Homomorphisms between fuzzy information systems

Changzhong Wang\textsuperscript{a,}\textsuperscript{*}, Degang Chen\textsuperscript{b}, Liangkuan Zhu\textsuperscript{c}

\textsuperscript{a} Department of Mathematics, Bohai University, Jinzhou 121003, PR China
\textsuperscript{b} School of Mathematics & Physics, North China Electric Power University, Beijing, 102206, PR China
\textsuperscript{c} College of Mechanical and Electrical Engineering, Northeast Forestry University, Harbin, 150040, PR China

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**Abstract**
The information system is one of the most important mathematical models in the field of artificial intelligence, and the concept of mapping is a useful tool for studying the communication between two information systems. In this work, the concepts of fuzzy relation mapping and inverse fuzzy relation mapping are first introduced and their properties are studied. Then, the notions of homomorphisms of information systems based on fuzzy relations are proposed, and it is proved that attribute reductions in the original system and image system are equivalent to each other under the condition of homomorphism.

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1. Introduction

The information system is one of the most important mathematical models in the field of artificial intelligence, and communication between information systems is a basic problem in granular computing [1]. Although in recent years many topics in information systems have been widely investigated [2–11], there are few researches that focus on communication between information systems [2–4,10,11].

The theory of rough sets, proposed by Pawlak, is a useful tool for studying information systems. According to the idea in [1], a rough approximation space is actually a granular information world. As for an information system, it can be seen as a combination of some approximation spaces on the same universe. The communication between two information systems, in mathematics, can be explained as a mapping between two information systems. The approximations and reductions in the original system can be regarded as encoding while the image system is seen as an interpretive system.

The notion of homomorphism as a kind of tool for studying the relationship between two information systems was introduced by Graymala-Busse in [2]. A homomorphism on information systems is useful for aggregating sets of objects, attributes, and descriptors of the original system [3,4,10,11]. In [3], Graymala-Busse depicted the conditions which make an information system selective in terms of an endomorphism of the system. In [4], Deyu Li and Yichen Ma discussed the features of superfluousness and reduces of two information systems under some homomorphisms. Wang et al. investigated some invariant properties of relation information systems under homomorphisms and proved that attribute reductions in the original system and image system are equivalent to each other under the condition of homomorphism [10,11]. However, all of the above studies are restricted to crisp binary relations.

Fuzzy rough sets [12], as a generalization of crisp rough sets, have powerful prospects in applications [9,13,14]. If we consider a fuzzy approximation space as a granular information world, we have to focus on the communication between two fuzzy information systems.
This work represents a new contribution to the development of the theory of communication between information systems. We define the concepts of homomorphisms between two information systems on the basis of fuzzy binary relations. Under the condition of homomorphism, some characters of relation operations in the original system and some structural features of the original system are guaranteed in the image system.

2. Fuzzy relation mappings and their properties

In this section, we first define the notions of fuzzy relation mappings by Zadeh’s extension principle [15], and then study their properties. Let U and V be two universes. The class of all fuzzy binary relations on U (respectively, V) will be denoted by \( \mathcal{F}(U \times U) \) (respectively, \( \mathcal{F}(V \times V) \)). Let us start with introducing the following concepts through Zadeh’s extension principle.

**Definition 2.1.** Let \( f : U \to V \), \( x \to f(x) \in V \), \( x \in U \). By the extension principle, \( f \) can induce a fuzzy mapping from \( \mathcal{F}(U \times U) \) to \( \mathcal{F}(V \times V) \) and a fuzzy mapping from \( \mathcal{F}(V \times V) \) to \( \mathcal{F}(U \times U) \), that is,

\[
\tilde{f} : \mathcal{F}(U \times U) \to \mathcal{F}(V \times V), \quad R \to \tilde{f}(R) \in \mathcal{F}(V \times V), \quad \forall R \in \mathcal{F}(U \times U);
\]

\[
\tilde{f}(R)(x, y) = \left\{ \begin{array}{ll}
\bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} R(u, v), & (x, y) \in f(U) \times f(U);
0, & (x, y) \notin f(U) \times f(U).
\end{array} \right.
\]

\[
\tilde{f}^{-1} : \mathcal{F}(V \times V) \to \mathcal{F}(U \times U), \quad T \to \tilde{f}^{-1}(T) \in \mathcal{F}(U \times U), \quad \forall T \in \mathcal{F}(V \times V);
\]

\[
\tilde{f}^{-1}(T)(u, v) = T(f(u), f(v)), \quad (u, v) \in U \times U.
\]

Then \( \tilde{f} \) and \( \tilde{f}^{-1} \) are called the fuzzy relation mapping and the inverse fuzzy relation mapping induced by \( f \), respectively. \( \tilde{f}(R) \) and \( \tilde{f}^{-1}(T) \) are called fuzzy binary relations induced by \( f \) on \( V \) and \( U \), respectively. When there is no confusion, we simply denote \( \tilde{f} \) and \( \tilde{f}^{-1} \) by \( f \) and \( f^{-1} \), respectively.

**Remark 1.** When fuzzy relations \( R \) and \( T \) have their values only from the set \( \{0, 1\} \), then the definitions of \( \tilde{f}(R) \) and \( \tilde{f}^{-1}(T) \) will be reduced to the definitions of the images of crisp binary relations in [10], respectively.

**Definition 2.2.** Let \( U \) and \( V \) be two universes, \( f : U \to V \) a mapping from \( U \) to \( V \), and \( R \in \mathcal{F}(U \times U) \). Let \( \text{x}_R = \{ y \in U : f(y) = f(x) \} \); then \( \{ \text{x}_R : x \in U \} \) is a partition on \( U \). For any \( x, y \in U \), if \( R(u, v) = R(s, t) \) for any two pairs \( (u, v), (s, t) \in \text{x}_R \times \text{x}_R \), then \( f \) is called consistent with respect to \( R \).

From Definition 2.2, an injection is trivially a consistent function.

**Remark 2.** If a fuzzy binary relation \( R \) has value 0 or 1, then for any \( x, y \in U \), \( R(u, v) = \text{constant} \) for any pair \( (u, v) \in [x]_R \times [y]_R \), where constant takes the value 0 or 1. Hence we have that \( [x]_R \subseteq R(x) \) for any \( x \in U \) and that \( R(x) \cap [y]_R = \emptyset \) for any \( x, y \in U \), where \( R(x) = \{ y \in U : f(x, y) \in R \} \) and \( R \) is a crisp binary relation (see [10]). Therefore, \( f \) will be reduced to both a type-1 and a type-2 consistent function in [10]. That is, the concept of a consistent function \( f \) in Definition 2.2 is actually an extension of the concepts of type-1 and type-2 consistent functions in [10].

**Proposition 2.3.** Let \( R, R_1, R_2 \in \mathcal{F}(U \times U) \). If \( f \) is consistent with respect to \( R, R_1 \) and \( R_2 \), respectively. Then:

1. \( f \) is consistent with respect to \( R \cap R_1 \);
2. \( f \) is consistent with respect to the complement of \( R \).

**Proof.** Straightforward. \( \square \)

The following theorem discusses properties of fuzzy binary relations under relation mappings \( f \) and \( f^{-1} \), respectively.

**Theorem 2.4.** Let \( f : U \to V \) and \( f \) be surjective, \( R \in \mathcal{F}(U \times U) \) and \( T \in \mathcal{F}(V \times V) \). Then:

1. If \( R \) (respectively, \( T \)) is reflexive, then \( f(R) \) (respectively, \( f^{-1}(T) \)) is reflexive.
2. If \( R \) (respectively, \( T \)) is symmetric, then \( f(R) \) (respectively, \( f^{-1}(T) \)) is symmetric.
3. If \( T \) is max – min transitive, then \( f^{-1}(T) \) is max – min transitive.
4. If \( f \) is consistent with respect to \( R \) and \( R \) is max – min transitive, then \( f(R) \) is max – min transitive.

**Proof.** (1) Let \( R \) be reflexive. Since \( f \) is surjective, it follows that for any \( y \in V \), there must exist \( x \in U \) such that \( f(x) = y \).

By the reflexivity of \( f \), we have \( R(x, x) = 1 \). From the definition of \( f(R), f(R)(y, y) = \bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} R(u, v) R(x, x) = f(R)(x, x) = 1 \).

Thus \( f(R) \) is reflexive.

Let \( T \) be reflexive. For any \( x \in U \), let \( f(x) = y \in V \). By the reflexivity of \( T \), we have \( T(y, y) = 1 \). Thus \( f^{-1}(T)(x, x) = T(f(x), f(x)) = T(y, y) = 1 \). Hence \( f^{-1}(T) \) is reflexive.
(2) Let $R$ be symmetric. For any $x, y \in V$, $f(R)(x, y) = \bigvee_{x \in f^{-1}(y)} \bigvee_{y \in f^{-1}(y)} R(u, v) = \bigvee_{y \in f^{-1}(y)} \bigvee_{x \in f^{-1}(x)} R(v, u) = f(R)(y, x)$ by the symmetry of $R$. Hence $f(R)$ is symmetric.

Let $T$ be symmetric. For any $u, v \in U$, $f^{-1}(T)(u, v) = T(f(u), f(v)) = T(f(v), f(u)) = f^{-1}(T)(v, u)$ by the symmetry of $T$. Hence $f^{-1}(T)$ is symmetric.

(3) Let $T$ be transitive, for any $x, y, z \in U$, $f^{-1}(T)(x, y) \wedge f^{-1}(T)(y, z) = T(f(x), f(y)) \wedge T(f(y), f(z)) \geq T(f(x), f(z)) = f^{-1}(T)(x, z)$ by the transitivity of $T$. Hence $f^{-1}(T)$ is transitive.

(4) For any $x, y, z \in V$, since $f$ is surjective, it follows that there must exist $u_0, v_0, t_0 \in U$ such that $f(u_0) = x, f(v_0) = y, f(t_0) = z$. Since $f$ is consistent with respect to $R$, we have that $R(u_0, v_0) = R(u, v), R(v_0, t_0) = R(v, t)$ for any $(u, v) \in f^{-1}(x) \times f^{-1}(y)$ and $(v, t) \in f^{-1}(y) \times f^{-1}(z)$. Hence $f(R)(x, y) \wedge f(R)(y, z) = \left( \bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} R(u, v) \right) \wedge \left( \bigvee_{v \in f^{-1}(y)} \bigvee_{t \in f^{-1}(z)} R(t, v) \right) = R(u_0, v_0) \wedge R(v_0, t_0) \geq R(u_0, t_0)$ by the transitivity of $R$. Similarly, $f(R)(x, z) = \bigvee_{u \in f^{-1}(x)} \bigvee_{t \in f^{-1}(z)} R(u, t) = R(u_0, t_0)$. Therefore $f(R)(x, y) \wedge f(R)(y, z) \leq f(R)(x, z)$, which implies $f(R)$ is transitive. \hfill \Box

**Remark 3.** In general, a fuzzy relation mapping $f$ can preserve the reflexivity and symmetry of a fuzzy relation, but does not preserve the transitivity of a fuzzy relation. The following theorem discusses the problem of fuzzy relation operations under a fuzzy relation mapping $f$.

**Theorem 2.5.** Let $f : U \to V, R_1, R_2 \in \mathcal{F}(U \times U)$; then:

1. $f(R_1 \cup R_2) = f(R_1) \cup f(R_2)$.
2. $f(R_1 \cap R_2) \subseteq f(R_1) \cap f(R_2)$; if $f$ is consistent with respect to $R_1$ and $R_2$ respectively, then the equality holds.

**Proof.** (1) $f(R_1 \cup R_2)(x, y) = \bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} \bigvee_{u \in f^{-1}(u)} \bigvee_{v \in f^{-1}(v)} R(u, v) = \bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} \bigvee_{u \in f^{-1}(u)} \bigvee_{v \in f^{-1}(v)} R_1(u, v) \vee R_2(u, v) = (f(R_1) \cup f(R_2))(x, y)$.

(2) For any $x, y \in V$,

$$f(R_1 \cap R_2)(x, y) = \bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} \bigvee_{u \in f^{-1}(y)} \bigvee_{v \in f^{-1}(v)} R_1(u, v) \wedge R_2(u, v) \leq \left( \bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} R_1(u, v) \right) \wedge \left( \bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} R_2(u, v) \right) = (f(R_1) \cap f(R_2))(x, y).$$

Now, we prove that if $f$ is consistent with respect to $R_1$ and $R_2$, respectively; then the equality holds.

Since $f$ is consistent with respect to $R_1$ and $R_2$ respectively, it follows from Proposition 2.3(1) that $f$ is consistent with respect to $R_1 \cap R_2$. According to Definition 2.2, for any $x, y \in U$, if $(u, v), (t, s) \in f^{-1}(x) \times f^{-1}(y)$, then $(R_1 \cap R_2)(u, v) = (R_1 \cap R_2)(t, s)$. In particular, let $t_0 \in f^{-1}(x), s_0 \in f^{-1}(y)$. Thus

$$f(R_1 \cap R_2)(x, y) = \bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} \bigvee_{u \in f^{-1}(u)} \bigvee_{v \in f^{-1}(v)} R_1(u, v) \wedge R_2(u, v)$$

and

$$(f(R_1) \cap f(R_2))(x, y) = f(R_1)(x, y) \wedge f(R_2)(x, y)$$

$$= \left( \bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} R_1(u, v) \right) \wedge \left( \bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} R_2(u, v) \right)$$

$$= R_1(t_0, s_0) \wedge R_2(t_0, s_0).$$

Therefore, we conclude the proof. \hfill \Box

**Corollary 2.6.** Let $f : U \to V, R_1, R_2, \ldots, R_n \in \mathcal{F}(U \times U)$; then:

1. $f\left( \bigcup_{i=1}^{n} R_i \right) = \bigcup_{i=1}^{n} f(R_i)$;
2. $f\left( \bigcap_{i=1}^{n} R_i \right) \subseteq \bigcap_{i=1}^{n} f(R_i)$; if $f$ is consistent with respect to each of the fuzzy relations $R_i$, then the equality holds.

**Proof.** It is similar to the proof of Theorem 2.4. \hfill \Box

**Theorem 2.7.** Let $f : U \to V, R \in \mathcal{F}(U \times U), T \in \mathcal{F}(V \times V)$; then:

1. $f^{-1}(T) \subseteq T$; if $f$ is surjective, then the equality holds.
2. $f^{-1}(f(R)) \supseteq R$; if $f$ is consistent with respect to $R$, the equality holds.
Prove. (1) For \((x, y) \in f(U) \times f(U) \subseteq V \times V, f^{-1}(x) \neq \emptyset \) \& \(f^{-1}(y) \neq \emptyset \). Thus
\[
f \left( f^{-1}(T) \right)(x, y) = \bigvee_{u \in f^{-1}(x)} \bigvee_{v \in f^{-1}(y)} f^{-1}(T)(u, v) = \bigvee_{u \in f^{-1}(y)} f^{-1}(T)(u, f(v)) = T(x, y).
\]

For \((x, y) \notin f(U) \times f(U)\) satisfying \((x, y) \in V \times V, f \left( f^{-1}(T) \right)(x, y) = 0\) by the Definition 2.1. Hence \(f \left( f^{-1}(T) \right)(x, y) \subseteq T(x, y)\).

(2) Since \(f^{-1}(f(R))(u, v) = f(R)(f(u), f(v)) = \bigvee_{g(x) = (u), g(y) = (v)} R(x, y) \geq R(u, v)\) for any \((u, v) \in U \times U\), we have \(f^{-1}(f(R)) \supseteq R\). If \(f\) is consistent with respect to \(R\), we have \(R(x, y) = R(u, v)\) for any \((x, y) \in f^{-1}(f(u)) \times f^{-1}(f(v))\), which implies \(f^{-1}(f(R))(u, v) = \bigvee_{g(x) = (u), g(y) = (v)} R(x, y) = R(u, v)\). Therefore, \(f^{-1}(f(R)) = R\). □

3. Homomorphism between fuzzy information systems and its properties

By means of the results of the above section, we introduce the notion of homomorphism to study communication between two fuzzy information systems, and investigate some properties of fuzzy information systems under the condition of homomorphism. Let us start with introducing the notions of fuzzy relation information systems.

**Definition 3.1.** Let \(U\) and \(V\) be finite universes, \(f : U \rightarrow V\) a mapping from \(U\) to \(V\), and \(\mathcal{R} = \{R_1, R_2, \ldots, R_n\}\) a family of fuzzy binary relations on \(U\); let \(f(R) = \{f(R_1), f(R_2), \ldots, f(R_n)\}\). Then the pair \((U, R)\) is referred to as a fuzzy relation information system, and the pair \((V, f(R))\) is referred to as an \(f\)-induced fuzzy relation information system of \((U, R)\).

By Corollary 2.6, we can introduce the following concept.

**Definition 3.2.** Let \((U, R)\) be a fuzzy relation information system and \((V, f(R))\) an \(f\)-induced fuzzy relation information system of \((U, R)\). If \(\forall R_i \in \mathcal{R}, f\) is consistent with respect to \(R_i\) on \(U\), then \(f\) is referred to as a homomorphism from \((U, R)\) to \((V, f(R))\).

**Remark 4.** After the notion of homomorphism is introduced, all the theorems and corollaries in which the equality “\(=\)” holds in the above section may be seen as properties of homomorphism.

**Definition 3.3.** Let \((U, R)\) be a fuzzy relation information system and \(P \subseteq \mathcal{R}\). The subset \(P\) is referred to as a reduct of \(R\) if \(P\) satisfies the following conditions:

1. \(\cap P = \cap R\);
2. \(\forall R_i \in P, \cap P \subseteq \cap (P - R_i)\).

**Theorem 3.4.** Let \((U, R)\) be a fuzzy relation information system, \((V, f(R))\) an \(f\)-induced fuzzy relation information system of \((U, R)\), \(f\) a homomorphism from \((U, R)\) to \((V, f(R))\) and \(P \subseteq \mathcal{R}\). Then \(P\) is a reduct of \(R\) if and only if \(f(P)\) is a reduct of \(f(R)\).

**Proof.** ⇒ Since \(P\) is a reduct of \(R\), we have \(\cap P = \cap R\). Hence \(f(\cap P) = f(\cap R)\). Since \(f\) is a homomorphism from \((U, R)\) to \((V, f(R))\), by Definition 3.2 and Corollary 2.6, we have \(\cap f(P) = \cap f(R)\). Assume that \(\exists R_i \in P\) such that \(\cap (f(P) - f(R_i)) = \cap f(P)\). Because \(f(P) - f(R_i) = f(P - R_i)\), we have that \(\cap (f(P) - f(R_i)) = \cap f(P - R_i) = f(\cap (P - R_i)) = f(\cap f(P))\). Similarly, by Definition 3.2 and Corollary 2.6, it follows that \(f(\cap (P - R_i)) = f(\cap R_i)\). Thus \(f^{-1}(f(\cap (P - R_i))) = f^{-1}(f(\cap R_i))\). By Definition 3.2 and Corollary 2.8, \(\cap (P - R_i) = R_i\). This is a contradiction to \(P\) being a reduct of \(R\).

⇐ Let \(f(P)\) be a reduct of \(f(R)\); then \(\cap f(P) = \cap f(R)\). Since \(f\) is a homomorphism from \((U, R)\) to \((V, f(R))\), by Definition 3.2 and Corollary 2.6, we have \(\cap f(P) = \cap f(R)\). Hence \(f^{-1}(f(\cap P)) = f^{-1}(f(\cap R))\). By Definition 3.2 and Corollary 2.8, \(\cap P = \cap R\). Assume that \(\exists R_i \in P\) such that \(\cap (P - R_i) = R_i\); then \(\cap f(P - R_i) = f(\cap R_i)\). Again, by Definition 3.2 and Corollary 2.6, we have \(\cap f(P - R_i) = f(\cap R_i)\). Hence \(\cap f(P - R_i) = f(\cap R_i)\). This is a contradiction to \(f(P)\) being a reduct of \(f(R)\). This completes the proof of this theorem. □

By Theorem 3.4, we immediately get the following corollary.

**Corollary 3.5.** Let \((U, R)\) be a fuzzy relation information system, \((V, f(R))\) an \(f\)-induced fuzzy relation information system of \((U, R)\), \(f\) a homomorphism from \((U, R)\) to \((V, f(R))\) and \(P \subseteq \mathcal{R}\). Then \(P\) is superfluous in \(R\) if and only if \(f(P)\) is superfluous in \(f(R)\).

**Remark 5.** According to Remarks 1 and 2 and the main results in this work, we can say that the current work will be reduced to the work in [10, 11] when all fuzzy binary relations considered have their values only from the set \([0, 1]\).
Let \((U, R)\) be a fuzzy relation information system, where \(U = \{x_1, x_2, \ldots, x_7\}\), \(R = \{R_1, R_2, R_3\}\). Let \(R_1, R_2\) and \(R_3\) be 'Table-4', 'Table-5' and 'Table-6' as described below respectively, simply denoted as T-4, T-5 and T-6 respectively. Denote \(R_1 \cap R_2 \cap R_3\) as T-7, described below.

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Let \(V = \{y_1, y_2, y_3, y_4\}\). Define a mapping \(f\) as follows:

\[
\begin{array}{ccccc}
  x_1, x_7 & x_2, x_6 & x_3, x_5 & x_4 \\
  y_1 & y_2 & y_3 & y_4 \\
\end{array}
\]

Thus \((V, f (R))\) is the \(f\)-induced fuzzy relation information system of \((U, R)\). It is very easy to verify that \(f\) is a homomorphism from \((U, R)\) to \((V, f (R))\).

We can see that \(f (R_1)\) is superfluous in \(f (R)\) is equivalent to \(R_1\) being superfluous in \(R\) and see that \(\{f (R_2), f (R_3)\}\) is a reduct of \(f (R)\) is equivalent to \(\{R_2, R_3\}\) being a reduct of \(R\). Therefore, we can reduce the original system by reducing the image system and reduce the image system by reducing the original system. That is, the attribute reductions of the original system and image system are equivalent to each other.

4. Conclusions

In this work, we point out that a fuzzy mapping between two universes can induce a fuzzy binary relation on one universe according to the given fuzzy relation on the other universe. For a fuzzy relation information system, we can consider it as a combination of some fuzzy approximation spaces on the same universe. The fuzzy mapping between fuzzy approximation spaces can be explained as a fuzzy mapping between the given fuzzy relation information systems. A homomorphism is a special fuzzy mapping between two fuzzy relation information systems. Under the condition of homomorphism, we discuss the characters of fuzzy relation information systems, and find out that the attribute reductions of the original system and image system are equivalent to each other. These results may have potential applications in knowledge reduction, decision making and reasoning about data, especially for the case of two fuzzy relation information systems. Our results also illustrate that some characters of a system are guaranteed in an explanation system, i.e., a system gains acknowledgement from another system.

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References


