Options pricing and hedging under canonical valuation have recently been demonstrated to be quite effective, but unfortunately are only applicable to European options. This study proposes an approach called canonical least-squares Monte Carlo (CLM) to price American options. CLM proceeds in three stages. First, given a set of historical gross returns (or price ratios) of the underlying asset for a chosen time interval, a discrete risk-neutral distribution is obtained via the canonical approach. Second, from this canonical distribution independent random samples of gross returns are taken to simulate future price paths for the underlying. Third, to those paths the least-squares Monte Carlo algorithm is then applied to obtain early exercise strategies for American options. Numerical results from simulation-generated gross returns under geometric Brownian motions show that the proposed method yields reasonably accurate prices for American puts. The CLM method turns out to be quite similar to the nonparametric approach of Alcock and Carmichael and simulations done with CLM provide...
additional support for their recent findings. CLM can therefore be viewed as an alternative for pricing American options, and perhaps could even be utilized in cases when the nature of the underlying process is not known. © 2009 Wiley Periodicals, Inc. Jrl Fut Mark 30:175–187, 2010

INTRODUCTION
Volatility, one of the most important parameters in modern options pricing, is unfortunately the only parameter that cannot be observed directly in the capital market. To bypass the problems of volatility estimation or volatility modeling as well as of making assumptions about the distribution of the underlying process altogether, Stutzer (1996, 2000) proposed the so-called canonical valuation method, a nonparametric approach. Canonical valuation derives a risk-neutral distribution from the observed time series data of the underlying price and then uses the resultant canonical probabilities to average directly the expiry payoffs to obtain the price of a European option. Two features of canonical valuation stand out. First, the risk-neutral distribution is derived directly from the realized empirical data, so it is unnecessary for us to be concerned about modeling the distribution of the underlying price. Second, the resultant risk-neutral distribution incorporates historical time-varying volatilities directly, thus the issue of volatility estimation is completely out of the picture.

Canonical valuation was first applied to S&P 500 index options, taking into consideration Crash-era data with encouraging results (Stutzer, 1996). The approach was found to outperform the Black model for CBOT bond futures options (Stutzer & Chowdhury, 1999). Compared with either the Black model or the canonical valuation method, a modified version of canonical valuation seemed to produce better prices for soybean futures options traded on CBOT (Foster & Whiteman, 1999, 2006). Recently Gray’s group investigated the canonical method and produced interesting and encouraging results. They reported that canonical valuation outperforms the Black–Scholes model in pricing options when the volatility is stochastic (Gray & Newman, 2005). They found that the outcomes of dynamic hedging under canonical valuation are superior to those under Black–Scholes (Alcock & Gray, 2005). Lastly, for Australia All Ordinaries Index options, the so-called constrained canonical valuation yields very good prices and superior hedging outcomes compared with the Black–Scholes model (Gray, Edwards, & Kalotay, 2007).

Given those exciting developments, it would be highly desirable and valuable if canonical valuation could be adapted to take into consideration the early exercise features of American options. This study tries to fill in such a gap by suggesting a possible solution of the problem.
While this study was under review, the author was informed of an important recent development. Alcock and Carmichael (2008) proposed a worthwhile extension of canonical valuation to price American options by directly simulating the paths of the underlying price with historical price ratios and by using a weighted least-squares regression to incorporate the early exercise features. Although their approach and the method proposed in this study seem to be quite close in spirit, the two methods differ greatly on how the historical price data is used, how the paths are simulated, and how the least-squares regression is done. Still, it is important to note that no significant methodological advance beyond Alcock and Carmichael (2008) is suggested here. Further, it is reassuring that simulations done in this study provide additional support for their findings.

This study first proposes the idea of combining the risk-neutral distribution from canonical valuation with the least-squares Monte Carlo (LSM) algorithm (Longstaff & Schwartz, 2001) to price American options, then presents the results of options prices obtained using risk-neutral distributions estimated from two sets of simulation-generated gross returns, and compares the prices with those of two known benchmarks. The study finishes with a few concluding remarks.

CANONICAL LSM

Canonical Risk-Neutral Distribution

Canonical valuation (see Appendix A1 for details) cannot be used to price American options, because possible future prices for the underlying asset before maturity are not projected. Instead, the risk-neutral probabilities at expiration are used directly to average the expiry payoffs of an option (see Equation (A4)).

To take the early exercise features of an American option into consideration, however, one needs intermediate prices of the underlying before the option’s maturity. In order to simulate a price path for the underlying of a \(T - t\) years option (for the meaning of \(t\) and \(T\), see Appendix A1), it is necessary to work with the underlying’s gross returns for a much shorter time interval. Therefore, instead of taking a sample of \(N\) historical \(T - t\) years price ratios (or gross returns) of the underlying, \(R_{T - t, k}, k = 1, 2, \ldots, N\), one could sample a set of \(n\) historical gross returns of the underlying, \(R_{\Delta t, k}, k = 1, 2, \ldots, n\), where \(\Delta t\) could be an hour, a day, a week, or any other appropriate time interval that is much shorter than the maturity of \(T - t\) years and yet provides enough steps in the path to represent plausible early exercise times.

1Private communication from Dr. Jamie Alcock (August 11, 2008).
As is done exactly in canonical valuation (see Appendix A1), a set of \(n\) discrete risk-neutral probabilities of the form

\[
\pi^*_\Delta t, k = \frac{\exp(\gamma^* R_{\Delta t, k} e^{-\Delta t})}{\sum_{k=1}^{n} \exp(\gamma^* R_{\Delta t, k} e^{-\Delta t})}
\]

is estimated, where \(r\) is the constant, continuously compounded risk-free interest rate and \(\gamma^*\) is the value of the Lagrange multiplier \(\gamma\) that minimizes the following sum:

\[
\sum_{k=1}^{n} \exp[\gamma(R_{\Delta t, k} e^{-\Delta t} - 1)].
\]

Note that discounting in Gray et al. (2007) is done by \((1 + r)^{-(T-t)}\) (see Appendix A1), while continuously compounding is used in Formulas (1) and (2). \(\pi^*_\Delta t, k\) in Equation (1) provides a risk-neutral distribution for the underlying from which independent random samples of gross returns for a shorter time interval \(\Delta t\) can be drawn and stock price paths then simulated. This distribution is termed the “canonical risk-neutral distribution” herein.

**Path Simulation for the Underlying**

An important distinction between canonical valuation and the approach in this study shall be noted here. Canonical valuation uses the \(T-t\) years risk-neutral measure to average directly the expiry payoffs of an option, whereas in this study the risk-neutral measure with a shorter time interval \(\Delta t\) is utilized to generate paths for the underlying price process.

Let \(U\) denote the uniform random variable on \([0,1]\), and \(D\) the cumulative distribution function (CDF) of an arbitrary random variable. It is known from probability theory that the CDF of \(D^{-1}(U)\) is \(D\). This useful result provides a simple computational procedure for taking samples from an arbitrary distribution, as long as its CDF is given. The CDF \(\Pi^*\) of the canonical risk-neutral measure as given in Equation (1) can be obtained simply by ordering the gross returns \(R_{\Delta t, k}\) and then adding up the corresponding probabilities \(\pi^*_\Delta t, k\) one by one. With a \(\Pi^*(R_{\Delta t, j}) \rightarrow R_{\Delta t, j}\) mapping, \(^2\) random samples of the gross return under the risk-neutral measure can then be simulated as follows. First, generate a random number from the uniform distribution. Second, match the value of the random number with the value of \(\Pi^*\) and read off the corresponding \(R_{\Delta t, j}\) as a random draw of the gross return. Note that the last step is equivalent to the operation of inverting the function \(\Pi^*\).

\(^2\)Index \(j\) is used to indicate the ordered set of the gross returns.
A path for the underlying price process is then generated by repeating the above procedure

\[ S_t, S_t \tilde{R}_1, S_t \tilde{R}_1 \tilde{R}_2, \ldots \]

where \( S_t \) is the price of the underlying asset at time \( t \) and \( \tilde{R}_i \) is the \( i \)th random sample of the gross return obtained as described above. As the uniform random numbers are generated independently, the sampled gross returns are identically and independently distributed under the risk-neutral measure. Therefore, the simulated risk-neutral paths all occur with the same probability and the method of Monte Carlo simulation can be applied readily here. Stutzer (1996, 2000) provided a theoretical justification and technical details for this point.

**Early Exercise Based on Least-Squares**

Now with the simulated risk-neutral paths for the underlying price, the LSM algorithm of Longstaff and Schwartz (2001) can be utilized to determine the early exercise strategies for each path in a Monte Carlo simulation (see Appendix A2). In other word, American options can be priced by combining the canonical risk-neutral distribution with the LSM algorithm. Such a combined approach will be called the canonical least-squares Monte Carlo (CLM) in this study.

In principle, if the proposed CLM approach is correct, simulations done with the canonical distribution derived from geometric Brownian motions (GBM) should yield the same prices for options as either the Black–Scholes formula (for European options and American calls)\(^3\) or the LSM simulation (for American puts). Further, the result should be independent of the growth rate used in the GBM model for the underlying price. Those issues will be investigated in the following section.

**SIMULATION TESTING**

**Numerical Example**

For ease of comparison, the parameters of an American put from Longstaff and Schwartz (2001) are used here (a minor change is that explicit dates are used in the current study):

- Strike price: 40.0
- Valuation date: January 1, 2007

\(^3\)For simplicity, dividends are not considered here.
Simulated Gross Returns and Risk-Neutral Distributions

Assume a GBM model for the underlying price

\[ dS = \mu S \, dt + \sigma S \, dB \]

where \( \mu \) is the constant growth rate, \( \sigma \) the constant volatility, and \( B \) a standard Brownian motion. The well-known solution of the above GBM is

\[ S_t = S_0 e^{(\mu - \frac{1}{2} \sigma^2) t + \sigma B_t} \]

which can be discretized\(^4\) (Clewlow & Strickland, 1998) into

\[ \frac{S_{t+\Delta t}}{S_t} = e^{(\mu - \frac{1}{2} \sigma^2) \Delta t + \sigma \sqrt{\Delta t} \varepsilon} \]

where \( \varepsilon \) indicates a random draw from the standard normal distribution. Note that the left-hand side of Equation (4) is nothing more than the gross return of the underlying price as defined in the second section.

As the option specified above matures in exactly one year, 365 one-day gross returns\(^5\) are generated computationally according to Equation (4) for both the risk-neutral case (\( \mu = r = 6.0\% \)) and the somewhat unrealistic case of a growth rate of 100\%.\(^6\) From those two sets of gross returns, risk-neutral distributions (see Appendix A3) are derived by using the approach described in the second section.

Prices of Options

Prices for European calls, American calls, European puts, and American puts are obtained via LSM using the risk-neutral distributions derived above. For each reported price below, three independent Monte Carlo runs are carried out

\(^4\)GBM can be approximately discretized as \((S_t - S_0) / S_t = \mu \Delta t + \sigma \sqrt{\Delta t} \varepsilon\) (Stutzer, 2000).

\(^5\)The number of one-day gross returns used could be 250 or 500 as well. In principle, better accuracy for the resultant distribution could be achieved by using more and more returns (but due to the restriction of the number of digits available in computing, more returns might not actually lead to better accuracy). With more gross returns as inputs, however, more paths might be needed to sample the distribution adequately, which means increased computation costs. What the best balance is between accuracy and computational efficiency is a topic worth future investigation.

\(^6\)Numerically, a lower growth rate may not show much difference from that of risk-neutral.
and the resultant prices averaged. In every Monte Carlo run, 100,000 paths are simulated, with each path made up of 365 one-day gross returns.\(^7\) For American options, each path is further divided into 73 possible exercising points occurring every five days.\(^8\) The least-squares algorithm uses the first five terms of the Laguerre polynomial plus a constant term as basis functions.

As a proof of concept, the CLM approach should give the correct results for European calls, American calls, and European puts,\(^9\) all of which can be valued by the well-known Black–Scholes formulas. As can be seen from Tables I–III, the prices from CLM are quite close to the Black–Scholes fair values for both the cases of 6 and 100% growth rates.\(^10\)

The proposed CLM approach prices European calls, American calls, and European puts quite accurately in the GBM special case. Can it price American puts accurately?

\(^7\)To draw a sample from the canonical risk-neutral distribution, a uniform random number is first generated and compared with the CDF. The gross return corresponding to the value of the CDF that is equal to or slightly greater than the uniform random number is chosen as the desired sample.

\(^8\)An alternative in this case is to derive the canonical risk-neutral distribution directly from five-day gross returns.

\(^9\)Dividends, not considered in this study, can be incorporated into CLM easily.

\(^10\)The differences in Columns 4 and 6 are calculated as \((\text{CLM} - \text{BlackScholes})/\text{BlackScholes}\) in Tables I–III and \((\text{CLM} - \text{LongstaffSchwartz})/\text{LongstaffSchwartz}\) in Table IV.
Table IV shows five prices for the American put with the stock price varying from 36 to 44. As no closed-form solution is available for American puts, prices from CLM are compared with those from LSM (Longstaff & Schwartz, 2001). The price differences between CLM and LSM are quite small, with the largest being only 1.1% for the at-the-money case, whereas the largest absolute difference is less than six cents. Similar to prices reported in Tables I–III, the CLM prices in Table IV are biased positively (or a little bit higher than those of Longstaff–Schwartz).

In summary, options—American or European, call or put—are priced fairly accurately by CLM when the underlying price process is modeled by GBM. Furthermore, CLM works independently of the underlying’s growth rate in a GBM model, as it should in a Black–Scholes world. It is therefore reasonable to conclude that the CLM approach can be used as an alternative and efficient tool to price American options.

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11Longstaff and Schwartz (2001) used 50 exercising times for each path. For prices of options, it appears that contrary to one’s intuition, the number of exercising times does not make a huge difference (Liu, 2008). Therefore, the results reported in this study with 73 steps are believed to be comparable to those of Longstaff and Schwartz.

12On average, it took only about 21 seconds for CLM implemented in C++ to compute each price reported in Tables II and IV on an IBM Thinkpad X32 running Windows XP with 512 MB memory and a 1.70 GHz Pentium Processor. Further, it took much less time to compute prices reported in Tables I and III.
Future research will be carried out to test CLM empirically, to check out the hedging effectiveness of CLM, and to extend CLM to multiple underlying assets.

CONCLUSIONS

CLM, a new approach for pricing American puts, extends the canonical valuation method to the pricing of American options by utilizing the canonical risk-neutral distribution of gross returns in the LSM algorithm.

As a proof of concept, canonical risk-neutral distributions for GBM are used to simulate price paths for the underlying. The CLM-calculated prices for European calls, European puts, and American calls are all quite close to the Black–Scholes prices, as they should be. More importantly, CLM prices American puts accurately, as determined by comparing to the results from LSM reported by Longstaff and Schwartz (2001). This lends some credibility to the hope that CLM will be useful in valuing American-style options in more realistic circumstances, when the functional form of the underlying price process is unknown.\(^{13}\)

APPENDIX

Canonical Valuation

This section follows closely Gray et al. (2007) with minor notation differences and certain omissions.

Suppose that the valuation time is \(t\), and a European option matures at time \(T\). To price the option, one needs the \(T - t\) years forward distribution of the underlying price. In the Black–Scholes framework, the underlying price is assumed to be lognormally distributed, whereas in canonical valuation, the empirical distribution of a sample set of the historical \(T - t\) years gross returns is used to nonparametrically represent the distribution function, which may not be lognormal.

Assume a sample of \(N\) historical \(T - t\) years price ratios (or gross returns) of the underlying asset, \(R_{T-t,k}, k = 1, 2, \ldots, N\), is given. Under the risk-neutral measure, the expected \(T - t\) years return of the underlying should be equal to the gross risk-free interest rate \(1 + r\)

\[
\sum_{k=1}^{N} \pi_{T-t,k} R_{T-t,k} = (1 + r)^{T-t}
\]  

\(^{13}\)The anonymous reviewer is thanked with gratitude for writing up almost this entire paragraph. The author is solely responsible for any errors, however.
where $\pi_{T-t}^{*}$ is the risk-neutral probability of $R_{T-t,k}$ that is to be determined. Following Stutzer (1996) and Gray et al. (2007), $\pi_{T-t}^{*}$ is best estimated from the principle of maximum entropy as

$$
\pi_{T-t,k}^{*} = \frac{\exp\left(\gamma^{*} \frac{R_{T-t,k}}{(1 + r)^{T-t}}\right)}{\sum_{k=1}^{N} \exp\left(\gamma^{*} \frac{R_{T-t,k}}{(1 + r)^{T-t}}\right)}
$$  \hspace{1cm} (A2)

where the Lagrange multiplier $\gamma^{*}$ is obtained by solving the following equation:

$$
\gamma^{*} = \arg \min_{\gamma} \sum_{k=1}^{N} \exp\left(\gamma \left[ \frac{R_{T-t,k}}{(1 + r)^{T-t}} - 1 \right] \right).
$$  \hspace{1cm} (A3)

Applying the risk-neutral measure obtained above to the payoffs of the option at maturity, one can use canonical valuation to price a European put as in the following, for example

$$
P_t = (1 + r)^{-(T-t)} \sum_{k=1}^{N} \pi_{T-t,k}^{*} \max[X - S_{T,k}, 0]
$$  \hspace{1cm} (A4)

where $X$ is the strike price of the put. The possible prices of the underlying at maturity in Equation (A4) are projected as

$$
S_{T,k} = S_t R_{T-t,k}, \quad k = 1, 2, \ldots, N
$$  \hspace{1cm} (A5)

where $S_t$ is the price of the underlying at time $t$.

### Least-Squares Monte Carlo

The relevant operational part of the LSM algorithm of Longstaff and Schwartz (2001) is given here. For the full details of LSM, please refer to their seminal study instead.

Assume that one can only exercise an American option at a set of $K$ fixed times $0 < t_1 < t_2 < \ldots < t_K = T$. Let $\omega$ denote all the paths in a Monte Carlo simulation, and $C(\omega, s; t, T)$ denote the option’s cash flows at time $s$, conditional on the option not being exercised up to time $t$ for $t < s \leq T$. At exercise time $t_k$, the value of holding the option without exercising immediately should be the following conditional expectation with respect to the risk-neutral measure $\Pi^{*}$ and the filtration $\mathcal{F}$:

$$
H(\omega; t_k) = E_{\Pi^{*}} \left[ \sum_{j=k+1}^{K} e^{-r(t_j-t_k)} C(\omega, t_j; t_k, T) \bigg| \mathcal{F}_{t_k} \right]
$$  \hspace{1cm} (A6)

where $r$ stands for the constant risk-free interest rate.
Unfortunately, the holding value cannot be calculated directly using Equation (A6), as those future cash flows on the right-hand side of the equation are in general not known at time $t_k$. LSM bypasses this problem by working backwards and approximating this value by a linear sum of a finite set of $M$ basis functions $L_j$

$$H(\omega; t_k) \equiv H_M(\omega; t_k) = \sum_{j=0}^{M} a_jL_j(S) \tag{A7}$$

where $S$ is the price of the underlying and the constant coefficients $a_j$ are to be determined by least-squares. Many types of basis functions, such as Laguerre, Hermite, and Legendre polynomials, can be used in the least-squares fitting.

The least-squares algorithm fits the underlying prices at time $t_k$ for all the in-the-money paths to their corresponding discounted time $t_{k+1}$ cash flows to obtain the coefficients $a_i$ in Equation (A7). With $a_i$ and $S$ for Path $i$, the holding value $H_M(\omega_i; t_k)$ for Path $i$ at time $t_k$ can then be estimated.

Denote $E(\omega_i; t_k)$ as the value of exercise at time $t_k$ for Path $i$. If $E(\omega_i; t_k)$ is larger than $H_M(\omega_i; t_k)$, it is optimal to exercise and $E(\omega_i; t_k)$ would be the option’s cash flow at time $t_k$ for Path $i$; otherwise, the option shall not be exercised and the cash flow discounted from time $t_{k+1}$ is kept as its cash flow at time $t_k$ for Path $i$. This is done one-by-one for all the paths at time $t_k$, and then the procedure is repeated for exercise time $t_{k-1}$.

The least-squares regression starts at exercise time $t_{K-1}$ and ends at $t_1$. From time $t_1$, LSM discounts the cash flows along all the paths to time 0, adds them up, and then divides the sum by the number of paths to obtain a price for the American option.

**Simulated Canonical Risk-Neutral Distribution**

Figures 1 and 2 display the canonical risk-neutral distributions simulated with data given in the study.

The risk-neutral probabilities are roughly equal across the various possible values of the gross return when the growth rate is 6%, as can be seen in Figure 1. This makes sense since as the input gross returns in this case are already generated from a risk-neutral distribution. On the other hand, the probabilities are higher for lower gross returns when the growth rate is 100%. This is again in line with our intuition, for a growth rate higher than risk-free implies a higher expected return than risk-free; in order to derive a risk-neutral measure, returns with lower values then need to be assigned with higher probabilities.\(^{14}\)

\(^{14}\)Among the 365 gross returns, 191 (or roughly 50% of the returns) are below one for the case of 6% growth rate, whereas only 154 below one for 100% growth rate.
It is somewhat surprising that even though the graphs of the probabilities for 6 and 100% growth rates look quite different (Figure 1), their corresponding CDF are almost indistinguishable (Figure 2). A moment of reasoning would convince us, however, that this result is also expected. As the vast majority of the gross returns of the two cases overlap, the two CDFs must be similar in order to yield approximately the same risk-neutral result.
BIBLIOGRAPHY


