stage}} \approx \frac{E_b}{W_{\text{stage}}} = \frac{\gamma_f}{\pi} \frac{E_z}{E_0} \left(\frac{n_e}{n_p}\right)^{-1} \approx \frac{E_b}{1 \text{[TeV]} \frac{10^{15} \text{[cm}^{-3}]}{n_p}} \left(\frac{\lambda_L}{1 \text{[\mu m]}}\right)^2. \quad (17)$$

where $\gamma_f = E_b/mc^2$ is the Lorentz factor at the final beam energy. The overall length of a linac consisting of periodic structures of an accelerator stage and a coupling section that installs both beam and laser focusing systems leads to be
The total laser power is bounded by conditions for avoiding bubble formation, \( \frac{k_p^2 r_L^2}{4} > a_L^2 (1 + a_L^2/2)^{-1/2} \), and strong self-focusing, \( P_L/\rho_c = k_p^2 r_L^2 a_L^2/32 \leq 1 \), where \( \rho_c = 2(m^2 c^5/e^4) \omega_0^2/\lambda_0^2 \approx 17(n_e/n_p) \) [GW]. These conditions put bounds to the spot size \( 2.5 \leq k_p r_L \leq 3.8 \) for \( a_L = 1.5 \). Accordingly, we choose \( k_p r_L = 3 \). For a given spot radius,

\[
r_L = \frac{\lambda_L}{2\pi} k_p r_L \left( \frac{n_i}{n_p} \right)^{1/2} = 510 [\mu m] \left( \frac{10^{15} [cm^{-3}]}{n_p} \right)^{1/2},
\]

(19)

the peak laser power becomes

\[
P_L = \frac{k_p^2 r_L^2 a_L^2}{32} \rho_c = 12 [PW] \left( \frac{1 [um]}{\lambda_L} \right)^2 \left( \frac{10^{15} [cm^{-3}]}{n_p} \right).
\]

(20)

With the FWHM pulse duration given by

\[
\tau_L = \frac{\sqrt{2\ln2} \lambda_L}{\pi} k_p r_L \left( \frac{n_i}{n_p} \right)^{1/2} \approx 930 [fs] \left( \frac{10^{15} [cm^{-3}]}{n_p} \right)^{1/2},
\]

(21)

the required laser pulse energy per stage is obtained as

\[
U_L = P \tau_L = 11 [kJ] \left( \frac{1 [um]}{\lambda_L} \right)^2 \left( \frac{10^{15} [cm^{-3}]}{n_p} \right)^{3/2}.
\]

(22)

III. BEAM DYNAMICS AND RADIATIVE DAMPING

Beams that undergo strong transverse focusing forces \( F_\perp = -mc^2 K^2 x_\perp \), in plasma waves exhibit the betatron oscillation, where \( x_\perp \) is the transverse amplitude of the betatron oscillation. From the equations of the axial and radial fields, Eqs. (7) and (8), the focusing constant \( K \) is given by

\[
K^2 \approx \frac{4k_p^2}{(k_p r_L)^2} E_z \langle \sin \Psi \rangle,
\]

(23)

where \( \Psi = k_p (z - v_x t) + \Psi_0 \) is the dephasing phase of the wakefield and \( \Psi_0 \) is the injection phase and \( \langle \sin \Psi \rangle \) is set to be the average value over the accelerating phase \( 0 \leq \Psi \leq \pi/2 \). It has been confirmed that this approximation is in excellent agreement with the numerical calculations on the beam dynamics with radiative damping [50]. In fact, since the focusing force is a periodic function as \( K^2 \propto \sin \Psi \) in the quasilinear regime and is constant as \( K^2 = k_p^2/2 \) inside a uniform ion channel in the blow-out and bubble regime [38], one can assume that the focusing constant is set to be the averaging value over the stage for solving the beam dynamics problem of large-scale laser-plasma accelerators consisting of many stages within good approximation. Assuming that electrons undergo no excitation force from direct laser fields and/or instabilities such as hose instability in plasma waves, the envelope equation of the rms beam radius \( \sigma_r \) is given by [51,52]

\[
\frac{d^2 \sigma_r}{dz^2} + \frac{K^2}{\gamma} \sigma_r - \frac{e_{n0}^2}{\gamma^2 \sigma_r^3} = 0,
\]

(24)

where \( e_{n0} \) is the initial normalized emittance. Assuming the beam energy \( \gamma \) is constant, this equation is solved with the initial conditions \( \sigma_r'(0) = (d\sigma_r/dz)|_{z=0} \) and \( \sigma_r(0) = \sigma_r(0) \) as

\[
\sigma_r^2(z) = C k^2 + 1 + \frac{C^2}{k^2} - \frac{4e_{n0}^2}{\gamma^2} \sin(kz + \phi_0),
\]

(25)

where \( k = 2K/\sqrt{\gamma} \) is the focusing strength, \( C = 2\sigma_r(0) + k^2 \sigma_r(0)/2 + 2e_{n0}^2/\gamma^2 \sigma_r(0) \) is the constant

\[
\tan \phi_0 = \frac{\sigma_r(0) - C/k^2}{2\sigma_r(0)/k}. \]

(26)

The beam envelope oscillates around the equilibrium radius \( \bar{\sigma}_r = \sqrt{C/k} \) with the betatron wavelength \( \pi/k \). For the condition \( C/k = 2e_{n0}/\gamma \) that leads to \( \sigma_r^2 = 2e_{n0}/k \gamma \) with \( \sigma_r(0) = 0 \), the beam propagates at the matched beam radius

\[
\sigma_r^2(0) = \frac{2e_{n0}}{k\gamma} = \frac{e_{n0}}{K\sqrt{\gamma}} \approx \frac{r_L e_{n0}}{2\sqrt{\gamma} E_0} \left( \frac{E_z}{E_0} \sin \Psi \right)^{-1/2}.
\]

(27)

The synchrotron radiation causes the energy loss of beams and affects the energy spread and transverse emittance via the radiation reaction force. The motion of an electron traveling along the \( z \) axis in the accelerating force \( eE_z \) and the radial force \( eE_y \) from the plasma wave evolves according to

\[
\frac{du_x}{cdt} = -K^2 x + \frac{F^{\text{RAD}}}{mc^2}, \quad \frac{du_y}{cdt} = k_p E_z + \frac{F^{\text{RAD}}}{mc^2},
\]

(28)

where \( F^{\text{RAD}} \) is the radiation reaction force and \( u = p/mc \) is the normalized electron momentum. The classical radiation reaction force [53] is given by

\[
F^{\text{RAD}}_{mc^2} = \frac{d}{dt} \left( \frac{\gamma d\mathbf{u}}{dt} \right) + \gamma u \left[ \frac{d(\gamma^2)}{dt} - \left( \frac{du}{dt} \right)^2 \right].
\]

(29)

where \( \gamma = (1 + u^2)^{1/2} \) is the relativistic Lorentz factor of the electron and \( \tau_R = 2r_c/3c \approx 6.26 \times 10^{-24} \) s. Since the scale length of the radiation reaction, i.e. \( c \tau_R = 2r_c/3 \approx 1.9 \) fm, is much smaller than that of the betatron motion, i.e. \( \sim \lambda_p \sqrt{\gamma} \), assuming that the radiation reaction force is a perturbation and \( u \gg u_z \), the equations of motion, Eq. (28), are approximately written as [50]
\[
\frac{du}{dt} \approx -cK^2x - c^2TrK^2u_x(1 + K^2\gamma x^2),
\]
\[
\frac{du}{dt} \approx \omega p \frac{E_z}{E_0} - c^2TrK^4\gamma^2x^2, \quad \frac{dx}{dt} = cu_x \approx \frac{c}{u_z}.
\]

Finally the particle dynamics is obtained from the following coupled equations:
\[
\frac{d^2x}{dt^2} + \left(\omega_p E_z + TrK^2\right)\frac{dx}{dt} + \frac{c^2K^2}{\gamma}x = 0,
\]
and
\[
\frac{d\gamma}{dt} = \omega p \frac{E_z}{E_0} - TrK^2\gamma^2x^2,
\]
where the second damping term proportional to \(TrK^2\gamma^2\) results in the linear damping of the betatron motion and the first one induces the nonlinear damping in conjunction with the energy evolution. The radiated power in the classical limit is given by [47]
\[
P_{RAD} = \frac{2e^2\gamma^2}{3c} \left[ \left(\frac{du}{dt}\right)^2 - \left(\frac{d\gamma}{dt}\right)^2 \right]
\]
\[
= \frac{2e^2\gamma^2}{3mc^2} \left[ |F_{ext}|^2 - |F_{ext} \cdot u/\gamma|^2 \right],
\]
using \(md\gamma/dt = F_{ext} \cdot u/\gamma\), where \(F_{ext}\) is the external force on the electron. As the force is transverse only, i.e., \(F_{ext} = F_\perp e_x\) and for a relativistic electron with \(u_x^2 \ll \gamma^2\), the radiated power can be written as
\[
P_{RAD} = \frac{2e^2\gamma^2}{3mc^2} F_\perp^2 = \frac{2}{3} e^2\gamma^2K^2x = mc^4TrK^4\gamma^2x^2,
\]
with \(F_\perp = -mc^2K^2x\).

Corresponding to the particle beam equations, Eqs. (31) and (32), with radiation damping, the envelope equation is written as
\[
\frac{d^2\sigma_r}{dz^2} + \left(\frac{k_p}{\gamma} \frac{E_z}{E_0} + TrK^2\right)\frac{d\sigma_r}{dz} + \frac{K^2}{\gamma} \sigma_r - \frac{\sigma_r}{\gamma^2\sigma_r} = 0,
\]
and
\[
\frac{d\gamma}{dz} = k_p \frac{E_z}{E_0} - 2TrK^4\gamma^2\sigma_r^2.
\]

The normalized emittance for highly relativistic particles is given by \(\varepsilon_n = \gamma e\) with the geometrical emittance \(\varepsilon\). For the present case, we use the following definition of the geometrical emittance:
\[
\varepsilon = \sqrt{\langle\delta x^2\rangle\langle\delta x'^2\rangle - \langle\delta x\delta x'\rangle^2} = \left| \sigma_r \frac{d\sigma_r}{dz} \right| = \frac{1}{2} \left| \frac{d\sigma_r^2}{dz^2} \right|,
\]
where \(\delta x = x - \langle x \rangle\), \(\delta x' = x' - \langle x' \rangle\), \(x' = dx/dz\), and \(\langle x \rangle = \sum x_i/N_b\) with the \(i\)th particle position \(x_i\) and the number of particles \(N_b\) [50]. Assuming that the evolution of the beam envelope, the energy, and the emittance is much slower than the betatron oscillation, one can calculate the average over the betatron wavelength \(\lambda_B = 2\pi/k_p\), as shown by \(\langle \delta x^2 \rangle\). As a result of these averages, we approximately use \(\langle \delta x\delta x' \rangle \ll \langle \delta x^2 \rangle\), \(\langle \delta x'^2 \rangle = \sigma_r^2\), and \(\langle \delta x^2 \rangle \sim k_p^2\sigma_r^4 \sim \sigma_r^2\). The above-mentioned emittance allows us to derive an analytic formula for the evolution of the normalized emittance without statistical treatment over the entire beam particles as follows. Considering the matched beam case, defined by Eq. (27) with the normalized emittance \(\varepsilon_n\), i.e., \(\sigma_r^2 = \varepsilon/k_p\), and using the following dimensionless variables and parameters \(\zeta = k_p z\), \(\Omega(\zeta) \equiv k_p \varepsilon_n\), \(\chi = E_z/E_0\) and \(K/k_p = \alpha \chi^{1/2}\) the coupled equations of the beam dynamics are obtained as
\[
\frac{d^2\Omega}{d\zeta^2} + p(\gamma) \frac{d\Omega}{d\zeta} + q(\gamma)\Omega = 0,
\]
and
\[
\frac{d\gamma}{d\zeta} = \chi - 2Tr\omega_p \alpha^3 \chi^{3/2} \gamma^{3/2}\Omega,
\]
with
\[
p(\gamma) = \frac{2\alpha \chi^{1/2}}{\gamma^{3/2}} - \frac{\chi}{\gamma} + Tr\omega_p \alpha^2 \chi + 6Tr\omega_p \alpha^3 \chi^{3/2} \gamma^{1/2}\Omega,
\]
and
\[
q(\gamma) = -\alpha \chi^{3/2} \gamma^{3/2} + Tr\omega_p \left( 2\alpha \chi^{3/2} \gamma^{1/2} - \frac{\alpha^2 \chi^2}{\gamma} \right)
\]
\[+ (6Tr\omega_p \alpha^4 \chi^2 - Tr\omega_p \alpha^3 \chi^{5/2} \gamma^{-1/2} + 2\alpha^2 \omega_p^2 \alpha^5 \chi^{5/2} \gamma^{1/2})\Omega + 2\alpha^2 \omega_p^2 \alpha^6 \chi^3 \gamma \Omega^2.\]

The evolution of energy spread is deduced from Eq. (37) as
\[
\frac{d}{d\zeta} \left( \frac{\delta\gamma}{\gamma} \right) = -\left[ \frac{\chi}{\gamma} + 3Tr\omega_p \alpha^3 \chi^{3/2} \gamma^{1/2}\Omega - 2Tr\omega_p \alpha^3 \chi^{3/2} \gamma^{5/2} \left( \frac{\Omega}{\gamma^2} - \frac{1}{\gamma} \frac{d\Omega}{d\gamma} \right) \right] \frac{\delta\gamma}{\gamma}.
\]
This becomes
\[
\frac{d}{d\zeta} \left( \frac{\delta\gamma}{\gamma} \right) = -\left[ \frac{\chi}{\gamma} + Tr\omega_p \alpha^3 \chi^{3/2} \gamma^{1/2}\Omega + 2Tr\omega_p \alpha^3 \chi^{3/2} \gamma^{7/2} \frac{d\Omega}{d\gamma} \right] \frac{\delta\gamma}{\gamma}.
\]
For given \(\chi = E_z/E_0\) and \(\alpha = (K/k_p)\chi^{-1/2}\), the normalized emittance, electron beam energy, and energy spread are obtained by integrating the coupled equations,
Eqs. (36)–(40), as a function of $\zeta = k_p \zeta$, provided with the initial conditions $\gamma_0$, $(d\delta\gamma/\gamma)_0$, $\Omega_0$, and $(d\Omega/d\zeta)_0$. Here we define the relative radiation loss rate $R$ as the ratio of the radiation loss rate to the acceleration gradient $\chi = E_z/E_0$, i.e.,

$$ R(\zeta) = 2\pi R_0 \alpha^3 \chi^{1/2} \gamma^{3/2} \Omega. \quad (41) $$

IV. NUMERICAL SOLUTIONS OF BETATRON RADIATION DAMPING

Harnessing the state-of-the-art high-energy lasers of the order of 10 kJ and the ongoing development of ultraintense lasers with the peak power of the order of 10 PW and the pulse duration of the order of 1 ps, we explore the feasibility of accelerating electron beams up to 1 PeV and evolution of the beam qualities such as emittance and energy spread at the final beam energy by solving the coupled equations, Eqs. (36)–(40), as a function of $\zeta$ numerically as shown in Sec. III. Table I shows the underlying parameters of the laser-plasma accelerator for the stage energy gain 1 TeV, which are used as the basis for the present study.

Here we assume the initial electron beam with energy of 1 GeV, relative energy spread of 1%, and bunch duration of 10 fs, which is externally injected from a high-quality electron beam injector into the laser-plasma accelerator stage. According to the scaling formulas described in Sec. II, the stage energy gain $W_{\text{stage}} = 1$ TeV can be achieved in the stage length $L_{\text{stage}} = 590$ m, which is operated at the plasma density $n_p = 10^{15}$ cm$^{-3}$. With the coupling length between consecutive stages, $L_{\text{coupl}} = 10$ m, the total length per stage is 600 m and the total linac length at the final beam energy $E_b$ turns out to be $L_{\text{total}}[\text{km}] = 0.6 E_b [\text{TeV}]$ disregarding the synchrotron radiation loss.

For matching both laser and electron beams to the laser-plasma accelerator stages, the ~10-m long coupling section that installs the laser and beam focusing systems may allow us to employ the conventional laser optics and magnets [50]. As mentioned in Appendix B, however, since the emittance growth is maximized at the beam energy of the order of 1 TeV, the radiative energy loss turns out significant and an additional linac length is required for recovering the radiation loss to reach the final beam energy.

Figure 1 shows the process of radiative self-cooling 1(a), the relative radiation loss rate $R$ 1(b), and the numerical solutions of energy $\gamma$, normalized transverse emittance $\epsilon_n$, and relative energy spread $\delta\gamma/\gamma$ 1(c) at the plasma density $n_p = 10^{15}$ cm$^{-3}$. As the beam energy $\gamma$ increases with constant accelerating gradient $E_z = 1.7$ GV/m in the energy region I, $\gamma_0 \leq \gamma \leq \gamma_M$, indicated in Fig. 1(c), the normalized emittance growth initially occurs as $\epsilon_n \propto \gamma^{1/2}$, while the relative energy spread reduces as $\delta\gamma/\gamma \propto \gamma^{-1}$. This behavior is attributed to conservation of the total phase space volume, given by the product of transverse and longitudinal emittances. From the WKB approximation, Eq. (B12), described in Appendix B, the emittance growth comes to balance with radiative damping and reaches the maximum at $\gamma_M$ given by Eq. (B15), where the maximum emittance is approximately given by Eq. (B16). Afterward in the energy region II, $\gamma_M \leq \gamma \leq \gamma_r$, the normalized emittance turns to damping as the relative radiation loss rate $R$ increases until reaching a constant rate

$$ R_T = 2\pi R_0 \alpha^3 \chi^{1/2} \Omega \gamma^{3/2} = \frac{2}{3} \quad (42) $$

at the transition energy $\gamma_T$ that corresponds approximately to $\sim 2\gamma_M$. As shown in Fig. 1(b), the relative radiation loss rate $R$ at the energies higher than $\gamma_T$ is $R \approx 2/3$, while we

| Table I: Example parameters of a 1 TeV laser-plasma accelerator stage. |
|------------------|------------------|
| Energy gain per stage $W_{\text{stage}}$ | 1 TeV |
| Injection beam energy $E_i$ | 10 MeV, 100 MeV, and 1 GeV |
| Plasma density $n_p$ | $1 \times 10^{15}$ cm$^{-3}$ |
| Plasma wavelength $\lambda_p$ | 1056 $\mu$m |
| Accelerating field $E_\perp$ ($\chi = E_z/E_0$) | 1.7 GV/m (0.56) |
| Focusing constant $K/k_p (\alpha)$ | 0.35 (0.47) |
| Stage length $L_{\text{stage}} (L_{\text{coupl}})$ | 590 m (10 m) |
| Charge per bunch $Q_b (N_b)$ | 1.3 nC ($8 \times 10^9$) |
| Matched beam radius and emittance $\sigma_r (e_m)$ | 168 $\mu$m (2622 $\mu$m rad) |
| Initial normalized emittance $\epsilon_{n0}$ | 30, 300, and 3000 $\mu$m rad |
| Initial relative energy spread $\delta\gamma/\gamma$ | 1% |
| Laser wavelength $\lambda_L$ | 1 $\mu$m |
| Laser intensity $I_L (a_L)$ | $3 \times 10^{18}$ W/cm$^2$ (1.5) |
| Laser pulse duration $\tau_L$ | 930 fs |
| Laser spot radius $r_L$ | 510 $\mu$m |
| Laser peak power $P_L$ | 12 PW |
| Laser energy per stage $U_L$ | 11 kJ |
| Plasma channel depth at $r_L \Delta n_c/n_p$ | 0.44 |
and normalized transverse emittance $\varepsilon_n$, respectively, where $R_T$ is the relative radiation loss rate.

The initial conditions of beam energy, normalized transverse emittance $\varepsilon_n$, and energy spread are $\varepsilon_{n0} = 3000\ \mu m$, and $\gamma_0 = 2000$, respectively. The accelerating gradient is $E_z/E_0 = 0.56$ and the focusing constant $K/k_p = 0.35$. The evolution of normalized transverse emittance is characterized by three energy regions: Region I ($\gamma_0 \leq \gamma_M$), Region II ($\gamma_M \leq \gamma \leq \gamma_T$), and Region III ($\gamma_T \leq \gamma$), respectively, where $\gamma_M$ is the energy at which normalized transverse emittance reaches the maximum and $\gamma_T$ is the energy at which the relative radiation loss rate $R_T$ reaches the maximum value $R_T = 2/3$, where the self-cooling due to betatron radiation starts.

As shown in Fig. 1, the WKB approximation is in excellent agreement with the numerical solution in the energy region I (the adiabatic region), while it starts in deviating from the numerical solution at the energy $\gamma_M$ where the normalized emittance is maximized. For $\gamma \geq \gamma_M$ in the energy regions II and III, the WKB solution deviates more and more from the numerical solution and eventually fails to approximate it at the PeV energies. The relative radiation loss rate reaches the maximum value $R_T$ at $\gamma_T$ in the energy region II named the transition region, where the radiation loss rate significantly increases up to 2/3 of the accelerating gradient. Since the assumptions of the WKB approximation become invalid above $\gamma \geq \gamma_M$, a discrepancy in the maximum value $R_T$ between two solutions becomes significant. However, the WKB solution can analytically show that the expression for the maximum value $R_T \approx 0.49$ includes no parameter dependence as indicated in Eq. (B18). Hence, one can infer that the more exact value $R_T = 2/3$ obtained from the numerical solution is regardless of the accelerating gradient, the focusing constant, plasma density, and the initial conditions.

It is noted that the $R_T$ is evaluated without regard to the initial conditions $\gamma_0, \varepsilon_{n0}$, and the plasma wakefield parameters $n_p, E_z/E_0$, and $K/k_p$. Figure 2 shows the evolutions of the normalized emittances $\varepsilon_n$, the relative energy spreads $\delta \gamma/\gamma$, and the relative radiation loss rate $R_T$ for the different initial conditions $\varepsilon_{n0}$ and $\gamma_0$. The evolution of normalized transverse emittance goes in the same way in the energy region III ($\gamma > \gamma_T$) regardless of the initial electron beam conditions. In Fig. 3, compared are three laser-plasma accelerator cases for the accelerating gradient $E_z/E_0$ and the focusing constant $K/k_p$, corresponding to the quasi-linear regime ($a_0 = \sqrt{2}$), the blow-out regime ($a_0 = 4$), and the bubble regime ($a_0 = 9$). In the bubble or blow-out regime [18,19], where plasma electrons radially expelled by the radiation pressure of the laser form a sheath with thickness of the order of the plasma skin depth $c/\omega_p$ outside the ion sphere remaining unshielded behind the laser pulse moving at relativistic velocity. In an electron-blow-out bubble, the transverse fields are composed of the electric fields from the ion sphere $E_{ion}/E_0 = k_pr/2$, the radial plasma current $E_{relm}/E_0 = -k_pr/4$ and the magnetic field.
From the radial plasma current \( B_{\text{tot}}/E_0 = -k_p r / 4 \). The total focusing field on a beam electron is \( E_r - B_{\text{tot}} = (1/2)mc^2k_p^2 r \). Thus, the focusing constant is \( K = k_p/\sqrt{2} \) for both laser- and beam-driven wakefields. In the bubble regime, the maximum accelerating field is given by \( E_z/E_0 = k_p R_B/2 \), where \( R_B \) is the bubble radius. Using the matched bubble condition [50], \( k_p R_B = k_p R_L = 2\sqrt{a_L} \), the accelerating field in the bubble regime is given by \( \chi = E_z/E_0 = \sqrt{a_L} \).

From the viewpoint of radiative cooling, however, one can see some differences only in Region II and minor differences in the final normalized emittance. The difference in the plasma wakefield regime has only minor effects on the self-cooling due to betatron radiation. Figure 4 shows the cooling behavior for different plasma densities of \( 10^{15} \text{ cm}^{-3} \), \( 10^{16} \text{ cm}^{-3} \), \( 10^{17} \text{ cm}^{-3} \), and \( 10^{18} \text{ cm}^{-3} \), respectively. Hence, it is confirmed that the constant radiation loss rate \( R \approx 2/3 \) in Region III for all aforementioned cases. Thus, we can estimate the normalized emittance in Region III as

\[
\varepsilon_n = \frac{1}{8\pi r_e^2 n_p} \chi^{3/2} \left( \frac{E_z}{E_0} \right)^3 \left( \frac{K}{k_p} \right)^{3/2} \left( \frac{n_p}{10^{15} \text{ cm}^{-3}} \right)^{-1} \left( \frac{E_b}{1 \text{ PeV}} \right)^{-3/2},
\]

where \( E_b \) is the electron beam energy.

In a plasma focusing channel, multiple Coulomb scattering between a beam electron and a plasma ion counteracts the radiation damping due to betatron radiation. The damping rate of the normalized emittance is given by

\[
\left( \frac{d\varepsilon_n}{dt} \right)_{\text{RAD}} \approx \frac{58}{\chi R} \left( \frac{E_z}{E_0} \right)^3 \left( \frac{K}{k_p} \right)^{3/2} \left( \frac{n_p}{10^{15} \text{ cm}^{-3}} \right)^{-1} \left( \frac{E_b}{1 \text{ PeV}} \right)^{-3/2},
\]

where \( R = 2\pi r_e n_p \chi^{3/2} \Omega^{3/2} \) is the relative radiation loss rate. The growth rate for the normalized emittance due to multiple Coulomb scattering [34] is given by

\[
\left( \frac{d\varepsilon_n}{dt} \right)_{\text{SCAT}} \approx \frac{c}{4} \left( \frac{\lambda_p}{R_N} \right) \left( \frac{K}{\gamma} \right)^{1/2} \ln \left( \frac{\lambda_p}{R_N} \right),
\]

where \( Z \) is the charge state of the ion, \( \lambda_p = (T_e/4\pi n_p e^2)^{1/2} \) is the Debye length for the plasma temperature \( T_e \), eV and
Solving this equation with respect to \( \gamma \) gives an estimate of the equilibrium energy \( \gamma_{\text{EQ}} \) at which the radiative damping reaches the balance with the emittance growth as

\[
\gamma_{\text{EQ}} = \frac{3}{4} \left( \frac{1 - R}{2 \pi r_p Z \ln(\Lambda_D / R_N)} \right)^{1/2} \frac{E_c}{E_0} \left( \frac{K}{k_{p'}} \right)^{-1} = 6 \times 10^6 \frac{E_c}{(Z \Lambda)^{1/2}} \frac{K}{k_{p'}} \left( \frac{n_p}{10^{15} \text{ cm}^{-3}} \right)^{-1/2},
\]

where \( R = 2/3 \) for \( \gamma \geq \gamma_r \) and \( \Lambda \) is calculated by

\[
\Lambda = \frac{1}{24.7} \ln \left( \frac{\Lambda_D}{R_N} \right) = 1 + 0.047 \log \left( \frac{T_e}{10 \text{ [eV]}} \left( \frac{n_p A^{2/3}}{10^{15} \text{ [cm}^{-3}] \right) \right).
\]

Thus, the equilibrium emittance is estimated as

\[
e_{\text{nEQ}} = 11 \text{[nm]} \left( Z \Lambda \right)^{3/4} \left( \frac{E_c}{E_0} \right)^{-1/2} \left( \frac{K}{k_{p'}} \right)^{-3/2} \left( \frac{n_p}{10^{15} \text{ [cm}^{-3}] \right)^{-1}.
\]

For example, with \( E_c/E_0 = 0.56 \), \( K/k_{p'} = 0.35 \), and \( n_p = 10^{15} \text{ cm}^{-3} \), the equilibrium emittance becomes \( e_{n\text{EQ}} = 71 \text{ nm at } \gamma_{\text{EQ}} \approx 9.6 \times 10^9 \text{ (4.9 PeV) in a hydrogen plasma with } Z = 1 \text{ and } A = 1.\)

The quantum mechanical consideration of radiation damping in the continuous focusing channel results in the minimum normalized emittance \( e_{\text{nmin}} = \lambda / 2 \approx 0.2 \text{ pm} \), where \( \lambda = h / mc \) is the Compton wavelength, which is the fundamental emittance limited by the uncertainty principle, in case no other excitation sources than radiation reaction are present. According to the emittance scaling Eq. (43) in Region III, this quantum limit may be achieved at the electron energy \( E_p = 2.4 \times 10^{10} \text{ eV for } n_p = 10^{15} \text{ cm}^{-3} \). These limitations imposed on the radiative cooling are shown in Fig. 4.

V. DISCUSSIONS AND CONCLUSIONS

The remarkable progress on plasma-based acceleration offers us prospects for exploring high-energy-frontier physics and high-energy astrophysical phenomena that are relevant to extreme high-energy particle beams in the TeV–PeV range, which conventional accelerators may be difficult to reach. Although further developments must be required in stability and controllability of plasma-based accelerators for high-energy physics applications such as \( e^+e^- \) linear colliders [42], a most crucial issue is the capability of producing extremely qualified beams to yield substantial events of fundamental particle reactions. For example, in the \( e^+e^- \) collider the event rate \( Y \) is estimated to be \( Y = L \sigma \), where \( L \) is the luminosity and \( \sigma \) is the collision cross section for the reaction \( e^+ e^- \rightarrow e^+ e^- \), which scales as \( \propto \gamma^{-2} \) for the center-of-mass (c.m.) energy \( E_{\text{cm}} = 2 \gamma mc^2 \) with the electron rest mass \( m \) and the speed of light in vacuum \( c \). The geometric luminosity \( L \) is given by

\[
L = \frac{f N^2}{4 \pi \sigma_x \sigma_y} = \frac{\gamma f N^2}{4 \pi E_p \beta^2},
\]

where \( f \) is the collision frequency, \( N \) is the number of particles per bunch, \( \sigma_x^* \) and \( \sigma_y^* \) are the horizontal and
vertical rms beam sizes at the interaction point (IP), respectively, assuming \( \sigma_z^* = \sigma_r^* = (\epsilon_n^* B^*/\gamma)^{1/2} \) for the normalized transverse emittance \( \epsilon_n^* \) and the beta function \( B^* \) at IP. Assuming \( fN^2/(4\pi B^*) \) keeps constant with respect to \( \gamma \), in order to keep the event rate constant as the c.m. energy increases, the normalized emittance must decrease as \( \epsilon_n^* \propto \gamma^{-1} \). Hence, for future linear colliders, proposed is radiative damping due to synchrotron radiation in the uniform bending fields of a few GeV \( e^-/e^+ \) storage ring or radiative cooling via laser-electron Compton scattering [55] at the energy around 100 MeV in a laser-electron storage ring [56] or at the beam energy around 5 GeV [57]. These cooling methods are based on radiative energy loss and after recovery of its loss only in the longitudinal direction at the relatively low energy stage before accelerating beams up to the final energies of the TeV-range.

We have investigated the electron dynamics in plasma-based accelerator numerically, including electron energy, the normalized transverse emittance, and energy spread. The self-cooling effect of the normalized emittance due to betatron radiation was studied. There has been a controversy that plasma-based accelerators in which electrons undergo strong accelerating and focusing forces fall into disfavor by reason of the emittance growth [58] and the synchrotron radiation loss [35] in the TeV-energy regime. Our study would give a long-pending question the answer that self-cooling of the normalized emittance scales as \( \epsilon_n \propto \gamma^{-3/2} \) in the TeV-range requires exhaustive considerations from physical and technological points of view. The following problems remaining to be solved will be further pursued as the future subjects. The detailed analyses on evolutions of plasma wakefields and beam dynamics arising from small-scale detrimental imperfections of laser-plasma accelerators have not been considered. The multidimensional nonlinear effects in the propagation of ultraintense laser pulses and electron beams such as the self-modulation and both laser- and beam-hose instabilities are also neglected. In consideration on a limit of normalized transverse emittance at the PeV-energy level, bremsstrahlung and \( e^-/e^+ \) pair production may become important at the higher plasma densities. However, since the inverse of the radiation length via them, i.e., the emittance growth rate, is much smaller than that due to Coulomb scattering from nuclei, we can neglect effects of bremsstrahlung and pair production caused by interactions of high-energy beam electrons with plasma. A rough estimate of the radiation length of hydrogen plasma [37] is given by

\[
X_{\text{RAD}} \approx 4 \alpha r^2 n_p \times 11.454
\]

\[
\approx 3.8 \times 10^{15} \text{ cm} \left( \frac{n_p}{10^{15} \text{ cm}^{-3}} \right).
\]

where \( \alpha \approx 1/137 \) is the fine structure constant. With Eq. (45), the corresponding interaction length of electron-ion collision in hydrogen plasma with temperature \( T_e = 10 \text{ eV} \) is roughly

\[
r_{\text{B}} = \left( \frac{\epsilon_n}{K_k} \right)^{1/2} = \frac{1}{k_p} \gamma \left( \frac{1}{2k_p} \right)^{1/2} \left( \frac{E_z}{E_0} \right)^{1/2}
\]

\[
\approx 14.8 \text{ [nm]} \left( \frac{K_k}{k_p} \right)^{-2} \left( \frac{E_z}{E_0} \right)^{1/2} \left( \frac{n_p}{10^{15} \text{ [cm}^{-3}] \right)^{-3/4}
\]

\[
\times \left( \frac{E_b}{1 \text{ [PeV]} \right)^{-1}.
\]

For the above-mentioned example, the matched betatron amplitude is 90 nm at 1 PeV. Our finding that the normalized emittance scales as \( \epsilon_n \propto \gamma^{-3/2} \) in the energies higher than the transition energy \( \gamma_T \) would satisfy the aforementioned luminosity requirement \( \epsilon_n \propto \gamma^{-1} \) for the high-energy-frontier collider.

As expected, the reduction of the relative energy spread scales as \( \delta \gamma/\gamma \propto \gamma^{-1} \) in the adiabatic region (Region I) and a sudden reduction occurs at the transition energy (Region II), followed by decreasing the relative energy spread that approximately scales as \( \propto \gamma^{-1} \) at the reduced level in the self-cooling region (Region III) compared to that of Region I. This means that an absolute value of the energy spread \( \delta \gamma \) is reduced, while it is approximately constant in Region I.

It goes without saying that the realization of the large-scale laser-plasma accelerator aiming at the unprecedented energy range requires exhaustive considerations from physical and technological points of view. The following problems remaining to be solved will be further pursued as the future subjects. The detailed analyses on evolutions of plasma wakefields and beam dynamics arising from small-scale detrimental imperfections of laser-plasma accelerators have not been considered. The multidimensional nonlinear effects in the propagation of ultraintense laser pulses and electron beams such as the self-modulation and both laser- and beam-hose instabilities are also neglected.
\[ X_{\text{collision}} \approx 4 r_p^2 n_p \ln \left( \frac{\lambda_D}{R_N} \right) = 1.3 \times 10^8 \, \text{[cm]} \left( \frac{10^{15} \, \text{cm}^{-3}}{n_p} \right). \]

This interaction length is comparable to the acceleration length in which electrons reach 1 PeV with synchrotron radiation loss at the plasma density of \( n_p = 10^{15} \, \text{cm}^{-3} \), while the bremsstrahlung radiation length, where the electron loses \( 1/e \) of its energy, is roughly 200 times longer than that.

Although we assume the betatron radiative damping of electrons, the derived coupled equations expressed in the dimensionless form will be applicable to the radiative damping of a particle beam with the higher rest mass. In the energy region III defined as the cooling region, the normalized emittance scales as \( \varepsilon_n \propto \pi^{-1} \gamma^{-3/2} \propto m \gamma^{-3/2} \), where \( m \) is the particle rest mass. Considering the betatron radiative damping of the proton beam, the relativistic factor \( \gamma_p \) of the proton beam that reaches to the same normalized emittance as that of the electron beam with the energy \( \gamma_e \) is given by \( \gamma_p = \left( \frac{m_p}{m_e} \right)^{3/2} \gamma_e = 150 \gamma_e \). Hence, the proton energy required for cooling the beam down to the same emittance as that of the electron becomes \( E_p = \left( \frac{m_p}{m_e} \right)^{3/2} E_e = 2.8 \times 10^6 E_e \), where \( m_p/m_e = 1836 \) is the proton/electron mass ratio. The normalized emittance at the electron energy 1 PeV is obtained at the proton energy \( -2.8 \times 10^{20} \, \text{eV} \), which is incidentally coincident with the observed energy limit of the highest energy cosmic rays [10].

In conclusion, we have explored feasibility of laser-plasma accelerators for reaching a 1-PeV level and beam dynamics on energy, energy spread, and normalized emittance in the TeV-PeV range from a purely physical point of view, apart from the detailed technological problems and small-scale nonlinear effects that might be suppressed or mitigated by robust properties of self-cooling due to betatron radiation. The results may provide us with an insight into long-pending questions in controversy about feasibility of plasma-based accelerators on the limits of acceleration and beam qualities due to strong betatron motion and synchrotron radiation arising beyond the TeV regime. The coupled equations describing beam dynamics have been derived regardless of any small-scale wakefield nonlinearities such as the linear and nonlinear regimes and any drive sources such as lasers and beams. Hence, the derived equations may be valid for cosmic acceleration process, if the accelerating and focusing forces are specified. Our results suggestively conjecture the acceleration and radiation processes in the astrophysical conditions as well as future ultrahigh-energy particle accelerators and may give some hints to explore the origin of ultrahigh-energy cosmic rays [43,59].

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**APPENDIX A: SOLUTION OF NORMALIZED EMITTANCE EVOLUTION WITHOUT RADIATION**

Assuming \( \tau_{\text{R}} \omega_p^2 \ll 1 \), and \( d\gamma/d\zeta = \chi \). Eq. (36) is written as

\[
\frac{d^2 \Omega}{d\gamma^2} + \frac{1}{\gamma} \left( \frac{2\alpha \gamma^{1/2}}{\chi^{1/2}} - 1 + \tau_{\text{R}} \omega_p \alpha^2 \gamma \right) \frac{d\Omega}{d\gamma} = 0.
\]

(A1)

In case of no radiation damping and no linear focusing force \( \alpha = 0 \), the equation is reduced to

\[
\frac{d^2 \Omega}{d\gamma^2} - \frac{1}{\gamma} \frac{d\Omega}{d\gamma} = 0.
\]

(A2)

This shows the conservation of the normalized emittance if the initial emittance growth is \( (d\Omega/d\gamma)_0 = 0 \). When the initial emittance growth \( (d\Omega/d\gamma)_0 = \Omega'_0 \), the normalized emittance is given by

\[
\Omega(\gamma) = \Omega_0 + \frac{1}{2} \Omega'_0 \left( \frac{\gamma^2}{\gamma_0^2} - 1 \right).
\]

(A3)

Letting

\[
\sqrt{\frac{\gamma}{\chi}} = s, \quad \gamma = \chi s^2, \quad \frac{ds}{d\gamma} = \frac{1}{2 \chi s^2}.
\]

(A4)

Eq. (A1) is rewritten as

\[
\frac{d^2 \Omega}{ds^2} + \left[ 4\alpha - \frac{3}{s} + 2\tau_{\text{R}} \omega_p \alpha^2 \chi (s + 6\alpha \chi \Omega s^2) \right] \frac{d\Omega}{ds} - 4 \left[ \frac{\alpha}{s} - \tau_{\text{R}} \omega_p \alpha^2 \chi (2\alpha s - 1 + s) \right] \Omega = 0.
\]

(A5)

Neglecting nonlinear terms of \( \Omega^2 \) and \( d\Omega^2/ds \),

\[
\frac{d^2 \Omega}{ds^2} + 2 \left( 2\alpha - \frac{3}{2s} + \tau_{\text{R}} \omega_p \alpha^2 \chi s \right) \frac{d\Omega}{ds} - 4 \left[ \frac{\alpha}{s} - \tau_{\text{R}} \omega_p \alpha^2 \chi (2\alpha s - 1) \right] \Omega = 0.
\]

(A6)
Disregarding radiation terms, Eq. (A6) becomes
\[
\frac{d^2 \Omega}{ds^2} + (4\alpha s - 3) \frac{d\Omega}{ds} - 4\alpha \Omega = 0. \tag{A7}
\]
With \( \Omega(s_0) = \Omega_0 \) and \( \Omega'(s_0) = 0 \) at \( s = s_0 \), Eq. (A7) has the exact solution as
\[
\Omega(s) = \Omega_0 \left[ 1 + \frac{1}{2\alpha s_0} + \frac{1}{8\alpha^2 s_0^2} s - \frac{3}{4\alpha s_0} \right] + \frac{1}{4\alpha s_0} s^2 + \frac{1}{\alpha s_0} s + \frac{3}{8\alpha^2 s_0^2} \times \exp \left[ -4\alpha s_0 \left( \frac{s}{s_0} - 1 \right) \right]. \tag{A8}
\]
Using Eq. (A4), the dimensionless normalized emittance is expressed as
\[
\Omega(\gamma) = \Omega_0 \left[ 1 + \frac{1}{2\alpha \gamma_0} + \frac{1}{8\alpha^2 \gamma_0^2} \left( \sqrt{\gamma} - \frac{3}{4\alpha \gamma_0} \right) \right] + \frac{1}{4\alpha \gamma_0} \sqrt{\gamma} \left( \frac{\gamma}{\gamma_0} + \sqrt{\gamma \gamma_0} + \frac{3\gamma}{8\alpha^2 \gamma_0^2} \right) \times \exp \left[ -4\alpha \gamma_0 \left( \gamma / \gamma_0 - 1 \right) \right]. \tag{A9}
\]
This means the normalized transverse emittance increases as \( \epsilon_n \propto \gamma^{1/2} \) when the energy increases. From Eq. (40), regardless of radiation, the energy spread is given by
\[
\frac{\delta \gamma}{\gamma} = \left( \frac{\delta \gamma}{\gamma_0} \right) \frac{\gamma_0}{\gamma}. \tag{A10}
\]
This behavior is attributed to conservation of the total phase space volume \( e_n e_n' (\delta \gamma / \gamma) = \) constant, given by the product of transverse and longitudinal emittances. That is in the case of an adiabatic process resulting from Liouville’s theorem in beam dynamics.

**APPENDIX B: WKB APPROXIMATION OF NORMALIZED EMITTANCE EVOLUTION WITH RADIATION**

As shown in Eq. (A9), disregarding radiation causes the emittance growth, regardless of fast damping of the second term due to the focusing force. However, notice that considering radiation results in significant emittance damping thanks to the nonlinear damping term in Eq. (A5), which counteracts the emittance growth so that the adiabatic increase balances with the radiative damping at the maximum emittance. In order to evaluate the maximum emittance, we use the WKB (Wentzel-Kramers-Brillouin) method \([60]\) to obtain the approximate solution of Eq. (A5) as follows. Writing Eq. (A5) as
\[
\frac{d^2 \Omega}{ds^2} + 2a(s) \frac{d\Omega}{ds} + b(s) \Omega = 0, \tag{B1}
\]
where
\[
a(s) = 2\alpha - \frac{3}{2s} + \tau_R \omega_p \alpha^2 \chi s + 6\tau_R \omega_p \alpha^3 \chi^2 s^2, \tag{B2}
\]
and
\[
b(s) = -\frac{4\alpha}{s} + 4\tau_R \omega_p \alpha^2 \chi (2\alpha s - 1) + 4\tau_R \omega_p \alpha^3 \chi^2 s (6\alpha s - 1) \Omega. \tag{B3}
\]
An approximate solution is given by
\[
\Omega(s) \sim \Phi(s) \exp \left[ -\int a(s) ds \right] = \Phi(s)^{3/2} \exp \left[ -2\alpha s - \frac{1}{2} \tau_R \omega_p \alpha^2 \chi s^2 - 6\tau_R \omega_p \alpha^3 \chi^2 \int s^2 \Omega ds \right]. \tag{B4}
\]
where
\[
\frac{d^2 \Phi}{ds^2} + k^2 P(s) \Phi(s) = 0, \tag{B5}
\]
and
\[
-k^2 P(s) = a^2 + \frac{da}{ds} - b
\]
\[
= 4\alpha^2 + \frac{15}{4s^2} - \frac{2\alpha}{s} + 2\tau_R \omega_p \alpha^2 \chi - 4\tau_R \omega_p \alpha^3 \chi s + \tau_R^2 \omega_p \alpha^4 \chi^2 s^2 + \cdots
\]
\[
= 4\alpha^2 \left( 1 - \frac{1}{4\alpha s} \right)^2 + \frac{7}{2s^2} + \cdots. \tag{B6}
\]
Here we first neglect all radiative terms with \( \tau_R \omega_p \) in Eq. (B6) compared to the nonradiative terms because of
\[
\tau_R \omega_p = \frac{2r_e}{3c} \omega_p = \frac{4\pi r_e}{3\lambda_L} \left( \frac{n_e}{n_p} \right)^{1/2}
\]
\[
= 1.1 \times 10^{-12} \left( \frac{n_p}{10^{15} \text{cm}^{-3}} \right)^{1/2},
\]
and secondly neglect a part of the \( 1/s^2 = \chi / \gamma \) term because of \( 1/s^2 \ll 1/s = (\chi / \gamma)^{1/2} \) for \( \gamma \gg 1 \). Letting
\[
k = 2\alpha, \quad P(s) = -\left( 1 - \frac{1}{4\alpha s} \right)^2,
\]
\( \Phi(s) \) is given by
\[ \Phi(s) \sim P(s)^{-1/4}[A' \exp(k|s|) + B' \exp(-k|s|)] \\
= \left(1 - \frac{1}{4\alpha}\right)^{-1/2} \left[A's^{-(1/2)} \exp(2\alpha s) + B's^{(1/2)} \exp(-2\alpha s)\right] \\
= \left(s - \frac{1}{4\alpha}\right)^{-1/2} \exp(2\alpha s)[A' + B's \exp(-4\alpha s)]. \\
\] (B7)

With the initial conditions \( \Omega(s_0) = \Omega_0 \) and \( d\Omega(s_0)/ds = 0 \), the solution of \( \Omega(s) \) is

\[ \Omega(s) \sim A\Omega_0 \exp\left[-\frac{1}{2} \tau_R \omega_p \alpha^2 \chi(s^2 - s_0^2) - 6\tau_R \omega_p \alpha^2 \chi^2 \int_{s_0}^{s} s^2 \Omega ds\right] \times \left[\frac{s}{s_0} + B \left(\frac{s}{s_0}\right)^2 \exp[-4\alpha(s - s_0)]\right]. \] (B8)

where

\[ A = \frac{8\alpha s_0 - 2 + \tau_R \omega_p \alpha^2 \chi s_0^2}{8\alpha s_0 - 1}, \quad \text{and} \]

\[ B = \frac{1 - \tau_R \omega_p \alpha^2 \chi s_0^2}{8\alpha s_0 - 2 + \tau_R \omega_p \alpha^2 \chi s_0^2}. \] (B9)

In Eq. (B8), the nonlinear damping integral is calculated as follows:

\[ I = \int_{s_0}^{s} s^2 \Omega ds \\
\approx A\Omega_0 \left[\frac{1}{2} \int_{s_0}^{s} s^3 ds + B \frac{\exp(4\alpha s_0)}{s_0^2}\right] \times \int_{s_0}^{s} s^4 \exp(-4\alpha s) ds \\
\approx \frac{1}{4} A\Omega_0 \left[\frac{s^4}{s_0^4} - 1\right] + B\frac{g(s_0)}{\alpha s_0^2} \left[1 - \frac{g(s)}{g(s_0)}\right] \times \exp[-4\alpha(s - s_0)]. \] (B10)

where

\[ g(s) = s^4 + \frac{1}{\alpha} s^3 + \frac{3}{4\alpha^2} s^2 + \frac{3}{8\alpha^3} s + \frac{3}{32\alpha^4}. \] (B11)

Finally, the WKB approximation is given by

\[ \Omega(s) \sim A\Omega_0 \left[\frac{s}{s_0} + B \left(\frac{s}{s_0}\right)^2 \exp[-4\alpha(s - s_0)]\right] e^{-D(s)}. \] (B12)

where

\[ D(s) = \frac{1}{2} \tau_R \omega_p \alpha^2 \chi(s^2 - s_0^2) + \frac{3}{2} \tau_R \omega_p \alpha^3 \chi^2 \Omega_0 \left[s^3 \left(\frac{s}{s_0}\right)^4 - 1\right] + B g(s_0) \left[1 - \frac{g(s)}{g(s_0)} \exp[-4\alpha(s - s_0)]\right]. \] (B13)

In Eq. (B13), the first term shows the linear damping proportional to \( \gamma \), and the second term causes the nonlinear damping proportional to \( \gamma^2 \).

Calculating \( d\Omega/ds = 0 \), one can estimate the maximum amplitude of \( \Omega(s_M) \), where the normalized emittance reaches the maximum value at \( s = s_M \) given by

\[ s_M = \left(\frac{s_0}{6\tau_R \omega_p \alpha^3 \chi^2 \Omega_0}\right)^{1/4}. \] (B14)

The corresponding \( \gamma \) and the maximum value \( \Omega_M \) are obtained as

\[ \gamma_M = \frac{\gamma_0^{1/4}}{\chi} \left(\frac{1}{6\tau_R \omega_p \alpha^3 \Omega_0}\right)^{1/2} \]
\[ = \frac{1}{4\tau_p \alpha} \left(\frac{1}{\pi \alpha n_p e^0}\right)^{1/2} \left(\frac{\gamma_0^{1/4}}{\chi}\right)^{1/2} \]
\[ = \frac{\gamma_0^{1/4}}{4\tau_p} \left(\frac{E_x}{E_0}\right)^{1/2} \left(\frac{K}{k_p}\right)^{-3/2} \left(\frac{1}{\pi \alpha n_p e^0}\right)^{1/2} \]
\[ = 1.58 \times 10^6 \gamma_0^{1/4} \left(\frac{E_x}{E_0}\right)^{1/2} \left(\frac{K}{k_p}\right)^{-3/2} \]
\[ \times \left(\frac{n_p}{10^{15} \text{ cm}^{-3}}\right) \left(\frac{e^0}{1 \mu m}\right)^{-1/2}, \] (B15)

and

\[ \Omega_M = 0.78 \Omega_0 \left(\frac{\chi}{\gamma_0}\right)^{1/2} \left(\frac{\gamma_0^{1/2}}{6\tau_R \omega_p \alpha^3 \chi^2 \Omega_0^{1/2}}\right)^{1/4} \]
\[ = 0.39 \Omega_0 \left(\frac{E_x}{E_0}\right)^{1/2} \left(\frac{K}{k_p}\right)^{-3/4} \left(\frac{1}{\pi \alpha n_p e^0}\right)^{1/4} \]
\[ = 980 \Omega_0 \gamma_0^{-3/8} \left(\frac{E_x}{E_0}\right)^{1/4} \left(\frac{K}{k_p}\right)^{-3/4} \]
\[ \times \left(\frac{n_p}{10^{15} \text{ cm}^{-3}}\right) \left(\frac{e^0}{1 \mu m}\right)^{-1/4}, \] (B16)

respectively. As shown in Fig. 1, this energy and the maximum value of normalized emittance are in good agreement with those obtained from the numerical solution.

Equation (38) shows that an energy loss due to synchrotron radiation may limit acceleration of electron beams when the radiation loss rate surpasses the accelerating gradient as the beam energy increases under the condition of

\[ R = 2\tau_R \omega_p \alpha^3 \chi^{1/2} \Omega \gamma^{3/2} > 1. \] (B17)
However, substituting Eq. (B12) into Eq. (B17) and evaluating $dR/ds = 0$, the maximum value $R_T$ at $s = s_T$ can be calculated as

$$R_T = R(s_T) \approx \frac{4}{3} \varepsilon_0^{-1} \approx 0.49,$$

(B18)

where

$$s_T = \left(\frac{2s_0}{3\pi R_0 \rho \alpha^2 \lambda^2 \Omega_0}\right)^{1/4}.$$

(B19)

The corresponding $\gamma$ is

$$\gamma_T \approx \frac{\gamma_0}{\chi} \left(\frac{2}{3\pi R_0 \rho \alpha^2 \lambda^2 \Omega_0}\right)^{1/2} = \frac{1}{2\tau_\epsilon} \left(\frac{1}{\pi \alpha^3 n_p \varepsilon_0}\right)^{1/2} \left(\frac{\gamma_0}{\chi}\right)^{1/4}$$

$$= \frac{\gamma_0^{1/4}}{2\tau_\epsilon} \left(\frac{E_z}{E_0}\right)^{1/2} \left(\frac{K}{k_p}\right)^{-3/2} \left(\frac{1}{\pi \alpha^3 n_p \varepsilon_0}\right)^{1/2}$$

$$= 3.17 \times 10^6 \gamma^{1/4} \left(\frac{E_z}{E_0}\right)^{1/2} \left(\frac{K}{k_p}\right)^{-3/2} \times \left(\frac{n_p}{10^{15} \text{ cm}^{-3}}\right)^{-1/2} \left(\frac{\varepsilon_0}{1 \text{ [µm]}^3}\right).$$

(B20)

Notice that $R_T$ is constant, regardless of the accelerating gradient, the focusing constant, and the initial conditions, and that $\gamma_T \sim 2\gamma_{\text{max}}$. It is found that the radiation loss rate is always smaller than the accelerating gradient, i.e. $R < 1$, over $\gamma \rightarrow \infty$, which implies that the beam energy can be increased without limit due to the radiative loss and the normalized emittance evolution [61].


