The distance spectral radius of digraphs

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Let \( D(G) \) denote the distance matrix of a strongly connected digraph \( G \). The eigenvalue of \( D(G) \) with the largest modulus is called the distance spectral radius of a digraph \( G \), denoted by \( \rho(G) \). In this paper, we first give sharp upper and lower bounds for the distance spectral radius for strongly connected digraphs; we then characterize the digraphs having the maximal and minimal distance spectral radii among all strongly connected digraphs; we also determine the extremal digraph with the minimal distance spectral radius with given arc connectivity and the extremal digraph with the minimal distance spectral radius with given dichromatic number.

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1. Introduction

Throughout this paper, we consider finite, simple strongly connected digraphs, i.e. without loops and multiple arcs. We use standard terminology and notation, and refer the reader to [2] for an extensive treatment of digraphs.

For a digraph \( G = (V, A) \), two vertices are called adjacent if they are connected by an arc, \(|V|\) is always called the order of \( G \) and \(|A|\) is always called the size of \( G \). We call \( G \) strongly connected if for every pair of vertices \( x, y \in V(G) \) there exists a directed path from \( x \) to \( y \). Let \( P_n \) and \( C_n \) denote the dipath and dicycle of \( n \) vertices, respectively. The complete digraph of order \( n \) is the digraph \( K_n^\ast \) in which every pair of vertices is connected by an arc. For a strongly connected digraph \( G \), let \( D(G) = (d_{ij}) \) be the distance matrix of a strongly connected digraph \( G \), where \( d(ij) = d_{ij}(v_i, v_j) \) is defined to be the length (i.e. the number of arcs) of the shortest dipath from \( v_i \) to \( v_j \). We call \( D_1 = \sum_{j=1}^{n} d_{ij} (i = 1, 2, \ldots, n) \) the distance degree of vertex \( v_i \). Clearly, we can assign the subscripts of vertices of \( V(G) \) such that \( D_1 \leq D_2 \leq \cdots \leq D_n \), so we may let \( \{v_1, v_2, \ldots, v_n\} \) be such a vertex ordering until stated otherwise. We call a digraph \( G \) distance regular if \( D_1 = D_2 = \cdots = D_n \). The matrix \( D(G) \) is nonnegative and irreducible when \( G \) is strongly connected. The eigenvalue of \( D(G) \) with the largest modulus is called the distance spectral radius of the digraph \( G \), denoted by \( \rho(G) \). The positive unit eigenvector corresponding to \( \rho(G) \) is called the Perron vector of \( D(G) \).

Theorem 1.1 (Perron and Frobenius [17]). Let \( A_{n \times n} \) be a nonnegative irreducible square matrix. Suppose that \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are all the eigenvalues of \( A \). Then:

(i) \( \rho(A) \) is a simple eigenvalue of \( A \) and \(|\lambda_i| \leq \rho(A) \) for any eigenvalue \( \lambda_i \) (\( 1 \leq i \leq n \));
(ii) there exists a positive unit eigenvector corresponding to \( \varrho(A) \), which is called the Perron vector of \( A \).

Let \( \overrightarrow{G} \) be a digraph. A vertex set \( F \subseteq V(\overrightarrow{G}) \) is acyclic if the induced subdigraph \( \overrightarrow{G}[F] \) is acyclic. A partition of \( V(\overrightarrow{G}) \) into \( k \) acyclic sets is called a \( k \)-coloring of \( \overrightarrow{G} \). The minimum integer \( k \) for which there exists a \( k \)-coloring of \( \overrightarrow{G} \) is the dichromatic number \( \chi(\overrightarrow{G}) \) of the digraph \( \overrightarrow{G} \). The above definition of the dichromatic number of a digraph was first introduced by Neumann-Lara [16]. The same notation was independently introduced much later by B. Mohar when considering the circular chromatic number of weighted (directed or undirected) graphs [14]. The dichromatic number of digraphs was further investigated in [3]. B. Mohar [15] gave a lower bound on the spectral radius for digraphs with given dichromatic number. Later, Lin et al. [10] characterized the unique digraph with the maximal spectral radius with given dichromatic number. Clearly, if \( G \) is an undirected graph, and \( \overrightarrow{G} \) is the digraph obtained from \( G \) by replacing each edge with the pair of oppositely directed arcs joining the same pair of vertices, then \( \chi(\overrightarrow{G}) \) is the same as the usual chromatic number of the undirected graph \( G \) since any two adjacent vertices in \( G \) induce a directed cycle of length 2. For a strongly connected digraph \( \overrightarrow{G} = (V, A) \), a set of arcs \( S \subseteq A \) is a cut set if \( \overrightarrow{G} - S \) is not strongly connected. A digraph \( D \) is \( k \)-arc strongly connected if \( D \) has no arc cut with less than \( k \) arcs. The largest integer \( k \) such that \( \overrightarrow{G} \) is \( k \)-arc strongly connected is the arc connectivity of \( \overrightarrow{G} \), denoted by \( \eta(\overrightarrow{G}) \). For two vertex sets \( A, B \subseteq V \) and \( A \cap B = \emptyset \), let \( [A, B] \) denote the arcs between \( A \) and \( B \), and \( \delta^- \) denote the minimum in-degree of \( D \).

The distance matrix is very useful in different areas including the design of communication networks, graph embedding theory as well as molecular stability. In [1], Balaban et al. proposed the use of the distance spectral radius as a molecular descriptor. In [5], it was successfully used to infer the extent of branching and model boiling points of alkane. Recently in [21,22], Zhou and Trinajstić provided upper and lower bounds for \( \varrho(G) \) in terms of the number of vertices, Wiener index and Zagreb index. Subhi and Powers in [19] proved that for \( n \geq 3 \) the path \( P_n \) has the maximal distance spectral radius among trees on \( n \) vertices. Stevanović and Ilić in [18] generalized this result, and proved that among trees with fixed maximum degree \( \Delta \), the broom has the maximal distance spectral radius. Furthermore, they proved that the star \( S_n \) is the unique graph with the minimal distance spectral radius among trees on \( n \) vertices. In [7], Ilić characterized \( n \)-vertex trees with given matching number \( m \) which minimize the distance spectral radius. Zhai [20] showed the extremal graph with the minimal distance spectral radius with given clique number. For other studies of distance spectral radius we suggest readers refer to [8,9,12]. Recently, Lin et al. [11] determined the unique digraph with the minimal distance spectral radius with given connectivity. However, there are only a few papers on digraphs.

The current paper is organized as follows. In Section 2, we give sharp upper and lower bounds for strongly connected digraphs. In Section 3, we characterize the maximal and minimal distance spectral radius among strongly connected digraphs. In Section 4, we determine the extremal digraph with the minimal distance spectral radius with given arc connectivity. In Section 5, we determine the extremal digraph with the minimal distance spectral radius with given dichromatic number.

2. Sharp upper and lower bounds for the distance spectral radius for digraphs

A reformulation of inequalities from the theory of nonnegative matrices [13, Chapter 2] yields the following lemma.

**Lemma 2.1.** Let \( \overrightarrow{G} \) be a strongly connected digraph with \( n \) vertices. Then

\[
D_1 \leq \varrho(\overrightarrow{G}) \leq D_n.
\]

Moreover, one of the equalities holds if and only if \( \overrightarrow{G} \) is a distance regular digraph.

Now we give sharp upper and lower bounds for \( \varrho(\overrightarrow{G}) \) in terms of the distance degree \( D_i \).

**Theorem 2.1.** Let \( \overrightarrow{G} \) be a strongly connected digraph with vertices \( v_1, v_2, \ldots, v_n \) of distance degrees \( D_1 \leq D_2 \leq \cdots \leq D_{n-1} \leq D_n \). Then \( \sqrt{D_1D_2} \leq \varrho(\overrightarrow{G}) \leq \sqrt{D_{n-1}D_n} \), and one of the equalities holds if and only if \( \overrightarrow{G} \) is a distance regular digraph.

**Proof.** Suppose that \( x = (x_1, x_2, \ldots, x_n) \) is the Perron vector of \( D(\overrightarrow{G}) \), \( x_i = \max_{1 \leq j \leq n} x_j \) and \( x_i = \max_{1 \leq j \leq n} x_j \). Then \( \varrho(\overrightarrow{G})x_i \leq D_i x_i \), \( \varrho(\overrightarrow{G})x_i \leq D_i x_i \).

Hence \( \varrho(\overrightarrow{G}) \leq \sqrt{D_1D_2} \leq \sqrt{D_{n-1}D_n} \).

If \( \overrightarrow{G} \) is a distance regular digraph, then \( \varrho(\overrightarrow{G}) = D_n \) by Lemma 2.1. For the converse, if \( \varrho(\overrightarrow{G}) = \sqrt{D_{n-1}D_n} \), then \( \varrho(\overrightarrow{G})x_i = D_i x_i \), and \( \varrho(\overrightarrow{G})x_i = D_i x_i \); hence \( x_1 = x_2 = \cdots = x_n \). Therefore, we have that \( \overrightarrow{G} \) is a distance regular digraph. Similarly, let \( x_j = \min_{1 \leq i \leq n} x_j \) and \( x_j = \min_{1 \leq i \leq n} x_j \). Then

\[
\varrho(\overrightarrow{G})x_j \geq D_j x_j, \quad \varrho(\overrightarrow{G})x_j \geq D_j x_j.
\]

Hence \( \varrho(\overrightarrow{G}) \geq \sqrt{D_1D_2} \geq \sqrt{D_1D_2} \). As above, the equality holds if and only if \( \overrightarrow{G} \) is a distance regular digraph. Therefore, we complete the proof. □
3. Maximal and minimal distance spectral radius for digraphs

A digraph \( \overrightarrow{G} \) is a minimizing (maximizing) digraph of \( \overrightarrow{g}_n \) if \( \overrightarrow{G} \in \overrightarrow{g}_n \) and \( \varrho(\overrightarrow{G}) = \min \{ \varrho(\overrightarrow{G}) | \overrightarrow{G} \in \overrightarrow{g}_n \} \) (\( \varrho(\overrightarrow{G}) = \max \{ \varrho(\overrightarrow{G}) | \overrightarrow{G} \in \overrightarrow{g}_n \} \) ), where \( \overrightarrow{g}_n \) is the set of all strongly connected digraphs on \( n \) vertices. In the following, we characterize the minimizing and maximizing digraphs of \( \overrightarrow{g}_n \).

**Theorem 3.1.** The complete digraph \( \overrightarrow{K}_n \) is the unique minimizing digraph among all digraphs in \( \overrightarrow{g}_n \). Furthermore, for any strongly connected digraph \( \overrightarrow{G} \) we have \( \varrho(\overrightarrow{G}) \geq n - 1 \) and the equality holds if and only if \( \overrightarrow{G} \equiv \overrightarrow{K}_n \).

**Proof.** It is well known that \( \varrho(\overrightarrow{G} + e) \leq \varrho(\overrightarrow{G}) \) for all \( \overrightarrow{G} \in \overrightarrow{g}_n \) and \( e \not\in A(\overrightarrow{G}) \). Then for every digraph \( \overrightarrow{G} \) of \( \overrightarrow{g}_n \) we have \( \varrho(\overrightarrow{G}) \geq \varrho(\overrightarrow{K}_n) \), with equality holding if and only if \( \overrightarrow{G} \equiv \overrightarrow{K}_n \). Since \( \varrho(\overrightarrow{K}_n) = n - 1 \), we thus complete the proof. \( \square \)

**Theorem 3.2.** The dicycle \( \overrightarrow{C}_n \) is the unique maximizing digraph among all digraphs in \( \overrightarrow{g}_n \). Furthermore, for any strongly connected digraph \( \overrightarrow{G} \) we have \( \varrho(D(\overrightarrow{G})) \leq \frac{n(n-1)}{2} \) and the equality holds if and only if \( \overrightarrow{G} \equiv \overrightarrow{C}_n \).

**Proof.** Since \( \overrightarrow{C}_n \) is a distance regular digraph, by **Lemma 2.1**, and does not contain a Hamiltonian dicycle, noted that \( D_1(\overrightarrow{C}_n) = \varrho(\overrightarrow{C}_n) = D_n(\overrightarrow{C}_n) = 1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2} \).

Since \( \varrho(\overrightarrow{G} + e) \leq \varrho(\overrightarrow{G}) \) for all \( \overrightarrow{G} \in \overrightarrow{g}_n \) and \( e \not\in A(\overrightarrow{G}) \), if \( \overrightarrow{G} \) has a Hamiltonian dicycle, we have \( \varrho(\overrightarrow{G}) \leq \varrho(\overrightarrow{C}_n) \), with equality holding if and only if \( \overrightarrow{G} \equiv \overrightarrow{C}_n \).

In the following, we suppose that \( \overrightarrow{G} \) does not contain a Hamiltonian dicycle. Note that \( D_n(\overrightarrow{G}) \leq \frac{n(n-1)}{2} \). If \( D_n(\overrightarrow{G}) < \frac{n(n-1)}{2} \), then \( \varrho(\overrightarrow{G}) < \varrho(\overrightarrow{C}_n) \).

If \( D_n(\overrightarrow{G}) = \frac{n(n-1)}{2} \), then \( \overrightarrow{G} \) contains a vertex \( v \) such that \( D_v = D_n = \frac{n(n-1)}{2} \); without loss of generality let \( v = v_1 \).

It is easy to see that \( \overrightarrow{G} \) contains a Hamiltonian dipath \( P \) initiating at \( v_1 \). Suppose that \( P = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n \) is the Hamiltonian dipath initiating at \( v_1 \). Then there is no arc \( v_i v_j \in A \) if \( j - i \geq 2 \) since \( D_{v_1} = \frac{n(n-1)}{2} \). Since \( \overrightarrow{G} \) is strongly connected and does not contain a Hamiltonian dicycle, there exists a dipath \( P' \) from \( v_n \) to \( v_1 \) and thus there exists some vertex, namely, \( v_1 (k \neq n) \), that is adjacent to \( v_1 \), that is \( v_k \in A \). Since \( v_k \) is on the Hamiltonian dipath \( P \), we have \( v_k v_{k+1} \in A \). Hence

\[
D_{v_k} \leq 1 + 1 + 2 + \cdots + n - 2 < 1 + 2 + \cdots + n - 1 = D_n(\overrightarrow{G})
\]

By **Lemma 2.1**, we have

\[
\varrho(\overrightarrow{G}) < \varrho(\overrightarrow{C}_n)
\]

Thus we complete the proof. \( \square \)

4. The minimal distance spectral radius for digraphs with given arc connectivity

We use \( I_p \) to denote the \( p \times p \) identity matrix and \( J_{p,q} \) to denote the \( p \times q \) matrix in which every entry is 1. We denote by \( \mathbb{I}_{p} \) the \( p \)-vector of entries 1. Let \( \overrightarrow{G}_1 \) \( \overrightarrow{G}_2 \) denote the digraph obtained from two disjoint digraphs \( \overrightarrow{G}_1 \), \( \overrightarrow{G}_2 \) with vertex set \( V(\overrightarrow{G}_1) \cup V(\overrightarrow{G}_2) \) and arc set \( E = (\overrightarrow{G}_1) \cup E(\overrightarrow{G}_2) \cup \{u, v | u \in V(\overrightarrow{G}_1), v \in V(\overrightarrow{G}_2)\} \). If \( 1 \leq k \leq n - 1 \), let \( \mathcal{K}_{n-k}^{k} \) denote the digraph \( \overrightarrow{G} = (\overrightarrow{K}_n \cup \overrightarrow{K}_{n-k}) \cup E \) where \( E = \{u, v | u \in V(\overrightarrow{K}_n), v \in V(\overrightarrow{K}_{n-k})\} \). Lin et al. [11] proved that the digraphs \( \overrightarrow{G} \equiv \mathcal{K}_{n-k}^{k} \) and \( \overrightarrow{G} \equiv \mathcal{K}_{k+1}^{k-n} \) attain the minimal distance spectral radius with given connectivity and number of vertices, and they also conjectured that those digraphs also attain the minimal distance spectral radius with given arc connectivity. In this section we will give a proof of this conjecture.

**Theorem 4.1.** Let \( n \geq 1 \) and \( k \geq 1 \). If \( G \) is a strongly connected digraph with \( n \) vertices and arc connectivity \( k \), then

\[
\varrho(\overrightarrow{G}) \geq \frac{n - 2 + \sqrt{(n+2)^2 - 4k - 8}}{2}
\]

with equality if and only if either \( \overrightarrow{G} \equiv \mathcal{K}_{n-k}^{k} \) or \( \overrightarrow{G} \equiv \mathcal{K}_{k+1}^{k-n} \).
5. The minimal distance spectral radius for digraphs with given dichromatic number

Let $\overrightarrow{G}$ be a digraph with $n$ vertices and arc connectivity $k \geq 1$. Suppose that $S$ is an arc cut of $\overrightarrow{G}$ with $|S| = k$. Then $\overrightarrow{G} - S$ has exactly two strongly connected components, say $G_1$ and $G_2$. Without loss of generality, we may assume that there are no arcs from $G_2$ to $G_1$ in $\overrightarrow{G} - S$. Next, we create a new digraph $\overrightarrow{H}$ by adding to $\overrightarrow{G}$ any legal arcs from $G_1$ to $G_1 \cup G_2$ or any legal arcs from $G_2$ to $G_2$ that were not present in $\overrightarrow{G}$. Since the distance matrix of $\overrightarrow{H}$ is irreducible, the addition of any such arc will give $\varphi(\overrightarrow{G}) > \varphi(\overrightarrow{H})$; see [6, Section 8.5, Problem 15]. We note that the arc connectivity of $\overrightarrow{H}$ remains equal to $k$.

The distance matrix of $\overrightarrow{H}$ has the form

$$
\begin{pmatrix}
I_p - I_p & J_{p,q} \\
J_q - I_q & X
\end{pmatrix}
$$

where $|V(G_1)| = p$, $|V(G_2)| = q$ with $n = p + q$ and $X$ is a zero–one matrix with exactly $k$ entries 1. It is clear that the first $p$ entries of the right Perron vector of $\overrightarrow{H}$ are all equal. So the Perron root $\varphi(\overrightarrow{H})$ is given by

$$
\begin{pmatrix}
I_p - I_p & J_{p,q} \\
J_q - I_q & X
\end{pmatrix}
\begin{pmatrix}
1_p \\
1_q
\end{pmatrix} = \varphi(\overrightarrow{H})
\begin{pmatrix}
1_p \\
1_q
\end{pmatrix}
$$

which we write out as

$$
(p - 1)1_p + 1_p(1_q'x) = \rho 1_p,
$$

(2)

\begin{align*}
X1_p + 1_q(1_q'x) - x &= \rho x. \tag{3}
\end{align*}

Premultiplying (2) by $1_p'$ and (3) by $1_q'$, we get

$$
p(p - 1) + p(1_q'x) = \varphi(\overrightarrow{H})p
$$

$$
k + (q - 1)(1_q'x) = \varphi(\overrightarrow{H})(1_q'x)
$$

or equivalently

$$
\begin{pmatrix}
p - 1 \\
k
\end{pmatrix}
\begin{pmatrix}1 \\
q - 1
\end{pmatrix} = \varphi(\overrightarrow{H})
\begin{pmatrix}1 \\
1_q'
\end{pmatrix}
$$

which evaluates to

$$
\varphi(\overrightarrow{H}) = \frac{n - 2 + \sqrt{(n - 2)^2 + 4pq + 4(n - 1) - 4k}}{2}. \tag{4}
$$

The minimal $\varphi(\overrightarrow{H})$ clearly occurs when $p$ and $q$ are as far apart as possible. We take $q = 1$ and $p = n - 1$. Substitution in (4) gives the desired result. It is clear that $\overrightarrow{H} \cong \kappa_{n-1}^k$ or $\overrightarrow{H} \cong \kappa_{n+1}^{k-1}$. □

5. The minimal distance spectral radius for digraphs with given dichromatic number

Let $\mathcal{G}_{n,k}$ denote the set of digraphs of order $n$ with the dichromatic number $k \geq 2$, $\mathcal{T}_{n,k}$ denote the digraphs with $V(\mathcal{T}_{n,k}) = V^1 \cup V^2 \cup \ldots \cup V^k$, where each $V^i$ ($i = 1, 2, \ldots, k$) is a transitive tournament and $|V^i| = |V^j| = |\{uv, vu\} \subseteq V^i, v \in V^j\}$, and $\mathcal{T}_{n,k}^*$ denote the digraph in $\mathcal{T}_{n,k}$ with $||V^i| - |V^j|| \leq 1$. The techniques used in this section are motivated by those in Lin et al. [10]. Before proceeding, we cite a known result as a lemma (see, for example, [4, Exercise 10.1.4]).

**Lemma 5.1.** Let $\overrightarrow{G}$ be a digraph with no dicycle. Then $\delta^- = 0$ and there is an ordering $v_1, v_2, \ldots, v_n$ of $V(\overrightarrow{G})$ such that for $1 \leq i \leq n$, every arc of $\overrightarrow{G}$ with head $v_i$ has its tail in $\{v_1, v_2, \ldots, v_{i-1}\}$.

The following well-known result can be found in [13]; we cite it as a lemma.

**Lemma 5.2.** If $A$ is a nonnegative matrix and $x \geq 0$ is a nonzero vector such that $Ax \geq \alpha x$ for some $\alpha \in R$, then $\rho(A) \geq \alpha$.

Let $\overrightarrow{G}$ be a digraph of order $n$ with $\chi(\overrightarrow{G}) = k \geq 2$. From the definition, $\overrightarrow{G}$ has $k$-color classes and each is an acyclic set. Suppose that the $k$-color classes are $V^1, V^2, \ldots, V^k$ having order $n_1, n_2, \ldots, n_k$, respectively. Without loss of generality, we suppose that $n_1 \leq n_2 \leq \cdots \leq n_k$. Note that $\varphi(\overrightarrow{G} + e) \leq \varphi(\overrightarrow{G})$ for all $\overrightarrow{G} \in \mathcal{G}_{n,k}$ and $e \not\in A(\overrightarrow{G})$. By Lemma 5.1, the digraph minimizing the distance spectral radius must be in $\mathcal{T}_{n,k}$. Next we prove that the digraph $\mathcal{T}_{n,k}^*$ has the minimal distance spectral radius among all digraphs in $\mathcal{T}_{n,k}$. 


We have Claim 1.\[x\] shall consider the case where

By induction on \(i = 1, 2, \ldots, k\) and ordered as \(n_1 \leq n_2 \leq \cdots \leq n_k\) (as shown in Fig. 1). Let \(\overrightarrow{G} = \overrightarrow{G} - (v_i^{(1)}v_j^{(1)} | t = 1, \ldots, n_i) + (v_i^{(1)}v_j^{(1)} | k = 1, \ldots, n_i - 1)\) where \(2 \leq n_i \leq n_j\). Then \(\omega(\overrightarrow{G}) < \omega(\overrightarrow{G}')\).

**Proof.** Let \(D\) denote the distance matrix of \(\overrightarrow{G}\). Since each \(V_i\) is a transitive tournament, we give a vertex ordering \(\{v_1, v_2, \ldots, v_{n_i}\}\) such that \(v_i v_j \in E\) for all \(s < t\).

Suppose that \(x = (x_1^1, x_1^2, \ldots, x_1^{n_1}, x_2^1, x_2^2, \ldots, x_2^{n_2}, \ldots, x_k^1, x_k^2, \ldots, x_k^{n_k})\) is a Perron vector of \(D\) where \(x_i^j\) corresponds to \(v_i^j\) for each \(1 \leq i \leq k\) and \(1 \leq j \leq n_i\).

**Claim 1.** \(x_1^1 < x_1^2 < \cdots < x_1^{n_1}\).

We have

\[
\omega x_i^j = S + x_{i+1}^j + 2 \sum_{t=j+2}^{n_i} x_t^j
\]

\[
\omega x_{i+1}^j = S + 2x_i^j + 2 \sum_{t=j+2}^{n_i} x_t^j,
\]

where \(S = \sum_{m \neq s} \sum_{l=1}^{n_m} x_l^m\). Hence we have

\[
(q + 1)x_{i+1}^j = (q + 2)x_i^j,
\]

which implies that \(x_i^j < x_{i+1}^j\). Therefore, Claim 1 holds.

**Claim 2.** \(x_1^1 = x_1^2 = \cdots = x_k^l\) for \(1 \leq i \leq n_k\).

By induction on \(i\), we first show that the Claim 2 holds when \(i = 1\):

\[
\omega x_1^1 = \sum_{t=2}^{n_1} x_t^1 + x_1^1 + \sum_{t=2}^{n_1} x_t^1 + S,
\]

\[
\omega x_1^1 = \sum_{t=2}^{n_1} x_t^1 + x_1^1 + \sum_{t=2}^{n_1} x_t^1 + S,
\]

where \(S = \sum_{m \neq s} \sum_{l=1}^{n_m} x_l^m\). Then

\[
\omega x_1^s - \omega x_1^t = x_1^s - x_1^t,
\]

which implies that \(x_1^s = x_1^t\) for \(1 \leq s \neq t \leq k\).

Now we suppose that it holds for all \(i < N \leq n_k\), that is, \(x_1^1 = x_2^2 = \cdots = x_k^l\) for each \(t \in \{1, 2, \ldots, N - 1\}\). Now we shall consider the case where \(i = N\), noting that

\[
\omega x_N^1 = 2 \sum_{t=1}^{N-1} x_t^1 + \sum_{t=N+1}^{n_N} x_t^1 + x_N^1 + \sum_{t=N+1}^{n_N} x_t^1 + S,
\]

\[
\omega x_N^N = 2 \sum_{t=1}^{N-1} x_t^N + \sum_{t=N+1}^{n_N} x_t^N + x_N^N + \sum_{t=N+1}^{n_N} x_t^N + S,
\]

Fig. 1. The digraph \(\overrightarrow{G}\).

**Theorem 5.1.** Let \(\overrightarrow{G}\) be a digraph of \(T_{n,k}\) with \(V(\overrightarrow{G}) = V^1 \cup V^2 \cup \cdots \cup V^k\), where \(|V^i| = n_i\) and each \(V^i\) is a transitive tournament \((i = 1, 2, \ldots, k)\) and ordered as \(n_1 \leq n_2 \leq \cdots \leq n_k\) (as shown in Fig. 1). Let \(\overrightarrow{G} = \overrightarrow{G} - (v_i^{(1)}v_j^{(1)} | t = 1, \ldots, n_i) + (v_i^{(1)}v_j^{(1)} | k = 1, \ldots, n_i - 1)\) where \(2 \leq n_i \leq n_j\). Then \(\omega(\overrightarrow{G}) < \omega(\overrightarrow{G}')\).
where \( S = \sum_{m \neq l, s} \sum_{i=1}^{m} x_{i}^s \). Since \( x_t^s = x_t^s \) for \( t = 1, 2, \ldots, N - 1 \), we hence have
\[
\varrho x_N^s = \varrho x_N^s - x_N^s,
\]
that is, \( x_{i}^s = x_{i}^s \) for all \( 1 \leq s \neq l \leq k \). Therefore, Claim 2 holds.

Let \( G' = G - \{v_t^i, v_j^l | t = 1, \ldots, n_i \} + \{v_t^i, v_j^l | k = 1, \ldots, n_i - 1 \} \) (see Fig. 2) where \( 2 \leq n_i \leq n_j \), where \( n_i \leq n_j \), and \( D' \) be the distance matrix of \( G' \). Obviously, \( G' \in T_{n,k} \).

Claim 3. \( D'x \geq Dx = \varrho (D)x \), where \( x \) is the Perron vector of \( D \) corresponding to \( \varrho (D) \).

Since \((Dx)_t = \sum_{i=1}^{n} d_{t,i}x_i \) and \((Dx)_t = \sum_{i=1}^{n} d_{t,i}x_i \), we have that: if \( v_t \neq v_i \), then \((Dx)_t = \varrho (D)_t = (D')_t \); if \( v_t = v_i \), then \((Dx)_t = S + 2 \sum_{i=1}^{n} x_i + \sum_{i=1}^{n-1} x_i \) and \((Dx)_t = S + 2 \sum_{i=1}^{n-1} x_i \). By Claims 1 and 2, we have \((Dx)_t = (D')_t \geq 0 \); therefore \( D'x \geq Dx \).

Then by Lemma 5.2, we get \( \varrho (G') > \varrho (G) \). Therefore, we complete the proof. \( \square \)

Note that the digraph achieving the minimal distance spectral radius must be in \( T_{n,k} \), and every digraph \( G \in T_{n,k} \) can be obtained from \( T_{n,k} \) by the steps of the above operation. Hence, we present our main theorem as follows:

Theorem 5.2. The digraph \( T_{n,k}^* \) is the unique digraph with the minimal distance spectral radius among all digraphs in \( \mathcal{G}_{n,k} \).

Let \( B_{n,k} \) be a digraph obtained by adding a directed path \( P_{n-k+2} = u_1u_2 \ldots u_{n-k+2} \) to a clique \( K_k \) such that \( V(K_k) \cap V(P) = \{u_1, u_{n-k+2}\} \). By Theorem 3.2, the digraph \( C_n \) attains the maximum distance spectral radius among all strongly connected digraphs. Obviously, \( C_n \) is a digraph with dichromatic number \( k \), then among all strongly connected digraphs with dichromatic number \( k \), the digraph \( C_n \) attains the maximal distance spectral radius. For the extremal digraph with the maximal distance spectral radius with dichromatic number at least \( 3 \), we may pose the following problem.

Problem 1. Among all strongly connected digraphs with dichromatic number \( k \geq 3 \), does the digraph \( B_{n,k} \) attain the maximal distance spectral radius?

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References