Mean–variance analysis of the newsvendor model with stockout cost

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Abstract

We study the risk-averse newsvendor model with a mean–variance objective function. We show that stockout cost has a significant impact on the newsvendor’s optimal ordering decisions. In particular, with stockout cost, the risk-averse newsvendor does not necessarily order less than the risk-neutral newsvendor. We illustrate this finding analytically for the case where the demand follows the power distribution.

Keywords: Supply chain; Newsvendor problem; Mean–variance; Stockout cost

1. Introduction

As a fundamental problem in stochastic inventory control, the newsvendor problem has been studied for a long time and applied in a broad array of business settings with the objective of expected profit maximization or expected cost minimization. However, modern supply chains are very complex and increasingly becoming more vulnerable to uncertainties. The assumption of risk-neutrality therefore seems to be inadequate for contemporary supply chain management. In view of this, a number of papers have been devoted to risk analysis of supply chain models. Recent studies include, but are not limited to, those by Lau [1], Bouakiz and Sobel [2], Choi et al. [3], Eeckhoudt et al. [4], Lau and Lau [5], Agrawal and Seshadri [6,7], Chen and Federgruen [8], Buzacott et al. [9], Chen et al. [10], Wang and Webster [11], Wu et al. [12], Bogataj and Bogataj [13], He and Zhang [14], Sounderpandian et al. [15], Agrawal and Ganeshan [16], Xiao and Qi [17], and Hua and Li [18]. For an extensive review of the literature on supply chain risk management or extensions of different objectives for the newsvendor problem, the reader is referred to Khouja [19], Tang [20] and Wu et al. [21].

In the newsvendor problem, if there is not enough stock to satisfy all the demand occurring in the selling season, the newsvendor may incur a stockout cost. Besides making a loss in marginal profit, the stockout cost may include such adverse effects on a firm as tarnishing the firm’s reputation and jeopardizing the loyalty of the firm’s customers, which can greatly impair the firm’s performance and profitability. For example, the Wall Street Journal [22] reported that IBM, as a result
of under-producing its Aptiva PC line, lost more than $100 million in potential revenue in 1994.

Although stockout cost plays an important role in the practice of supply chain management, it is often ignored or has not been studied in depth in risk analysis of supply chain models. Chen and Federgruen [8] studied the newsvendor problem using the mean–variance framework. Without stockout cost, the variance function of the stochastic profit is a monotone increasing function of order quantity, so the mean–variance trade-off can be carried out efficiently. However, if stockout cost is considered, the variance function will lose this monotonicity property and the mean–variance trade-off becomes much more complicated. Buzacott et al. [9] studied a class of commitment-option supply contracts under the mean–variance framework. They used the mean–variance criterion as the objective function, which is a newsvendor type of problem without stockout cost, and obtained a similar monotone increasing property of the variance function. They further emphasized that monotonicity is a fundamental result for this type of stochastic planning models. Eeckhoudt et al. [4] examined the effects of risk and risk aversion on a risk-averse and prudent newsvendor without considering the stockout cost. They pointed out that risk aversion will lead to a reduced initial newspaper order. Lau [1] considered the risk-averse newsvendor problem with mean–standard deviation tradeoff under two cases: with and without a stockout cost. He proved that the risk-neutral newsvendor order quantity is an upper bound on the optimal order quantity of the risk-averse newsvendor without stockout cost. He also stated without proof that a similar result still holds when stockout cost is considered.

Our study is most related to Wang and Webster [11], but with several major differences. Both studies consider the newsvendor problem with stockout cost based upon objectives different from profit maximization and find some results different from the risk-neutral newsvendor model. The differences are as follows: (1) Wang and Webster [11] used loss aversion to model the newsvendor problem, while we use mean–variance tradeoff. Loss aversion belongs to a class of utility function, while mean–variance tradeoff belongs to the return-risk framework. Generally speaking, utility maximization is mainly used in theoretical study, while mean–variance tradeoff is widely applied in both theoretical study and practice. We select mean–variance tradeoff mainly based on the following two considerations.

First, since there are various utility functions, it is not easy to construct a proper one convenient for analysis. Second, return-risk models usually have a much more intuitive explanation than the utility maximization approach. Here we must point out that, unlike many other utility functions, the loss aversion used in [11] is also intuitively appealing. (2) Wang and Webster [11] studied the risk-averse newsvendor problem within the loss aversion framework, while our paper is motivated mainly from previous studies. Thus, we also carry out comparisons with previous risk-averse newsvendor problems and present a counterexample to one result presented in the literature.

If stockout cost is considered in the newsvendor model, the properties of the variance function and the mean–variance tradeoff may be very different from those of the model without stockout cost. Moreover, some results obtained in the previous literature may no longer be valid. Motivated by this observation, we study the risk-averse newsvendor model presented in Chen and Federgruen [8] but with stockout cost consideration. We derive an explicit form of the variance of the profit function and obtain its properties. We show that the variance of the profit function is no longer a monotone increasing function. Furthermore, under the assumption that the demand function follows the power distribution, we work out the set of optimal ordering quantities. Contrary to the traditional result in the literature that the risk-averse newsvendor (without stockout cost) always orders less than the risk-neutral newsvendor order quantity, our findings show that this may not be the case when stockout cost is considered in a situation with mean–variance tradeoff. We also give a counterexample to one result presented in Lau [1].

The rest of this paper is organized as follows. In Section 2 we analyze the newsvendor problem under study using the mean–variance approach. In Section 3, under the assumption that demand follows the power distribution, we derive the new properties and results due to the stockout cost. We give conclusions in Section 4.

2. Mean–variance analysis with stockout cost

Let \( Q \) be the newsvendor’s order quantity. Let \( D \) be the future stochastic demand during the selling season. Let \( F \) be the cumulative distribution function and \( f \) the probability density function of demand, respectively. We assume that \( F \) is a continuous and strictly increasing function and \( f \) is a nonnegative function.

The purchasing cost of the product is \( c \) per unit, the selling price is \( r \) per unit, the salvage value of any unsold
product is \( p \) per unit, and the stockout cost of unsatisfied demand is \( p \) per unit. To avoid unrealistic and trivial cases, we assume that \( 0 < s < c < r \) and \( 0 < p \). Throughout the paper, we use the following notation: for any numbers \( a \) and \( b \), \( a^+ = \max\{a, 0\} \) and \( a^\wedge b = \min\{a, b\} \).

Let \( \pi(Q) \) be the newsvendor’s random profit, namely

\[
\pi(Q) = r(Q \wedge D) + s(Q - D)^+ - p(D - Q)^+ - cQ.
\]

(1)

Let \( II(Q) \) be the mean profit, namely

\[
II(Q) = E[\pi(Q)] = -(r + p - s) \int_0^Q F(x) \, dx + (r + p - c)Q - pE[D],
\]

(2)

where \( E[D] \) is the mean of the random demand \( D \).

The risk-neutral newsvendor problem is given by

\[
\max_{Q \geq 0} \{E[\pi(Q)]\}.
\]

(3)

The optimal solution \( Q^* \) for problem (3) is called the newsvendor order quantity. It is straightforward to verify that the expected profit function is a concave function of \( Q \). By using the first-order optimality conditions, we obtain the newsvendor order quantity \( Q^* \) as follows:

\[
F(Q^*) = \frac{r + p - c}{r + p - s}.
\]

The mean–variance analysis was first proposed by Markowitz [23] to measure the risk associated with the return of assets. It uses a parameter \( \alpha (\alpha \geq 0) \) to characterize a decision maker’s risk averseness, which is a quantitative balance between the mean profit and the risk associated with its variance. \( \alpha = 0 \) denotes the special case of maximizing the mean profit function only. An increase in \( \alpha \) indicates the decision maker’s increasing willingness to sacrifice the mean profit to avoid the risk of its variance. Note that, for any given \( \alpha \), a solution is optimal in the sense that we cannot improve the mean profit without bearing more risk, or reduce the risk without decreasing the mean profit.

Under the mean–variance framework, the objective function of the newsvendor problem is given by

\[
\max_{Q \geq 0} \{E[\pi(Q)] - \alpha Var[\pi(Q)]\},
\]

where \( Q \) is the order quantity, \( \alpha \) is the parameter denoting the decision maker’s risk attitude, \( \pi(Q) \) is the random profit given by (1), \( E[\pi(Q)] \) is the mean of the random profit given by (2), and \( Var[\pi(Q)] \) is the variance of the random profit given by (4).

Note that the variance of the random profit is given by

\[
\text{Var}[\pi(Q)] = E[(\pi(Q))^2] - (E[\pi(Q)])^2.
\]

Substituting (1) and (2) into (4), the variance function of the newsvendor’s profit can be written as

\[
\text{Var}[\pi(Q)] = -(r + p - s)^2 \left( \int_0^Q F(x) \, dx \right)^2 + 2Q(r - s)(r + p - s)\left( \int_0^Q F(x) \, dx \right) - 2(r + p - s)(r - p - s) \int_0^Q xF(x) \, dx + p^2 \text{Var}[D],
\]

(4)

where \( \text{Var}[D] \) is the variance of the random demand \( D \).

As shown in Appendix A.1, the expected profit function is a concave function of \( Q \) and asymptotically linear with a slope \( (s - c) < 0 \).

We remark that the variance may in general be unbounded and hence the presence of a unique optimum is not guaranteed. However, as shown in Appendix A.2, this can be guaranteed under the mild assumption that the random demand \( D \) has a finite second moment.

We can obtain the optimal order quantity by applying a one-dimensional search algorithm when the closed-form solution cannot be obtained. Furthermore, the newsvendor order quantity can be used as an initial solution for the search algorithm.

We remark that compared with the results presented in Lau [1], the explicit form of the variance function expressed as (4) in this paper has computational advantages. In Lau [1], computation of the value of the variance function needs some central moments, their derivatives and partial moments. Our results show that we only need to compute \( \int_0^Q F(x) \, dx \) and \( \int_0^Q xF(x) \, dx \), which can be easily determined via numerical integration methods or which may even have closed-form expressions.

3. Special case: power distributed demand

To derive structural results and generate managerial insights into the optimal decisions of the risk-averse newsvendor problem, we present in the following specific results for the case where demand follows the power distribution. The power distribution was also analyzed by Chen and Federgruen [8], and we compare our results with theirs.
Without loss of generality, we choose the interval $[0, 1]$ with probability distribution function $F_D(x)$ and probability density function $f_D(x)$ as follows:

$$F_D(x) = \begin{cases} 
1, & x > 1, \\
x^k, & 0 \leq x \leq 1, \\
0, & x < 0,
\end{cases}$$

$$f_D(x) = \begin{cases} 
kx^{k-1}, & 0 \leq x \leq 1, \\
0 & \text{otherwise}.
\end{cases}$$

The mean and variance of the power distribution are $k/(k+1)$ and $k/[(k+2)(k+1)^2]$, respectively.

Based on the assumption that the demand function follows the power distribution, the mean profit becomes

$$\mu(Q) = \left\{ \begin{array}{ll}
\frac{(r+p-s)Q^k}{k+1} + (r+p-c)Q - p k - s (r+c)Q, & 1 < Q, \\
(s-c)Q + k(r-s), & 0 \leq Q \leq 1, \\
0, & k > k^*.
\end{array} \right.$$

The first-order derivative of the variance function of (6) with respect to $Q$ is given by

$$\frac{d \text{Var}[\pi(Q)]}{dQ} = \left\{ \begin{array}{ll}
\frac{2(r+p-s)Q^k}{k+1} - (r+p-s)Q^{k+1} \\
+(k+1)p+(r-s)(Q-pk), & 0 \leq Q \leq 1, \\
0, & \text{otherwise}.
\end{array} \right.$$

As shown in Appendix A.3, we can use these expressions to establish the following theorem.

**Theorem 3.1.** For power distributed demand, there exists one unique minimizer $Q^*_p$ for $\text{Var}[\pi(Q)]$ on $(0, 1)$, where $\text{Var}[\pi(Q)]$ is decreasing in $[0, Q^*_p]$ and increasing in $[Q^*_p, 1]$. Moreover, there exists a critical value $k^*$ with $0 < k^* < 1$ and the newsvendor’s optimal order quantity is distinguished by three cases as follows:

1. If $0 < k < k^*$, then $Q^* < Q^*_p$ and the optimal order quantity is in the interval $[Q^*, Q^*_p]$.
2. If $k = k^*$, then $Q^* = Q^*_p$ and the optimal order quantity is exactly $Q^*$.
3. If $k > k^*$, then $Q^* > Q^*_p$ and the optimal order quantity is in the interval $[Q^*_p, Q^*]$.

Theorem 3.1 leads to the insightful result that the risk-averse newsvendor may order less than the risk-neutral newsvendor if the stockout cost is positive, which will never happen if the stockout cost is zero. This result disproves a claim by Lau [1].

### 4. Conclusions

We studied the risk-averse newsvendor model with a mean–variance objective function. We showed that stockout cost has a significant impact on the newsvendor’s optimal ordering decisions. In particular, contrary to the situation without stockout cost, it is no longer guaranteed that the risk-averse newsvendor orders less than the risk-neutral newsvendor. For the special case that the demand follows the power distribution, we derived the exact conditions under which this will happen.

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Appendix A

A.1. Concavity and slope of the expected profit function

The first-order derivative of $\Pi(Q)$ of (2) with respect to $Q$ is given by

$$
\frac{d\Pi(Q)}{dQ} = -(r + p - s)F(Q) + (r + p - c).
$$

The second-order derivative of $\Pi(Q)$ of (2) with respect to $Q$ is given by

$$
\frac{d^2\Pi(Q)}{dQ^2} = -(r + p - s)f(Q).
$$

From the assumptions in Section 2, we know that $f(Q) \geq 0$, $0 < s < r$ and $0 < p$; hence, $d^2\Pi(Q)/dQ^2 \leq 0$.

Therefore, $\Pi(Q)$ is a concave function of $Q$.

Furthermore, when $Q \to +\infty$, we obtain

$$
\lim_{Q \to +\infty} \Pi(Q)/Q = s - c < 0.
$$

A.2. Boundedness of the variance function

**Proof.** First, we study the case where $Q \to 0$, and we obtain

$$
\lim_{Q \to 0} \text{Var}[\pi(Q)] = \text{Var}[\pi(0)] = p^2 \text{Var}[D],
$$

which is obviously finite under the assumption that the variance of demand is finite.

Second, we study the case where $Q \to +\infty$. Under the assumption that the random demand $D$ has a finite second moment, we have $\int_0^{+\infty} x^2 dF(Q) < +\infty$.

So, we have $\lim_{Q \to +\infty} \int_0^{+\infty} x^2 dF(x) = 0$. Note that $0 \leq Q \int_0^{+\infty} x dF(x) \leq \int_0^{+\infty} x^2 dF(x)$; hence, $\lim_{Q \to +\infty} Q \int_0^{+\infty} x dF(x) = 0$.

From the above analysis, we obtain that

$$
\lim_{Q \to +\infty} \left( Q^2 F(Q) - Q \int_0^Q F(x) \, dx \right)
$$

is given by

$$
- E[D] \int_0^Q F(x) \, dx.
$$

Therefore, we have

$$
\lim_{Q \to +\infty} \left( Q^2 F(Q) - Q \int_0^Q F(x) \, dx \right)
$$

is given by

$$
\lim_{Q \to +\infty} \left( Q^2 F(Q) - Q F(Q) \right).
$$

Finally, we have

$$
\lim_{Q \to +\infty} \left( Q^2 F(Q) - Q F(Q) \right) = p^2 \text{Var}[D] - (r + p - s)^2
$$

and

$$
\lim_{Q \to +\infty} \left( Q^2 F(Q) - Q F(Q) \right) = p^2 \text{Var}[D] - (r + p - s)^2.
$$
A.3. Proof of Theorem 3.1

Then, for the same as the sign of with respect to the second-order derivative function of the function is a bounded function in the function is continuous, we obtain that the variance function is a bounded function in $Q \in [0, \infty)$. □

A.3. Proof of Theorem 3.1

Proof. First, we define a function $W(Q)$ as follows:

$$W(Q) := -(r + p - s)Q^{k+1} + [(k+1)p + (r-s)]Q - pk.$$  \hspace{1cm} (11)

Then, for $Q \in [0, 1]$, the sign of $d\text{Var}[\pi(Q)]/dQ$ is the same as the sign of $W(Q)$. Note that the first-order derivative function of $W(Q)$ of (11) with respect to $Q$ is given by

$$\frac{dW(Q)}{dQ} = -(k+1)(r+p-s)Q^k + (k+1)p + (r-s).$$

The second-order derivative function of $W(Q)$ of (11) with respect to $Q$ is given by

$$\frac{dW^2(Q)}{dQ^2} = -k(k+1)(r+p-s)Q^{k-1} \leq 0.$$  \hspace{1cm} (12)

From (12), we know that $W(Q)$ is a concave function. Furthermore, we have

$$W(0) = -kp < 0, \quad W(1) = 0,$$

$$\frac{dW(Q)}{dQ}\bigg|_{Q=0} = (k+1)p + (r-s) > 0,$$

$$\frac{dW(Q)}{dQ}\bigg|_{Q=1} = -k(r-s) < 0.$$  

Thus, there must exist a unique root $Q^0_p \in (0, 1)$ that satisfies the equation $W(Q) = 0$. So, the signs of $W(Q)$ and $d\text{Var}[\pi(Q)]/dQ$ change exactly once in $Q^0_p$ and the sign changes are from negative to positive. Hence, $Q^0_p$ is the unique minimizer for $\text{Var}[\pi(Q)]$ in $Q \in [0, 1]$, where $\text{Var}[\pi(Q)]$ is decreasing in $[0, Q^0_p]$ and increasing in $[Q^0_p, 1]$. Therefore, the newsvendor’s optimal order quantity is within an interval bounded by $Q^*$ and $Q^0_p$.

Note that $(Q^*)^k = (r + p - c)/(r + p - s)$, and

$$W(Q^*) = -(r + p - c)Q^* + [(k+1)p + (r-s)]Q^* - pk = (kp + c - s)Q^* - pk.$$  

Therefore, we obtain $Q^* < (r+p)/c$. In other words, $Q^* < (r+p)/c$, and $(r+p-c)/(r+p-s) < (r+p)/c$. It is straightforward to verify that

$$\left(\frac{kp}{kp+c-s}\right)^k \left(\frac{kp}{kp+c-s}\right)^k \times \ln \left(1 - \frac{c-s}{kp+c-s}\right) + \frac{c-s}{kp+c-s} < 0.$$  

Moreover, we have the following results:

$$\lim_{k \to 0} \left(\frac{kp}{kp+c-s}\right)^k = 1 > \frac{r+p-c}{r+p-s},$$

$$\left(\frac{kp}{kp+c-s}\right)^k \bigg|_{k=1} = \frac{p}{p+c-s} < \frac{r+p-c}{r+p-s}.$$  

Thus, there exists a critical value $k^*$ with $0 < k^* < 1$ and the newsvendor’s optimal order quantity is
distinguished by three cases as follows:

(1) If $0 < k < k^*$, then $Q^* < Q^0_P$ and the optimal order quantity is in the interval $[Q^*, Q^0_P]$.
(2) If $k = k^*$, then $Q^* = Q^0_P$ and the optimal order quantity is exactly $Q^*$. 
(3) If $k > k^*$, then $Q^* > Q^0_P$ and the optimal order quantity is in the interval $[Q^0_P, Q^*]$. □

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