Estimating risk of foreign exchange portfolio: Using VaR and CVaR based on GARCH–EVT-Copula model

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A B S T R A C T  
This paper introduces GARCH–EVT-Copula model and applies it to study the risk of foreign exchange portfolio. Multivariate Copulas, including Gaussian, \( t \) and Clayton ones, were used to describe a portfolio risk structure, and to extend the analysis from a bivariate to an \( n \)-dimensional asset allocation problem. We apply this methodology to study the returns of a portfolio of four major foreign currencies in China, including USD, EUR, JPY and HKD. Our results suggest that the optimal investment allocations are similar across different Copulas and confidence levels. In addition, we find that the optimal investment concentrates on the USD investment. Generally speaking, \( t \) Copula and Clayton Copula better portray the correlation structure of multiple assets than Normal Copula.

1. Introduction

Value-at-Risk (VaR), a statistical measure of downside risk of financial portfolio, is one of the most important measures of market risk. It has been widely used for financial risk management by financial institutions, regulators and portfolio managers. The greatest advantage of VaR is that it can summarize risks in a single number. In recent years, Conditional VaR (CVaR) has emerged as a supplement to VaR. It measures the expected loss given that the loss is greater than or equal to VaR at certain confidence levels (such as 95\%). It provides a more conservative measure of losses relative to VaR.

VaR typically deals with the low-probability events in the tails of asset return distribution. Extreme value theory (EVT) focuses directly on the tails and therefore could potentially give us better estimates and forecasts of risk. But applying EVT to the return series is inappropriate as they are not independently and identically distributed. Thus following the approach of Frey and McNeil [1], we use GARCH model to fit the return series and apply EVT to the residuals rather than to the return series. Several researchers home and abroad have made great progress using this method to measure VaR (e.g. Refs. [2–4] among others).

On the other hand, recent research in Econophysics has demonstrated time and again that economic and financial systems are self-adaptive complex systems that are open and time evolving. The complexity involved in pricing derivatives and effective risk management is a manifestation of the nonlinear dynamic characteristics of equity, interest rate and exchange rate market. It is not uncommon to see that a small disturbance in one market leads to dramatic changes in the entire system. The negative impact of an unexpected event could change the correlation structure between risks—for example, risks that were previously thought to be uncorrelated could become correlated, etc. This is because financial markets often respond to external disturbances in a nonlinear way. So, when measuring the risk of a multi-asset portfolio, how to model the relation between the risk factors has become an active line of research in the finance field in recent years.
Copulas are widely used along this line of research, because it is able to extract the dependence structure from the joint probability distribution function and, simultaneously, to isolate such dependence structure from the univariate marginal distributions. Copulas represent a new era in risk analysis and multivariate time-series analysis, and a breakthrough in financial risk measurement methodology. In addition to elliptical copulas, which are symmetric around the mean, non-elliptical copulas can be used to model joint down movements of multiple risk factors with greater probability than elliptical copulas. Besides, Copulas can be used to accurately describe the nonlinear, asymmetrical structure between multiple risk factors.

Our approach proposes the use of copulas for the representation of spatial dependencies. In this regard, copulas have proven to be a good tool for representing dependency between input variables in both simple and complex system models. In system reliability models, they are an alternative if the common causes are not known or not intended to be modeled.

The literature on the subject is quite rich (e.g. Refs. [5–10] among others). For example, Geman and Kharoubi [11] introduces copula functions to have a better representation of the dependence structure of oil Futures with index equites. Fernandez [12] illustrates how tail-dependency tests may be misleading as a tool to select a copula that closely mimics the dependency structure of the data, using US stock data. In addition, Fernandez [5] presents a model to select the optimal hedge ratios of a portfolio composed of an arbitrary number of commodities, using copula to account for returns co-movement.

However, most empirical studies have focused on the equity risk using Gaussian Copula and $t$ Copula. Among the few studies that did focus on the foreign exchange risk, the analysis was limited to two dimension. For example, Patton [13] applies Copula to the modeling of the time-varying joint distribution of the Mark–Dollar and Yen–Dollar exchange rate returns, and finds that the conditional dependence between these exchange rates is asymmetric. Patton [14] considers an extension of the theory of Copulas to allow for conditional dependence structure of these exchange rates, and shows that the Mark–Dollar and Yen–Dollar exchange rates are more correlated when they are depreciating against the Dollar than when they are appreciating. Tursunalieva [15] extends Patton’s work and shows that the SJC-Copula captures the tail dependence between the exchange rates in pre-euro and post-transition periods, while the rotated Gumbel Copula captures the dependence during the transition period.

In China, Wu and Ye [16] were the first to analyze a two-asset foreign exchange portfolio using Archimedean Copula. Sun and Bai [17] considers the EUR/CNY and JPY/CNY exchange rates, using a variety of Archimedean Copulas to fit the data and derive the optimal Copula to capture correlation between the two variables. Wang and Chen [18] uses EVT to measure the foreign exchange risk, and builds Copula-EVT models to calculate the VaR for the exchange rate portfolio of EUR/CNY and JPY/CNY by Monte Carlo simulation. On the other hand, some researchers combine GARCH model and Copula to dynamically study the correlation and risk of financial assets [19–22].

Applying copulas to financial risk management has demonstrated great potentials and much progress have been made in the recent literature. However, when studying multivariate Copula, most researchers focused on Gaussian Copula and $t$ Copula. The Archimedean Copula is rarely used because of implementation issues. In this paper, we fit a Clayton Copula to a foreign exchange portfolio.

The contributions of our paper are two-fold. First, in addition to Gaussian Copula and $t$ Copula, we use the multivariate Archimedean Copula, in particular the Clayton Copula, to analyze the asymmetric dependence structure and measure the complex nonlinear relations among financial asset returns. Secondly, we extend the analysis from a bivariate to an $n$-dimensional asset allocation problem. Specifically, we use GARCH nested copulas with EVT margins to estimate the risk of a four-dimensional foreign exchange portfolio.

The rest of the paper is organized as follows. Section 2 describes the marginal distribution using GARCH–EVT model and provides a brief overview of Copula theory. It also introduces the parameter estimation of Copula. Section 3 presents the simulation algorithm using $n$-variable Copula to describe the correlation structure and analyze the portfolio risk. Section 4 applies GARCH–EVT–Copula to measure risk in a four-dimensional foreign exchange portfolio, and select the optimal copula to model the dependency structure of the data set. Section 5 concludes.

2. GARCH–EVT model and Copula parameter estimates

2.1. GARCH–EVT model

Recognizing the heteroskedasticity and volatility clustering nature of time series, we first use GARCH (1, 1) to model the time series. The model is:

\[
\begin{align*}
    r_t &= \mu + \varepsilon_t = \mu + \sigma_t z_t \\
    \sigma_t &= \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2
\end{align*}
\]  \hspace{1cm} (1)

where $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta < 1$; $r_t$ is the actual return, $\mu$ is the expected return and $\sigma_t$ is the volatility of the returns on day $t$.

The randomness in the model comes through the stochastic variables, which are the residuals or the innovations of the process. Conventionally these residuals are assumed to follow a normal distribution. However, the conditional distribution of GARCH models has been shown to have a heavier tail than that of a normal distribution, which provides better approximation to actual financial time series.
Furthermore, EVT appears to be an appropriate approach for modeling the tail behavior. However, applying EVT to the random variable $r_t$ is inappropriate as $r_t$ is not independently and identically distributed. Therefore, our marginal model is built on GARCH–EVT model, in which we use GARCH model to fit the historical return data, then we model the standard innovation by two distributions: by the GPD distribution in the lower or upper tail and by the empirical distribution in the remaining part. The distribution of innovation is followed by (see the details in Ref. [23]):

$$F(z) =\begin{cases} \frac{N_u}{N} \left\{ \frac{1 + \frac{\xi u^l - z}{\beta^l}}{1 + \frac{\xi^R u^R - z}{\beta^R}} \right\}^{-1/\xi^l} & \cdots z < u^l \\ \phi(z) \cdots u^l < z < u^R \\ 1 - \frac{N_u}{N} \left\{ \frac{1 + \frac{\xi u^l - z}{\beta^l}}{1 + \frac{\xi^R u^R - z}{\beta^R}} \right\}^{-1/\xi^R} & \cdots z > u^R \end{cases}$$

(2)

where $\beta, \xi$ are the scale and shape parameters, respectively, $u^R (u^l)$ is the upper (lower) threshold.

In order to estimate $\beta, \xi$ exactly we have to choose the proper threshold. The choice of the optimal threshold can be tricky because there is a tradeoff between high precision and low variance. If we choose thresholds that are too low, we might obtain biased estimates because the limit theorems do not apply any more. On the other hand, if we choose high thresholds, it will generate estimates with high standard errors due to limited number of observations. A large body of research on statistics used an exponential regression model to select the optimal threshold [24–26]. Especially, Fernandez [12] gives a satisfactory application of such methods based on US stock data. On the other hand, Frey and McNeil [1] shows that the GPD method appears to be more efficient than the Hill method. The GPD method is also shown to be more stable with respect to choice of extreme data. Neftci [27] chooses 1.65 as the threshold. Du–Mouchel [28] propose choosing the exceedances to be the 90th (or 10th) percentile of the sample. It is a compromise between the practical need for enough observations to reliably estimate the parameter $\xi$ and the theoretical desire to describe the behavior of CDF. In this study, we choose the exceedances to be the 10th percentile of the sample and use the sample MEF plot and Hill plot to determine an appropriate threshold.

2.2. Copula parameter estimates

The following theorem is known as Sklar’s Theorem [29]. It is the most important theorem about copula functions because it is used in many practical applications.

**Theorem.** Let $F$ be an $n$-dimensional c.d.f. with continuous margins $F_1, F_2, \ldots, F_n$. Then it has the following unique copula representation:

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)).$$

(3)

From Sklar’s Theorem we see that, for continuous multivariate distribution, the univariate marginal distribution and the multivariate dependence structure can be separated. Assuming that the dependence structure does not change with time, we select three types of dependence structures, namely Gaussian Copula, $t$ Copula and Clayton Copula to estimate the joint distribution. In the following section, we describe how to estimate the parameters of each Copula.

(1) $n$-variable Gaussian Copula.

Let $\phi_\Sigma$ be the standardized multivariate normal distribution with correlation matrix $\Sigma$. The Gaussian Copula can be defined as follows:

$$C^G(u_{1t}, u_{2t}, \ldots, u_{nt}) = \phi_\Sigma(\phi^{-1}(u_{1t}), \phi^{-1}(u_{2t}), \ldots, \phi^{-1}(u_{nt}))$$

(4)

where $\Phi^{-1}$ denotes the inverse of the normal cumulative distribution function. From formula (2) we know that the marginal distribution is $F_t(z)$. Based on the historic data $\{z_{1t}, z_{2t}, \ldots, z_{nt}\}$, $t = 1, 2, \ldots, T$, we set:

$$u_t = (u_{1t}, u_{2t}, \ldots, u_{nt}) = (F_1(z_{1t}), F_2(z_{2t}) \cdots F_n(z_{nt}))$$

$$\xi_t = (\phi^{-1}(u_{1t}), \phi^{-1}(u_{2t}), \ldots, \phi^{-1}(u_{nt})).$$

(5)

Therefore we can get the simple expression: $C^G(u_t) = \phi_\Sigma(\xi_t)$. And using MLE method, $\Sigma$ is estimated as follows:

$$\hat{\Sigma} = \frac{1}{T} \sum_{i=1}^T \xi_i \xi_i'.$$

(6)

To capture the fat tail property, we introduce the multivariate student’s $t$ Copula which shows more observations in the tails than the Gaussian.

(2) $n$-variable $t$ Copula.

Let $t_v, \Sigma$ be the standardized multivariate $t$ distribution with correlation matrix $\Sigma$ and $v$ degrees of freedom. The $t$ Copula can be defined as follows:

$$C^t(u_{1t}, u_{2t}, \ldots, u_{nt}) = t_v, \Sigma(t_v^{-1}(u_{1t}), t_v^{-1}(u_{2t}), \ldots, t_v^{-1}(u_{nt}))$$

(7)

where $t_v^{-1}$ denotes the inverse of the Student’s $t$ cumulative distribution function.
Step (2) Simulating returns via Clayton Copula areas follows:

\[ \xi_t = \left( t_{v_1}^{-1}(u_{1t}), t_{v_2}^{-1}(u_{2t}), \ldots, t_{v_n}^{-1}(u_{nt}) \right). \]

Therefore we have: \( C_t(u_t) = t_v(\xi_t) \). The parameter matrix \( \Sigma \) is also estimated using the MLE method which is the same as the Gaussian Copula, and it can be calculated as follows:

1. The initial matrix is the correlation coefficient matrix of multivariate normal Copula function estimated from (6).
2. We can get the correlation coefficient matrix \( \hat{\Sigma}_{n+1} \) of multivariate t Copula through the following iterative calculation method:

\[ \hat{\Sigma}_{k+1} = \frac{1}{T} \left( \sum_{t=1}^{T} \frac{\xi_t \xi_t^\prime}{1 + \frac{1}{v} \hat{\Sigma}_k \xi_t} \right), \]

(9) for \( k = 1, 2, \ldots \).

3. Repeat the above process until \( \hat{\Sigma}_{n+1} = \hat{\Sigma}_n \), so using MLE we can get the correlation coefficient matrix \( \Sigma \) of t Copula which can be \( \hat{\Sigma} = \hat{\Sigma}_n \).

In addition to the fat tails, we can try to check whether asymmetry exists in the tails which the t Copula cannot detect. We use the Clayton Copulas, one of the Archimedean families better known in capturing left tail dependence.

(3) \( n \)-variable Clayton Copula.

The Clayton Copula was first proposed by Clayton [30]. The Cumulative distribution functions are defined by the following:

\[ C_{\text{Clayton}}^{\text{Clayton}}(u_1, u_2, \ldots, u_n) = \left( u_1^\theta + u_2^\theta + \cdots + u_n^\theta - n + 1 \right)^{-\frac{1}{\theta}}, \quad \theta \geq 0. \]

(10) Based on the historic data \( \{z_{1t}, z_{2t}, \ldots, z_{nt}\}, t = 1, 2 \cdots T \) and formula (2), we have \( u_t = F_t(z_{it}) \). Thus, we can directly estimate the parameter \( \theta \) of \( n \)-variable Clayton Copula using MLE method.

3. Simulation algorithm and portfolio risk analysis

3.1. Simulation algorithm

(1) Simulating returns via Gaussian Copula.

Step 1. For \( \hat{\Sigma} \) derived from formula (6), we do Cholesky decomposition and then have \( \Sigma = A^\prime A \);

Step 2. Draw independent \( n \)-dimensional vectors from the multiple normal distribution \( x = (x_1, x_2, \ldots, x_n)^\prime, x_i \sim N(0, 1) \).

Let \( y = A^\prime x \), then \( z = (F_1^{-1}(\phi(y_1)), F_2^{-1}(\phi(y_2)), \ldots, F_n^{-1}(\phi(y_n))) \), where \( F_i^{-1}, i = 1, 2, \ldots, n \) is the inverse of distribution of \( F_i \) in formula (2);

Step 3. Repeating the above steps and simulating \( M \) times, we can get the vector \( \{z_{1m}, z_{2m}, \ldots, z_{nm}\}^\prime, m = 1, 2 \cdots M \).

Then restoring it into formula (1) we can get \( M \) returns at the time \( t + 1 \). The returns residuals’ joint distribution is this Gaussian Copula. The returns can be defined by \( r_{T+1} = (r_{1m}, r_{2m}, \ldots, r_{nm}) = (\mu_1 + z_1\sigma_1, \mu_2 + z_2\sigma_2, \ldots, \mu_n + z_n\sigma_n) \);

where \( \sigma_{i,T+1,\mu_i}, i = 1, 2, \ldots, n \) are calculated by GARCH (1, 1) model.

(2) Simulating returns via t Copula.

Step 1. For \( \hat{\Sigma} \) derived from formula (9) we do Cholesky decomposition to get \( \Sigma = A^\prime A \);

Step 2. Generate \( m \) random vectors which are independently and identically distributed, such as \( x = (x_1, x_2, \ldots, x_n)^\prime, x_i \sim N(0, 1) \).

Let \( y = A^\prime x \), then we produce random vector \( z \), subject to Chi-square distribution and independent of \( x \). Then we set:

\[ z = \left( F_1^{-1}(t(\frac{\nu_1}{\sqrt{\nu_1}})), F_2^{-1}(\phi(t(\frac{\nu_2}{\sqrt{\nu_2}}))), \ldots, F_n^{-1}(\phi(t(\frac{\nu_n}{\sqrt{\nu_n}}))) \right) \]

where \( F_i^{-1}, i = 1, 2, \ldots, n \) is the inverse of distribution of \( F_i \) in formula (2);

Step 3. Repeat the above step \( M \) times, we can get the vector \( \{z_{1m}, z_{2m}, \ldots, z_{nm}\}^\prime, m = 1, 2 \cdots M \).

Then restoring it into formula (1) we can get \( M \) returns at the time \( t + 1 \). The returns residuals’ joint distribution is this t Copula. The returns can be defined by \( r_{T+1} = (r_{1m}, r_{2m}, \ldots, r_{nm}) = (\mu_1 + z_1\sigma_1, \mu_2 + z_2\sigma_2, \ldots, \mu_n + z_n\sigma_n) \);

(3) Simulating returns via Clayton Copula.

A hurdle for practical implementation of any multivariate Archimedean Copula was the absence of an efficient method to generate returns. Marshall and Olkin [31] proposed an alternative method, which is computationally straightforward than the conditional distribution approach. It requires generating an additional variable.

Considering the inverse function of Gamma(\( \frac{1}{\theta}, 1 \))’s LT transformation \( \tilde{c}^{-1} = t^{-\theta} - 1 \), it is closely related to the generator of Clayton Copula: \( \phi(t) = \frac{1}{\theta}(t^{-\theta} - 1) \). The two expressions are only different by a factor of \( \frac{1}{\theta} \), which would not affect generating Clayton Copula. As a result, Clayton Copula is an LT-Archimedean Copula and the algorithms of simulating returns via Clayton Copula are as follows:

Step 1. Generate \( M \) random variables \( Y_m, m = 1, 2, \ldots, M \) which satisfy the function Gamma(\( \frac{1}{\theta}, 1 \)), where the parameter \( \theta \) is estimated using MLE;
Step 2. Simulate independent uniform random variables $y_1, y_2, \ldots, y_n$.

Step 3. Let $u_{im} = \text{Ga}(\text{in}(y_i))$, $i = 1, 2, \ldots, n$, $m = 1, \ldots, M$, and then the joint distribution of $u = (u_1, u_2, \ldots, u_n)$ is the $n$-dimensional Clayton Copula with the parameter $\theta$, and $z_{im} = F_i^{-1}(u_{im})$, $i = 1, 2, \ldots, n$, $m = 1, \ldots, M$;

Step 4. Similar to the last step of simulating via Gaussian Copula or t Copula above, we can get $M$ returns at the time $t + 1$ and the returns residuals’ joint distribution is this $t$ Copula. The returns can be defined by $r_{t+1} = (r_{1m}, r_{2m}, \ldots, r_{nm})^T = (\mu_1 + z_{1m}\sigma_{1,t+1}, \mu_2 + z_{2m}\sigma_{2,t+1}, \ldots, \mu_n + z_{nm}\sigma_{n,t+1})^T$.

3.2. Portfolio risk analysis

(1) VaR and CVaR of the equally weighted portfolio.

Using the algorithms above, we can simulate returns of $n$-dimensional time series at $t + 1$, using the three Copula functions to describe the correlation structure. Now suppose we want to compute the empirical VaR and CVaR of an equally weighted portfolio. Assuming an equally weighted portfolio with $n$ assets, then the weight in each asset is the same, that is, $w = \left[\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right]^T$. Then, using the sample of $M$ returns from simulation described above, we can get $M$ total returns of the portfolio: $\text{Returns} = r_{t+1} \ast w = (r_{1m}, r_{2m}, \ldots, r_{nm})^T \ast w$. As a result, we can get the empirical distribution and thus the VaR and CVaR of the equally weighted portfolio.

(2) Optimal portfolio with minimum risk.

The discussion above focused on estimating the VaR and CVaR of an equally weighted portfolio. However, for commercial banks and individual investors, one of the major concerns is to minimize the risk of the investment portfolio. To address this concern, the following algorithm computes the optimal weights of each asset that minimizes the portfolio risk.

Assume that the weight in individual asset within a portfolio is $w = \left[w_1, w_2, \ldots, w_n\right]^T$, where $0 \leq w_i \leq 1$ and $w_1 + w_2 + \cdots + w_n = 1$. Apparently, there are many weight vectors that satisfy these conditions. For each given investment weight vector $w$, we can get $M$ returns of the portfolio; that is, $\text{Returns} = r_{t+1} \ast w = (r_{1n}, r_{2n}, \ldots, r_{nm})^T \ast w$. We can then compute the VaR of these portfolios at a given confidence level. After getting all VaR values corresponding to all the weight vectors, we can find the minimum VaR. This enables us to get the weight in each asset that minimizes the VaR; that is, $	ext{min VaR}_n = \text{risk}(b_1, b_2, \ldots, b_n)$. In addition, we can compute the CVaR value at this weight vector derived from the minimum risk portfolio.

4. Empirical studies in foreign exchange market

In this section we choose four foreign exchange rates including USD/CNY, EUR/CNY, JPY/CNY and HKD/CNY to demonstrate the application of GARCH–EVT-Copula model. As we all know, China has started implementing the new exchange rate policy since July 21, 2005. The data series used here are from July 25, 2005 to July 25, 2008. (The data source is from http://www.safe.gov.cn/model_safe/index.html). We use the daily logarithmic returns defined as $r_t = \ln P_t - \ln P_{t-1}$, where $P_t$ is the middle price of exchange rate at time $t$. The data series give us 732 observations for each exchange rate.

4.1. Summary statistics

The descriptive statistics of daily logarithmic returns are presented in Table 1. Obviously, all series have heavy tails and they do not follow normal distribution. For example, all series have non-zero skewness, and the skewness of USD/CNY and HKD/CNY are especially large. In addition, all series have kurtosis greater than three.

Take USD/CNY as an example. The ADF test result indicates that there is no unit root and that the series follow a stationary stochastic process. The ARCH LM test shows that USD/CNY is a stationary stochastic process, but with heteroskedasticity. The results on the other three exchange rates are similar. As a result, we can use GARCH (1, 1) model as in Eq. (1) to study the exchange rate movement. The auto-correlogram of USD/CNY return series and the square of the returns are shown in Figs. 1 and 2.

4.2. GARCH–EVT-Copula application

We apply GARCH–EVT-Copula model to study the foreign exchange risk. First, we fit GARCH–EVT model to each of the exchange rate series and get the marginal distribution of residuals. Secondly, we use Multivariate Gaussian, $t$ and Clayton Copulas to describe the correlation of the residuals and obtain the Copula parameters. Finally, we use Monte Carlo
Fig. 1. The auto-correlogram of USD/CNY returns.

Fig. 2. The auto-correlogram of USD/CNY returns’ square.

Table 2
GARCH estimation results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>USD/CNY</th>
<th>EUR/CNY</th>
<th>JPY/CNY</th>
<th>HKD/CNY</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>$0.0000$</td>
<td>$0.0000$</td>
<td>$0.0000$</td>
<td>$0.0000$</td>
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<td>$\alpha_0$</td>
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<td>$0.0345$</td>
<td>$0.0870$</td>
<td>$0.3875$</td>
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<tr>
<td>$\beta_1$</td>
<td>$0.5966$</td>
<td>$0.9557$</td>
<td>$0.8901$</td>
<td>$0.5717$</td>
</tr>
</tbody>
</table>

simulation to generate returns based on the correlation structure specified in each of the Copulas. We then compute the risk of the portfolio and derive the weight of the optimal portfolio at the given confidence level. We describe the details of the procedures below.

(1) GARCH–EVT model.

First, we use GARCH model to fit the returns and eliminate heteroskedasticity. The model parameters are estimated using MLE method and the result of GARCH estimation is given in Table 2.

The Durbin–Watson Statistic show that there is no auto-correlation among the residuals. Thus, the specification is tenable. The validity of the AR equation is verified from the correlogram. Correlations at all lags have been found to be insignificant, implying that the return series are stationary, a necessary condition to use the GARCH model. The GARCH specification also takes care of volatility clustering. However, as argued in the previous sections, this is not enough as the descriptive statistics of the standard residuals clearly show that the conditional distribution has a heavier tail than that of a normal distribution.

As described in earlier sections, we would employ the POT method using GPD for tail estimation and choose the exceedances to be the 10th percentile of the sample. We use the sample MEF plot and Hill plot together to determine an appropriate threshold. Figs. 3–6 show the sample MEF plot and Hill plot of USD/CNY, EUR/CNY, JPY/CNY and HKD/CNY. We derive the thresholds of the four CNY exchange rates in Table 3. The other parameters can be obtained using the method of maximum likelihood.

We plot the graph of GPD fitting USD/CNY exchange rate’s residuals (Fig. 7) and the graph of upper tail estimation (Fig. 8). From Fig. 8, the diagrams indicate that GPD fits the residuals of USD/CNY exchange rate returns perfectly.
Fig. 3. Sample MEF plot and Hill plot for USD/CNY.

Fig. 4. Sample MEF plot and Hill plot for EUR/CNY.

Fig. 5. Sample MEF plot and Hill plot for JPY/CNY.
(2) Correlation description using Copula and the portfolio risk analysis.

After obtaining the parameters of GPD and the residuals $z_{it}, i = 1, 2, 3, 4, t = 1, 2, \ldots, T$, we substitute $z_{it}$ into formula (2) and get the marginal distribution $u_i = f(z_i)$. According to the parameter estimation method in Section 2.1 we can get the parameters of Copulas. That is the correlation matrix $\Sigma$ of Gaussian Copula, $\Sigma$ and degree of freedom $v$ of $t$ Copula, the parameter $\theta$ of Clayton Copula.

Applying the simulation algorithm described in Section 2.2 above, we can simulate the returns at time $t + 1$ based on correlation structure specified in the three types of Copula. With an equally weighted portfolio of four foreign exchanges (USD, EUR, JPY and HKY), we can calculate VaR and CVaR of the portfolio (Table 4). In addition, we can find the optimal portfolio weight that minimizes the portfolio risk. The result is shown in Table 6.

Table 4 indicates that under the same confidence level, VaR (CVaR) calculated from $t$ Copula and Clayton Copula is less than that from Gaussian Copula. That is because $t$ and Clayton Copula consider the tail correlation but the Gaussian Copula does not.

On the other hand, in order to select the optimal copula to model the dependency structure of the data set, we investigate the copula goodness-of-fit approach based on Rosenblatt’s transformation [32]. We also adopt Anderson–Darling test.
Table 4
Portfolio risk under an equally weighted foreign exchange portfolio.

<table>
<thead>
<tr>
<th>Confidence</th>
<th>Risk value</th>
<th>Gaussian Copula</th>
<th>t Copula</th>
<th>Clayton Copula</th>
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<tr>
<td>0.90</td>
<td>VaR</td>
<td>0.0029</td>
<td>0.0024</td>
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<tr>
<td></td>
<td>CVaR</td>
<td>0.0043</td>
<td>0.0037</td>
<td>0.0035</td>
</tr>
<tr>
<td>0.95</td>
<td>VaR</td>
<td>0.0038</td>
<td>0.0032</td>
<td>0.0031</td>
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<tr>
<td></td>
<td>CVaR</td>
<td>0.0053</td>
<td>0.0046</td>
<td>0.0042</td>
</tr>
<tr>
<td>0.99</td>
<td>VaR</td>
<td>0.0061</td>
<td>0.0048</td>
<td>0.0049</td>
</tr>
<tr>
<td></td>
<td>CVaR</td>
<td>0.0081</td>
<td>0.0070</td>
<td>0.0060</td>
</tr>
</tbody>
</table>

Table 5
The result of A–D testing for three class Copula families.

<table>
<thead>
<tr>
<th>Copula function</th>
<th>Rejection rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>0.0579</td>
</tr>
<tr>
<td>Student T</td>
<td>0.0140</td>
</tr>
<tr>
<td>Clayton</td>
<td>0.3772</td>
</tr>
</tbody>
</table>

The result is shown in Table 5. The result shows that the rejection rates of Normal Copula, t Copula and Clayton Copula are 0.0579, 0.0140, and 0.3772, respectively. As we can see, t Copula has the lowest rejection rate. Therefore, based on our analysis, t Copula best describes the dependency structure of foreign exchanges in this portfolio.
Table 6 presents the optimal portfolio weight under minimum portfolio risk, and the VaR (CVaR) under different Copulas and confidence levels. From Table 6 we can see:

- The optimal portfolio weights are similar across different Copulas and confidence levels. The investment concentrates on USD, although USD has consistently depreciated against CNY since China started the exchange rate reform in July 2005. Nevertheless, USD remains to be the most important currency in Chinese national foreign exchange reserves, and will continue to be the major currency of governmental and private investment. This helps explain why USD accounts for the largest weight in the foreign exchange portfolio with minimum risk, during our sample period of July 2005–2008. With higher confidence levels, the weight of USD investment becomes smaller while the weights of the other foreign currencies become larger. This is because with higher confidence levels, investors will be more risk tolerant and therefore willing to take more risk to achieve higher expected return.

- Comparing the risk value calculated by the three Copulas, we find that VaR obtained from $t$ Copula is less than that from Gaussian Copula. This is because although $t$ copula has heavy tail, it is able to capture the tail correlation. The extreme events of asset returns can offset each other due to low correlation, hence the smaller VaR. We reach similar conclusions when computing VaR and CVaR of the equally weighted foreign exchange portfolio.

- When $\alpha = 0.9$, VaR using Clayton Copula is larger than those using $t$ Copula and Gaussian Copula. However, with higher confidence levels, Clayton Copula better captures the correlations of the foreign exchanges and results in VaR that is similar to VaR by $t$ Copula. As a result, under the higher confidence level, we can choose $t$ and Clayton Copulas to calculate the portfolio risk. These findings can help foreign exchange investors better manage the foreign exchange risk.

5. Conclusions

In this paper we introduce the GARCH–EVT-Copula model and use it to measure the risk of a multi-dimensional foreign exchange portfolio. When modeling the correlation structure we not only apply the conventional Gaussian and $t$ Copulas, but also the multivariate Clayton Copula which was rarely used because of the complexity involved in programming.

We find that for an equally weighted foreign exchange portfolio, VaR (CVaR) calculated from $t$ and Clayton Copulas is less than that computed from the Gaussian Copula. This is because $t$ and Clayton Copulas model the tail correlation but the Gaussian Copula does not. On the other hand, when minimizing portfolio risk, optimal portfolio weights are similar across different Copulas and different confidence levels. The optimal investment tends to concentrate in the investment of USD. It is consistent with the fact that USD remains to be the most important currency in Chinese national foreign exchange reserves, and that it will continue to be the major currency in both governmental and private investments. Meanwhile, with higher confidence levels, weight in USD decreases while weight in EUR and JPY increases. In addition, we show that $t$ Copula best describes the dependency structure between foreign exchanges in our portfolio. It also captures portfolio risk better than Clayton and Gaussian Copulas. These findings can help foreign exchange investors better manage the foreign exchange risk.

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