Uniform asymptotic estimates for ruin probabilities of renewal risk models with exponential Lévy process investment returns and dependent claims

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This paper investigates the ruin probabilities of a renewal risk model with stochastic investment returns and dependent claim sizes. The investment is described as a portfolio of one risk-free asset and one risky asset whose price process is an exponential Lévy process. The claim sizes are assumed to follow a one-sided linear process with independent and identically distributed step sizes. When the step-size distribution is heavy tailed, we establish some uniform asymptotic estimates for the ruin probabilities of this renewal risk model. Copyright © 2012 John Wiley & Sons, Ltd.

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1. Introduction

Throughout this paper, let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be a filtered complete probability space on which all stochastic quantities are defined. The filtration \((\mathcal{F}_t)_{t \geq 0}\) is right continuous, and all stochastic processes defined in this paper are adapted to the filtration. Let \(\{X_n, n \geq 1\}\) be a one-sided linear process defined as

\[
X_n = \sum_{j=1}^{n} \varphi_{n-j} \varepsilon_j + \varphi_n \varepsilon_0, \quad n \geq 1,
\]

where \(\{\varphi_n, n \geq 0\}\) and \(\varepsilon_0\) are nonnegative constants with \(\varphi_0 > 0\), and \(\{\varepsilon_j, j \geq 1\}\) are independent, identically distributed, and nonnegative random variables with common distribution function \(F\) satisfying \(\overline{F}(x) = 1 - F(x) > 0\) for all \(x > 0\). In actuarial science, classical risk models are usually based on independence assumptions. However, because of the increasing complexity of insurance and reinsurance products, actuaries have been paying an increasing amount of attention to the modeling of dependent risks. Linear processes are a type of widely used dependence structures in insurance risk theory (e.g., [1–4]). In this paper, we use model (1) to describe insurance claim sizes. The locations of the successive claims, \(\tau_n, n = 1, 2, \ldots\), constitute a renewal counting process

\[
N_t = \sum_{n=1}^{\infty} \mathbb{1}_{[\tau_n \leq t]}, \quad t \geq 0,
\]

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where \( \mathbb{I}_A \) denotes the indicator function of an event \( A \). For later notational convenience, we write \( \tau_0 = 0 \), and to avoid triviality, we assume that \( \tau_1 \) is not degenerate at zero. The total amount of claims up to time \( t \) can be written as

\[
S_t = \sum_{n=1}^{N_t} X_n, \tag{2}
\]

where the summation over an empty set of indices is considered to be 0. Define the insurance risk process as

\[
R_t = x + ct - S_t, \quad t \geq 0,
\]

where \( x > 0 \) is the initial surplus and \( c > 0 \) is the constant premium rate.

Suppose that an insurer can invest his or her surplus into a portfolio consisting of one risk-free asset and one risky asset. The price process of the risk-free asset satisfies

\[
\tilde{X}_0(t) = e^{rt}, \quad t > 0, \quad \tilde{X}_0(0) = 1,
\]

where \( r > 0 \) is the risk-free rate. The price process of the risky asset satisfies

\[
\tilde{X}_1(t) = e^{L(t)}, \quad t > 0, \quad \tilde{X}_1(0) = 1,
\]

where \( \{L(t), t \geq 0\} \) is a Lévy process starting from 0 (for the general theory of Lévy processes, see [5] and [6]).

Suppose that the insurer continuously invests a constant fraction \( \theta \in [0, 1] \) of his or her surplus in the risky asset and keeps the remaining surplus in the risk-free asset. This is the so-called constant investment strategy and commonly used in mathematical finance and actuarial science (e.g., [7–10]).

With this constant investment strategy \( \theta \in [0, 1] \), we define the portfolio investment return process as the solution of the following stochastic differential equation (SDE):

\[
d\tilde{X}_\theta(t) = \tilde{X}_\theta(t-)-d\tilde{L}_\theta(t), \quad t > 0, \quad \tilde{X}_\theta(0) = 1,
\]

where \( d\tilde{L}_\theta(t) = (1-\theta)r \, dt + \theta \, d\tilde{L}(t) \) with \( \tilde{L}(t) \) defined by \( d\tilde{X}_1(t) = \tilde{X}_1(t-) \, d\tilde{L}(t) \). By Proposition 8.22 in [5] or Theorem 37, Chapter II, in [11], the SDE (5) admits the solution

\[
\tilde{X}_\theta(t) = e^{L_\theta(t)}, \quad t > 0, \quad \tilde{X}_\theta(0) = 1,
\]

where

\[
L_\theta(t) = \tilde{L}_\theta(t) - \frac{1}{2}[\tilde{L}_\theta, \tilde{L}_\theta]_t + \sum_{0<s \leq t} \left[ \log \left( 1 + \Delta \tilde{L}_\theta(s) \right) - \Delta \tilde{L}_\theta(s) + \frac{1}{2} \left( \Delta \tilde{L}_\theta(s) \right)^2 \right]
\]

with \( \Delta \tilde{L}_\theta(s) = \tilde{L}_\theta(s) - \tilde{L}_\theta(s-) \) and \([\tilde{L}_\theta, \tilde{L}_\theta]_t\) being the quadratic variation process of \( \tilde{L}_\theta \).

Assume that the processes \( \{\varepsilon_j, j \geq 1\}, \{\tau_n, n \geq 1\}, \) and \( \{L(t), t \geq 0\} \) are mutually independent. Now following the method used by Klüppelberg and Kostadinova [9], we define the integrated risk process as the result of the insurance business and the net gains of the investment, that is, the solution to the SDE:

\[
dU_\theta(t) = c \, dt - dS_t + U_\theta(t- \, d\tilde{L}_\theta(t), \quad t > 0, \quad U_\theta(0) = x. \tag{7}
\]

Notice that the independence between \( \{S_t, t \geq 0\} \) and \( \{L_\theta(t), t \geq 0\} \) implies that the two processes have no common jumps almost surely. Hence, following the proof of Lemma 2.2 of Klüppelberg and Kostadinova [9], we can verify that the solution to the SDE (7) is

\[
U_\theta(t) = e^{L_\theta(t)} \left( x + \int_0^t e^{-L_\theta(v)}(c \, dv - dS_v) \right), \quad t > 0, \quad U_\theta(0) = x. \tag{8}
\]

In view of (1), we remark that the solution \( U_\theta \) is a generalization of the surplus processes considered by Kalashnikov and Norberg [12], Paulsen [13], Yuen et al. [14, 15], and Cai [16].

As usual, define the ruin time of the renewal risk model (8) as

\[
\Gamma(x) = \inf\{t > 0 : U_\theta(t) < 0 \mid U_\theta(0) = x\}
\]
with \( \inf \emptyset = \infty \) by convention. Then, the probability of ruin up to a finite time \( t \geq 0 \) is

\[
\Psi_{\theta}(x, t) = \mathbb{P}(\Gamma(x) \leq t),
\]

and the probability of ultimate ruin is

\[
\Psi_{\theta}(x) = \mathbb{P}(\Gamma(x) < \infty).
\]

In this paper, we investigate the ruin probabilities of the renewal risk model (8) with the constant investment strategy and the claim-size model (1). When the common distribution function of the step sizes in (1) is heavy tailed, we establish some asymptotic formulas that hold uniformly over certain time regions.

The rest of this paper consists of three sections. Section 2 introduces some notations and states the main results of the paper, Section 3 provides some important lemmas, and Section 4 proves the main results.

### 2. Notations and main results

Throughout this paper, let \((\gamma, \sigma^2, \nu)\) be the Lévy triplet of \(\{L(t), t \geq 0\}\), where \(\gamma \in \mathbb{R}\), \(\sigma \geq 0\) are two constants and \(\nu\) is a Lévy measure satisfying \(\nu(\{0\}) = 0\) and \(\int_{-\infty}^{\infty} \min\{x^2, 1\} \nu(dx) < \infty\). By Lemma 2.5 in [8], \(\{L_{\theta}(t), t \geq 0\}\) in (6) is also a Lévy process with Lévy triplet \((\gamma_{\theta}, \sigma_{\theta}^2, \nu_{\theta})\) given by

\[
\gamma_{\theta} = \gamma + (1 - \theta) \left( r + \frac{\sigma^2}{2} \right),
\]

\[
\sigma_{\theta}^2 = \sigma^2; \quad \nu_{\theta}(A) = \nu \left( \{ x \in \mathbb{R} : \log(1 + \theta (e^x - 1)) \in A \} \right)
\]

for any Borel set \(A \subset \mathbb{R}\). Define the Laplace exponent of \(\{L_{\theta}(t), t \geq 0\}\) as

\[
\psi_{\theta}(v) = \log \mathbb{E}[e^{-vL_{\theta}(1)}], \quad v \in \mathbb{R},
\]

If \(\psi_{\theta}(v)\) is finite, then

\[
\psi_{\theta}(v) = -\gamma_{\theta} v + \frac{\sigma_{\theta}^2}{2} v^2 + \int_{-\infty}^{\infty} \left( e^{-vx} - 1 + v x \mathbb{1}_{|x| \leq 1} \right) \nu_{\theta}(dx),
\]

and

\[
\mathbb{E}e^{-vL_{\theta}(t)} = e^{t\psi_{\theta}(v)} < \infty, \quad t \geq 0
\]

[6, Theorem 25.17]. For every fixed \(\theta \in (0, 1)\), Lemma 4.1 and its proof in [9] show that \(\psi_{\theta}(v)\) is finite for any \(v \geq 0\); and if \(0 < \mathbb{E}L(1) < \infty\) and either \(\sigma > 0\) or \(\nu((-\infty, 0)) > 0\), then there exists a unique \(\kappa_{\theta} > 0\) such that \(\psi_{\theta}(\kappa_{\theta}) = 0\). In addition, Lemma A.1 in [9] asserts that if \(0 < \mathbb{E}L(1) < \infty\), then \(0 < \mathbb{E}L_{\theta}(1) < \infty\). Thus, for every fixed \(\theta \in (0, 1)\), if \(0 < \mathbb{E}L(1) < \infty\) and either \(\sigma > 0\) or \(\nu((-\infty, 0)) > 0\), then by the convexity of \(\psi_{\theta}(v)\) with \(\psi_{\theta}(0) = 0\) and \(\psi_{\theta}(0) = -\mathbb{E}L_{\theta}(1) < 0\), we obtain

\[
\psi_{\theta}(v) < 0 \quad \text{for any } 0 < v < \kappa_{\theta}.
\]

Define the renewal function of the renewal counting process \(\{N_t, t \geq 0\}\) as

\[
\lambda_t = \mathbb{E}N_t = \sum_{n=1}^{\infty} \mathbb{P}(\tau_n \leq t).
\]

In particular, if \(\{N_t, t \geq 0\}\) is a homogeneous Poisson process with intensity \(\lambda > 0\), then \(\lambda_t = \lambda t\) for all \(t \geq 0\). Other examples of the renewal counting process \(\{N_t, t \geq 0\}\) allowing explicit forms of the renewal function \(\lambda_t\) can be found in [17].
Denote $\Lambda = \{t : 0 < \lambda_I < \infty\}$. With $\underline{t} = \inf\{t : \lambda_I > 0\} = \inf\{t : P(t_1 \leq t) > 0\}$, it is clear that

$$\Lambda = \begin{cases} [\underline{t}, \infty], & \text{if } P(t_1 = t) > 0, \\ (\underline{t}, \infty], & \text{if } P(t_1 = t) = 0. \end{cases}$$

For notational convenience, let $\Lambda_{T_0} = \Lambda \cap [T_0, T]$ and $\Lambda_{T_0} = \Lambda \cap [T_0, \infty]$ for arbitrarily fixed $t < T_0 < T < \infty$.

Hereafter, all limit relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$ satisfying

$$0 \leq l^- \leq \liminf_{x \to \infty} \frac{a(x)}{b(x)} \leq \limsup_{x \to \infty} \frac{a(x)}{b(x)} \leq l^+ \leq \infty,$$

we write $a(x) = o(b(x))$ if $l^+ = 0$; $a(x) = O(b(x))$ if $l^+ < \infty$; $a(x) \asymp b(x)$ if $0 < l^- \leq l^+ < \infty$; $a(x) \preceq b(x)$ if $l^+ \leq 1$; and $a(x) \sim b(x)$ if $l^+ = l^- = 1$. For two positive bivariate functions $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ satisfying

$$0 \leq L^- \leq \liminf_{x \to \infty} \inf_{t \in E} \frac{a(x, t)}{b(x, t)} \leq \limsup_{x \to \infty} \sup_{t \in E} \frac{a(x, t)}{b(x, t)} \leq L^+ \leq \infty, \quad E \neq \emptyset,$$

we say the relation $a(x, t) \asymp b(x, t)$ holds uniformly for all $t \in E$ if $0 < L^- \leq L^+ < \infty$; the relation $a(x, t) \preceq b(x, t)$ holds uniformly for all $t \in E$ if $L^+ \leq 1$; the relation $a(x, t) \gtrsim b(x, t)$ holds uniformly for all $t \in E$ if $L^- \geq 1$; and the relation $a(x, t) \sim b(x, t)$ holds uniformly for all $t \in E$ if $L^- = L^+ = 1$.

Now we recall several classes of heavy-tailed distribution functions. We say a distribution function $F$ has a regularly varying tail with tail index $-\alpha < 0$, denoted by $F \in \mathcal{R}_{-\alpha}$, if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ and

$$\overline{F}(xy) \sim y^{-\alpha}\overline{F}(x) \quad \text{for any } y > 0.$$ 

We say a distribution function $F$ belongs to the class $\mathcal{D}$ (has dominated variation) if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ and

$$\overline{F}(xy) = O(\overline{F}(x)) \quad \text{for any } 0 < y \leq 1.$$ 

We say a distribution function $F$ belongs to the subexponential class $\mathcal{S}$ if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ and

$$\overline{F}^{*n}(x) \sim n\overline{F}(x) \quad \text{for any integer } n \geq 2,$$

where $\overline{F}^{*n}$ denotes the $n$-fold convolution of $F$. We say a distribution function $F$ belongs to the class $\mathcal{L}$ (is long tailed) if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ and

$$\overline{F}(x + l) \sim \overline{F}(x) \quad \text{for any } l > 0.$$ 

It is well known that

$$\mathcal{R}_{-\alpha} \subset \mathcal{D} \cap \mathcal{S} \subset \mathcal{L}.$$ 

For more details about heavy-tailed distributions and their applications to insurance and finance, see [18–20].

Now we follow Tang and Tsitsiashvili [21] to introduce two significant indices of a general random variable. Let $X$ be a random variable concentrated on $(-\infty, \infty)$ with a distribution function $F$. For any $y > 0$, we set

$$\overline{F}_*(y) = \liminf_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \quad \text{and} \quad \underline{F}_*(y) = \limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)}$$

and then define

$$\underline{J}_F^+ = J^+(X) = \inf \left\{ -\frac{\log\overline{F}_*(y)}{\log y} : y > 1 \right\} = -\lim_{y \to \infty} \frac{\log\overline{F}_*(y)}{\log y},$$

$$\overline{J}_F^- = J^-(X) = \sup \left\{ -\frac{\log\underline{F}_*(y)}{\log y} : y > 1 \right\} = -\lim_{y \to \infty} \frac{\log\underline{F}_*(y)}{\log y}.$$ 

Here, $\underline{J}_F^+$ and $\overline{J}_F^-$ are called the upper and lower Matuszewski indices of the nonnegative and nondecreasing function $f(x) = (\overline{F}(x))^{-1}, x \geq 0$ [18, Chapter 2.1]. Especially, if $F \in \mathcal{D}$, then $\underline{J}_F^+ < \infty$; and if $F \in \mathcal{R}_{-\alpha}$ with $\alpha > 0$, then
\( \mathcal{F}^- = \mathcal{F}^+ = \alpha \). For a distribution function \( F \in D \) and arbitrarily fixed \( p > \mathcal{F}^+ \), by Proposition 2.2.1 in [18], there exist positive constants \( C_p \) and \( D_p \) such that

\[
\frac{\mathcal{F}(y)}{\mathcal{F}(x)} \leq C_p \left( \frac{x}{y} \right)^p
\]  

(13)

holds uniformly for all \( x \geq y \geq D_p \). Fixing the variable \( y \) in (13) leads to

\[
x^{-p} = o(\mathcal{F}(x)) \quad \text{for any } p > \mathcal{F}^+.
\]  

(14)

In this paper, we only consider the case \( \theta \in [0, 1] \) because insurers are not allowed to invest all their wealth into risky assets. Now we are ready to state the main results.

**Theorem 1**

Consider the renewal risk model introduced earlier with \( \theta \in [0, 1] \). If the coefficients \( \{\varphi_n, n \geq 0\} \) satisfy \( \varphi_0 > 0 \) and \( \sup_{n \geq 0} \varphi_n < \infty \), and the common distribution function \( F \) of the step sizes \( \{\varepsilon_j, j \geq 1\} \) belongs to the class \( D \cap S \), then for arbitrarily fixed \( t < T_0 \leq T < \infty \), it holds uniformly for all \( t \in \Lambda_{T_0, T} \) that

\[
\Psi_\theta(x, t) \sim \int_{0-}^t P(\sum_{k=0}^\infty \varphi_k e^{-L_\theta(t+\tau_k)} 1_{[t+\tau_k \leq t]} > x) \, d\lambda_s.
\]  

(15)

**Theorem 2**

Consider the renewal risk model introduced earlier with \( \theta \in [0, 1] \) and \( \{\varphi_n, n \geq 0\} \) satisfying \( \varphi_0 > 0 \) and \( \sup_{n \geq 0} \varphi_n < \infty \). Assume that \( 0 < \mathbb{E}L(1) < \infty \) and either \( \sigma > 0 \) or \( \nu((-\infty, 0)) > 0 \), and for every fixed \( \theta \in (0, 1) \), let \( \kappa_\theta > 0 \) be the unique value satisfying \( \psi_\theta(\kappa_\theta) = 0 \). For the case \( \theta = 0 \), if the common distribution function \( F \) of \( \{\varepsilon_j, j \geq 1\} \) belongs to the class \( \mathcal{R}_{\alpha} \) for some \( 0 < \alpha < \infty \), then for arbitrarily fixed \( t < T_0 < \infty \), the relation

\[
\Psi_\theta(x, t) \sim \mathcal{F}(x) \int_{0-}^t E(\sum_{k=0}^\infty \varphi_k e^{-L_\theta(t+\tau_k)} 1_{[t+\tau_k \leq t]} )^\alpha \, d\lambda_s
\]  

(16)

holds uniformly for all \( t \in \Lambda_{T_0} \); for the case \( \theta \in (0, 1) \), if \( F \in \mathcal{R}_{\alpha} \) for some \( 0 < \alpha < \kappa_\theta \), then relation (16) also holds uniformly for all \( t \in \Lambda_{T_0} \).

**Corollary 1**

Under the conditions of Theorem 2, we have

\[
\Psi_\theta(x) \sim \mathcal{F}(x) \frac{E e^{\tau_1 \psi_\theta(\alpha)}}{1 - e^{\tau_1 \psi_\theta(\alpha)}} \left( \sum_{k=0}^\infty \varphi_k e^{-L_\theta(\tau_k)} \right)^\alpha.
\]  

(17)

**Remark 1**

From relation (16), we see that in finite time, the insurance risk always dominates the financial risk because the tail probability of \( \{\varepsilon_j, j \geq 1\} \) determines the exact decay rate of the ruin probabilities. The same conclusion can be drawn for the case of infinite time provided that \( \alpha < \kappa_\theta \) (see relation (17) for details). However, for the special case that the claim sizes are independent and identically distributed, Theorem 4.4 in [9] shows that the financial risk finally dominates the insurance risk when the common distribution function of the claim sizes has a regularly varying tail with tail index \( -\alpha < 0 \) satisfying \( \alpha > \kappa_\theta \).

**Remark 2**

The renewal risk model (8) includes the special case that the claim sizes are independent and identically distributed. This can be seen from (1) by letting \( \varphi_0 = 1 \) and \( \varphi_n = 0 \) for all \( n \geq 1 \). For the special renewal risk model, many researchers investigated the asymptotic behaviors of its ruin probabilities. For example, Paulsen [13] obtained an asymptotic formula of the ultimate ruin probability similar to (17) for the special case that \( \{N_t, t \geq 0\} \) is a homogeneous Poisson process and that \( \{L(t), t \geq 0\} \) is a Brownian motion with positive drift. Tang et al. [22] extended this work to the case that \( \{N_t, t \geq 0\} \) is a renewal counting process and that \( \{L(t), t \geq 0\} \) is a Lévy process, and established a uniform asymptotic formula of ruin probabilities similar to (16). Heyde and Wang [10] obtained a result for the finite-time ruin probability similar to (15) but with a fixed-time horizon provided that \( \theta \in (0, 1) \) and \( \{N_t, t \geq 0\} \) is a homogeneous Poisson process. For the case \( \theta = 0 \), Wang [23] derived some asymptotic formulas of the finite-time ruin probability similar to (15) but also with a fixed-time horizon when the premiums follow a general process and the claims arrive according to an arbitrary
counting process. We point out that all the studies mentioned earlier are based on the assumption that the claim sizes are independent and identically distributed.

**Remark 3**
Model (1) includes many commonly used linear processes such as the claim processes considered in Chapter 13 in [3] and [4], and the annual gain process considered in [2]. In addition, the coefficient condition in Theorems 1 and 2 can be easily satisfied by many linear processes with expression (1). As an example, we consider the case that \( \{X_n, n \geq 1\} \) is described as

\[
X_n = \phi_1 X_{n-1} + \varepsilon_n + \phi_2 \varepsilon_{n-1}, \quad 0 \leq \phi_1, \phi_2 \leq 1, \quad n \geq 1,
\]

where \( X_0, \varepsilon_0 \geq 0 \) are constants and \( \{\varepsilon_n, n \geq 1\} \) are independent, identically distributed, and nonnegative random variables. Notice that (18) can be rewritten as

\[
X_n = \varepsilon_n + (\phi_1 + \phi_2) \sum_{j=1}^{n-1} \phi_1^{n-j-1} \varepsilon_j + \phi_1^{n-1} \phi_2 \varepsilon_0 + \phi_1^n X_0.
\]

Trivially, \( \{X_n, n \geq 1\} \) is of form (1), and the coefficients in (19) are bounded from above by the constant \( \max\{1, \phi_1 + \phi_2\} \).

**Remark 4**
According to Corollary 1, we can use relation (17) to estimate the ultimate ruin probability. To do so, we need the explicit form of the Laplace exponent \( \psi_\theta(\cdot) \) of \( \{L_0(t), t \geq 0\} \). For some commonly used risky investment return processes \( \{e^{L(t)}, t \geq 0\} \), the explicit forms of \( \psi_\theta(\cdot) \) can be obtained from relations (9)–(11). We refer the reader to [9] and [10] for some important examples. Here, we give another example. Let \( \{L(t), t \geq 0\} \) be a generalized tempered stable process with Lévy triplet \((\mu, 0, v)\) satisfying \( \mu \in \mathbb{R} \) and

\[
\nu(x) = \frac{c_-}{|x|^{1+a_-} e^{-b_- |x|} 1_{[x<0]}} + \frac{c_+}{|x|^{1+a_+} e^{-b_+ x} 1_{[x>0]}}
\]

with \( a_- < 2, a_+ < 2, b_- > 0, b_+ > 0, c_- > 0, \) and \( c_+ > 0 \). This process was introduced by Koponen [24] and is used in financial modeling. The Laplace exponent of this process is

\[
\psi(s) = -\mu s + \Gamma(-a_+) \theta_+^{a_+} c_+ \left\{ \left( 1 + \frac{s}{b_+} \right)^{a_+} - 1 - \frac{s a_+}{b_+} \right\}
\]

\[
+ \Gamma(-a_-) \theta_-^{a_-} c_- \left\{ \left( 1 - \frac{s}{b_-} \right)^{a_-} - 1 + \frac{s a_-}{b_-} \right\}
\]

when \( a_+ \neq 0 \) and \( a_- \neq 1 \). For the case \( a_+ = 0 \) or \( a_- = 1 \), see Proposition 4.2 in [5] for the explicit expression of \( \psi(s) \). By relations (9)–(11), we have

\[
\psi_\theta(\alpha) = -\gamma_0 \alpha + \int_{-\infty}^{\infty} (e^{-\alpha x} - 1 + \alpha x) \nu_\theta(dx) = -(1 - \theta) \alpha - \theta \alpha \mu
\]

\[
+ \int_{x>\log(1-\theta)} [(1 + \theta (e^x - 1))^{-\alpha} - 1 + \alpha \log(1 + \theta (e^x - 1))] \nu(dx)
\]

with \( \nu \) specified in (20).

3. Lemmas

Hereafter, \( C \) always represents a positive constant without relation to \( x \) and \( t \) and may vary in different places. To prove the main results of this paper, we need a series of preliminaries. We first recall the following five important results. The following lemma will be used in the proof of Lemma 6.

**Lemma 1**
Let \( \varepsilon \) and \( \Theta \) be two independent and nonnegative random variables with \( \varepsilon \) distributed by \( F \). If \( F \in \mathcal{D} \), \( \Theta \) is not degenerate at zero and \( \mathbb{E} \Theta^p < \infty \) for some \( p > \mathfrak{J}^+_F \), then \( \mathbb{P}(\varepsilon \Theta > x) \asymp F(x) \).
The following lemma will be used in the proofs of Lemmas 7 and 9.

**Lemma 2**
Let \( \varepsilon \) and \( \Theta \) be two independent and nonnegative random variables with \( \varepsilon \) distributed by \( F \). If \( F \in \mathcal{D} \), then for arbitrarily fixed \( \delta > 0 \) and \( p > \mathbb{E}^{+} \), there exists a positive constant \( C \) without relation to \( \Theta \) and \( \delta \) such that for all large \( x \),

\[
P(\varepsilon \Theta > \delta x \mid \Theta) \leq C F(x) [\delta^{p} \Theta^{p} + \mathbb{I}_{[\Theta < \delta]}].
\]

**Proof**
See Lemma 3.2 in [10].

The following lemma will be used in the proofs of Lemmas 6 and 11.

**Lemma 3**
Let \( \varepsilon \) and \( \Theta \) be two independent and nonnegative random variables with \( \varepsilon \) distributed by \( F \). If \( F \in \mathcal{D} \) with \( \mathbb{E}^{+} > 0 \), then for arbitrarily fixed \( \delta > 0 \) and \( 0 < p_1 < \mathbb{E}^{+} < p_2 < \infty \), there exists a positive constant \( C \) without relation to \( \Theta \) and \( \delta \) such that for all large \( x \),

\[
P(\varepsilon \Theta > \delta x \mid \Theta) \leq C F(x) (\delta^{-p_1} \Theta^{p_1} + \delta^{-p_2} \Theta^{p_2}).
\]

**Proof**
See Lemma 4.1.5 in [25].

The following lemma will play a crucial role in the proof of Lemma 7.

**Lemma 4**
Let \( \varepsilon, j \leq j \leq k \) be \( k \) independent and identically distributed random variables with common distribution function \( F \in \mathcal{S} \). Then, for arbitrarily fixed \( 0 < a < b < \infty \), the relation

\[
P\left( \sum_{j=1}^{k} \varepsilon \Theta_{j} > x \right) \sim \sum_{j=1}^{k} F\left( \frac{x}{\theta_{j}} \right)
\]

holds uniformly for all \( (c_1, \ldots, c_k) \in [a, b] \times \cdots \times [a, b] \).

**Proof**
See Proposition 5.1 in [26].

The following lemma will be used in the proof of Lemma 11.

**Lemma 5**
Let \( \varepsilon, j \leq j \leq k \) be \( k \) independent and identically distributed random variables with common distribution function \( F \), and let \( \Theta, j \leq j \leq k \) be other \( k \) nonnegative and not-degenerate-at-zero random variables independent of \( \varepsilon, j \leq j \leq k \). If \( F \in \mathcal{R}_{-\alpha} \) for some \( 0 < \alpha < \infty \) and \( \mathbb{E} \Theta_{j}^{p} < \infty \) for some \( p > \alpha \) and all \( 1 \leq j \leq k \), then

\[
P\left( \sum_{j=1}^{k} \varepsilon \Theta_{j} > x \right) \sim F(x) \sum_{j=1}^{k} \mathbb{E} \Theta_{j}^{p}.
\]

**Proof**
See Theorem 2.2 in [25].

We next establish some lemmas that are crucial for the proofs of the main results. To do so, we first introduce the discounted net loss process \( \{ V_{\theta}(t), t \geq 0 \} \) defined as

\[
V_{\theta}(t) = x - e^{-L_{\theta}(t)} U_{\theta}(t) = \int_{0}^{t} e^{-L_{\theta}(v)} (dS_{v} - cdv), \quad t \geq 0,
\]

(21)
and express \( \{V_\theta(t), t \geq 0\} \) as randomly weighted sums. Actually, by substituting (1) and (2) into (21), we can get that for any \( t \geq 0 \),

\[
V_\theta(t) = \sum_{n=1}^{\infty} X_n e^{-\theta(t_n)} \mathbb{I}_{[t_n \leq t]} - c \int_0^t e^{-\theta(v)} \, dv
\]

or

\[
= \sum_{n=1}^{\infty} e^{-\theta(t_n)} \mathbb{I}_{[t_n \leq t]} \sum_{j=1}^n \varphi_{n-j} e^j + \sum_{n=1}^{\infty} e^{-\theta(t_n)} \mathbb{I}_{[t_n \leq t]} \varphi_n \epsilon_0 - c \int_0^t e^{-\theta(v)} \, dv
\]

or

\[
= \sum_{j=0}^{\infty} \epsilon_j \sum_{n=j}^{\infty} \varphi_{n-j} e^{-\theta(t_n)} \mathbb{I}_{[t_n \leq t]} + \epsilon_0 \sum_{n=1}^{\infty} \varphi_n e^{-\theta(t_n)} \mathbb{I}_{[t_n \leq t]} - c \int_0^t e^{-\theta(v)} \, dv
\]

or

\[
= \sum_{j=0}^{\infty} \epsilon_j W_j(t) - c Z_t,
\]

where

\[
Z_t = \int_0^t e^{-\theta(v)} \, dv
\]

and

\[
W_0(t) = \sum_{n=1}^{\infty} \varphi_n e^{-\theta(t_n)} \mathbb{I}_{[t_n \leq t]}, \quad W_j(t) = \sum_{n=j}^{\infty} \varphi_{n-j} e^{-\theta(t_n)} \mathbb{I}_{[t_n \leq t]}, \quad j \geq 1.
\]

For later use, denote the infinite-time versions of \( \{W_j(t), j \geq 0\} \) by

\[
W_0 = \sum_{n=1}^{\infty} \varphi_n e^{-\theta(t_n)}, \quad W_j = \sum_{n=j}^{\infty} \varphi_{n-j} e^{-\theta(t_n)}, \quad j \geq 1.
\]

We point out that \( W_j \) is finite almost surely for any \( j \geq 0 \) (see Lemma 11 for details). It is clear that \( W_j(t) \) in (24) increases to \( W_j \) almost surely as \( t \to \infty \) for any \( j \geq 0 \).

For the sake of convenience, we introduce the following three notations, which are frequently used in this section. For any nonnegative numbers \( \{t_n, n \geq 0\} \), denote

\[
w_{j,k} = \sum_{n=j}^{k} \varphi_{n-j} e^{-\theta(t_n)}, \quad j = 0, 1, \ldots, k, \quad k = 0, 1, \ldots.
\]

It should be noted that given \( \tilde{v}_{j,k} = (t_j, \ldots, t_k) \), \( w_{j,k} \) is a nonnegative random variable. For any \( t > 0 \), denote

\[
w^-_t = \varphi_0 e^{-\sup_{s \in [0,t]} \theta(s)} \quad \text{and} \quad w^+_t = \sup_{n \geq 0} (\varphi_n) e^{-\inf_{s \in [0,t]} \theta(s)}.
\]

According to (34), \( w^-_t \leq w^+_t < \infty \) almost surely for any \( t > 0 \). For arbitrarily fixed \( t > 0 \) and integers \( j, k \) such that \( 0 \leq j \leq k \), the two-sided inequality

\[
w^-_t \leq w_{j,k} \leq (k+1)w^+_t
\]

is valid for all \( \tilde{v}_{j,k} \in [0,t] \times \cdots \times [0,t] = [0,t]^{k-j+1} \) and will be frequently used in this section.

The following two lemmas are essential in establishing Lemma 8. To prove them, we need two properties of heavy-tailed distribution classes. For a distribution function \( F \in \mathcal{R}_- \) for some \( 0 < \alpha < \infty \) and arbitrarily fixed \( 0 < a < b < \infty \), by Theorem 1.5.2 in [18], we can obtain

\[
\lim_{x \to \infty} \sup_{y \in [a,b]} \left| \frac{\tilde{F}(xy)}{F(x)} - y^{-\alpha} \right| = 0.
\]

For a distribution function \( F \in \mathcal{L} \) and any \( l > 0 \), we can get that the relation

\[
\frac{F(xy)}{F(x)} \sim F(xy + ly) \quad \text{as} \quad y \to \infty.
\]
holds uniformly for all $y \in [a, b]$ for arbitrarily fixed $0 < a < b < \infty$. In fact,

$$1 \geq \limsup_{x \to \infty} \sup_{y \in [a, b]} \frac{F(xy + ly)}{F(xy)} \geq \liminf_{x \to \infty} \inf_{y \in [a, b]} \frac{F(xy + lb)}{F(xy)} = \liminf_{x \to \infty} \frac{F(x + lb)}{F(x)} = 1.$$

**Lemma 6**

Under the conditions of Theorem 1 with \{w_{j,k}, 0 \leq j \leq k\} defined in (26), for arbitrarily fixed $T > 0$ and integers $j, k$ such that $1 \leq j \leq k$, the relation

$$\Pr(\varepsilon_j w_{j,k} > x) \asymp \frac{\varepsilon_j w_{j,k}}{\varepsilon_j w_{j,k} + 1}$$

holds uniformly for all $\tilde{t}_{j,k} \in [0, T]^{k-j+1}$. Specifically, if $F \in \mathcal{R}_\alpha$ for some $0 < \alpha < \infty$, then it holds uniformly for all $\tilde{t}_{j,k} \in [0, T]^{k-j+1}$ that

$$\Pr(\varepsilon_j w_{j,k} > x) \sim \frac{\varepsilon_j w_{j,k}}{\varepsilon_j w_{j,k} + 1}.$$

**Proof**

We first prove the lower bound version of relation (31). For every fixed $T > 0$, because $\phi_0 > 0$, we can take $\delta \in (0, 1)$ such that $\Pr(w_T > \delta) = \Pr(\phi_0 e^{-\sup_{s \in [0, T]} L_\theta(s)} > \delta) > 0$. By inequalities (13) and (28), we have, for all $\tilde{t}_{j,k} \in [0, T]^{k-j+1}$ and all large $x$,

$$\Pr(\varepsilon_j w_{j,k} > x) \geq \Pr(\varepsilon_j w_T > x) \equiv \Pr(\varepsilon_j \delta > x) \geq C \Pr(\varepsilon_j w_T > \delta).$$

(33)

We next prove the upper bound version of relation (31). From the lemma of Willekens [27], we know that for all $T > 0$ and all $u > u_0 > 0$,

$$\Pr(-\inf_{s \in [0, T]} L_\theta(s) > u) \leq \Pr(-\sup_{s \in [0, T]} L_\theta(s) > -u_0).$$

For the fixed $T > 0$, take large $u_0 > 0$ such that $\Pr(-\sup_{s \in [0, T]} L_\theta(s) > -u_0) > 0$. Then, for any $p > 0$ and all $u > u_0 > 0$,

$$\Pr(\varepsilon_j w_{j,k} > x) \geq \Pr(\varepsilon_j w_T > x) \geq \Pr(\varepsilon_j \delta > x) \geq C \Pr(\varepsilon_j w_T > \delta).$$

(34)

Thus, by (28) and Lemma 1, we have, for all $\tilde{t}_{j,k} \in [0, T]^{k-j+1}$ and all large $x$,

$$\Pr(\varepsilon_j w_{j,k} > x) \leq \Pr(\varepsilon_j (k+1)w_T^+ > x) \leq C \Pr(\varepsilon_j w_T^+ > x).$$

(35)

By (33) and (35), relation (31) holds uniformly for all $\tilde{t}_{j,k} \in [0, T]^{k-j+1}$.

We next prove relation (32). For arbitrarily fixed $0 < b < B < \infty$, we have

$$\Pr(\varepsilon_j w_{j,k} > x) = \Pr(\varepsilon_j w_{j,k} > x, b \leq w_{j,k} \leq B) + \Pr(\varepsilon_j w_{j,k} > x, w_{j,k} < b)$$

$$+ \Pr(\varepsilon_j w_{j,k} > x, w_{j,k} > B)$$

$$: = I_1(x, b, B) + I_2(x, b) + I_3(x, B).$$

For $I_1(x, b, B)$, by relation (29), it holds uniformly for all $\tilde{t}_{j,k} \in [0, T]^{k-j+1}$ that

$$I_1(x, b, B) \sim \Pr(\varepsilon_j w_{j,k} > x) \leq C \Pr(\varepsilon_j w_{j,k} > x) \Pr\left\{w_{j,k}^{\alpha} [w_{j,k} < b] + w_{j,k}^{\alpha} [w_{j,k} > b]\right\}.$$
Likewise, for $I_3(x, B)$, we can get that for all large $x$,

$$I_3(x, B) \leq C \mathbb{F}(x) \mathbb{E} \left\{ w^{p_1}_{j,k} \mathbb{I}_{[w_{j,k} > B]} + w^{p_2}_{j,k} \mathbb{I}_{[w_{j,k} > B]} \right\}.$$ 

Thus, to prove relation (32), it suffices to prove that for any $p > 0$,

$$\limsup_{b \to 0} \limsup_{B \to \infty} \sup_{i_j,k \in [0,T]^{k-j+1}} \mathbb{E} \left\{ w^{p}_{j,k} \left( \mathbb{I}_{[w_{j,k} < b]} + \mathbb{I}_{[w_{j,k} > B]} \right) \right\} = 0. \quad (36)$$

In fact, from (28), we can get that for any $p > 0$,

$$\mathbb{E} \left\{ w^{p}_{j,k} \left( \mathbb{I}_{[w_{j,k} < b]} + \mathbb{I}_{[w_{j,k} > B]} \right) \right\} \leq \mathbb{E} \left\{ (k+1)^p \mathbb{I}_{[w_{1} < b]} + \mathbb{I}_{[(k+1)w_{1} > B]} \right\}. $$

Because $\mathbb{P}(w_{1} < b) = \mathbb{P}(\varphi_0 e^{-\sup_{x \in [0,T]} L_\phi(x)} < b) \to 0$ as $b \to 0$ and (34) holds for any $p > 0$, we see that the last expectation previously mentioned tends to 0 as $b \to 0$ and $B \to \infty$. On the other hand, from (26), we can derive that for all $i_{j,k} \in [0, T]^{k-j+1}$,

$$\mathbb{E} w^{\alpha}_{j,k} \geq \varphi_0^\alpha \mathbb{E} \left[ e^{-\alpha L_\phi(i_j)} \right] = \varphi_0^\alpha e^{\psi_0(\alpha)} \geq \varphi_0^\alpha \min \left\{ e^{T \psi_0(\alpha)}, 1 \right\} > 0. $$

Thus, relation (36) holds, and we conclude the proof. \qed

**Lemma 7**

Under the conditions of Theorem 1 with $Z_t$ and $\{w_{j,k}, 0 \leq j \leq k\}$ defined in (23) and (26), respectively, for arbitrarily fixed $T > 0$, $c_0 \geq 0$ and integer $k \geq 1$, the relation

$$\mathbb{P} \left( \sum_{j=0}^{k} \epsilon_j w_{j,k} - c_0 Z_t > x + \epsilon_0 \varphi_0 \right) \sim \sum_{j=1}^{k} \mathbb{P} (\epsilon_j w_{j,k} > x) \quad (37)$$

holds uniformly for all $\bar{t}_{0,k} = (t_0, t_1, \ldots, t_k) \in [0, T]^{k+1}$ and all $t \in [0, T]$.

**Proof**

We first prove the upper bound version of relation (37). For arbitrarily fixed $0 < b < 1 < B < \infty$, denote

$$A_1 = \bigcap_{i=0}^{k} \{b \leq w_{i,k} \leq B\}, \quad A_2 = \bigcup_{i=0}^{k} \{w_{i,k} > B\}, \quad A_3 = \bigcup_{i=0}^{k} \{w_{i,k} < b\}. \quad (38)$$

Recall that $\epsilon_0$ is a nonnegative constant and $Z_t \geq 0$. It is clear that

$$\mathbb{P} \left( \sum_{j=0}^{k} \epsilon_j w_{j,k} - c_0 Z_t > x + \epsilon_0 \varphi_0 \right) \leq \sum_{m=1}^{3} \mathbb{P} \left( \sum_{j=0}^{k} \epsilon_j w_{j,k} > x, A_m \right). \quad (39)$$

By Lemma 4 and relation (30), it holds uniformly for all $\bar{t}_{0,k} \in [0, T]^{k+1}$ that

$$\mathbb{P} \left( \sum_{j=0}^{k} \epsilon_j w_{j,k} > x, A_1 \right) \leq \mathbb{P} \left( \sum_{j=0}^{k} \epsilon_j w_{j,k} > x - \epsilon_0 B, A_1 \right) \sim \sum_{j=1}^{k} \mathbb{P} (\epsilon_j w_{j,k} > x - \epsilon_0 B, A_1) \sim \sum_{j=1}^{k} \mathbb{P} (\epsilon_j w_{j,k} > x, A_1) \leq \sum_{j=1}^{k} \mathbb{P} (\epsilon_j w_{j,k} > x). \quad (40)$$
Take $p > \frac{1}{2}$. By (28), Lemma 2, and Chebyshev’s inequality, we can get that for all $\bar{t}_{0,k} \in [0, T]^{k+1}$ and all large $x$,

$$
P\left(\sum_{j=0}^{k} \varepsilon_j w_{j,k} > x, A_2\right) \leq \sum_{i=0}^{k} \sum_{j=0}^{k} P\left(\varepsilon_j w_{j,k} > \frac{x}{k+1}, w_{i,k} > B\right)
$$

$$
\leq (k+1) \sum_{j=1}^{k} P\left(\varepsilon_j w_{j}^+ > \frac{x}{(k+1)^2}, w_{j}^+ > \frac{B}{k+1}\right) + (k+1) P\left(\varepsilon_0 w_{0}^+ > \frac{x}{(k+1)^2}\right)
$$

$$
\leq C k^2 \bar{F}(x) E \left\{(k+1)^2 p_0 (w_{j}^+)^p \left[I_{[w_{j}^+ > \frac{B}{k+1}]} + I_{[w_{j}^+ > \frac{B}{k+1}]}\right] + C k^2 p_0^2 x^{-p} E(w_{j}^+)^p\right\}. \quad (41)
$$

Likewise, we can get that for all $\bar{t}_{0,k} \in [0, T]^{k+1}$ and all large $x$,

$$
P\left(\sum_{j=0}^{k} \varepsilon_j w_{j,k} > x, A_3\right) \leq \sum_{i=0}^{k} \sum_{j=0}^{k} P\left(\varepsilon_j w_{j,k} > \frac{x}{k+1}, w_{i,k} < b\right)
$$

$$
\leq C k^2 \bar{F}(x) E \left\{(k+1)^2 p_0 (w_{j}^-)^p \left[I_{[w_{j}^- > \frac{B}{k+1}]} + I_{[w_{j}^- > \frac{B}{k+1}]}\right] + C k^2 p_0^2 x^{-p} E(w_{j}^-)^p\right\}. \quad (42)
$$

Recall that the lemma of Willekens [27] and $E e^{-p L_\omega(T)} = e^{T \psi_\omega(p)} < \infty$ imply that $E(w_{j}^+)^p < \infty$ (see (34) for details). Then, by the fact that $P(w_{j}^- < b) \to 0 (b \to 0)$ and by relation (14), we obtain

$$
\limsup_{B \to \infty} \limsup_{b \to 0} \limsup_{x \to \infty} \sup_{\bar{t}_{0,k} \in [0, T]^{k+1}} \frac{\sum_{m=2}^{3} \sum_{j=0}^{k} P\left(\varepsilon_j w_{j,k} > x, A_m\right)}{\bar{F}(x)} = 0. \quad (43)
$$

Combining (39), (40) and (43) with relation (31) gives that, uniformly for all $\bar{t}_{0,k} \in [0, T]^{k+1}$ and all $t \in [0, T]$,

$$
P\left(\sum_{j=0}^{k} \varepsilon_j w_{j,k} - c_0 Z_t > x + \varepsilon_0 \psi_0\right) \leq \sum_{j=1}^{k} P\left(\varepsilon_j w_{j,k} > x\right). \quad (44)
$$

We next prove the lower bound version of relation (37). Using the notations of $A_m, m = 1, 2, 3$, in (38), we can derive that for arbitrarily fixed $D > 0$,

$$
P\left(\sum_{j=0}^{k} \varepsilon_j w_{j,k} - c_0 Z_t > x + \varepsilon_0 \psi_0\right) \geq P\left(\sum_{j=0}^{k} \varepsilon_j w_{j,k} > x + \varepsilon_0 \psi_0 + c_0 D, A_1, Z_t \leq D\right)
$$

$$
\geq P\left(\sum_{j=0}^{k} \varepsilon_j w_{j,k} > x + \varepsilon_0 \psi_0 + c_0 D, A_1\right) - P\left(\sum_{j=0}^{k} \varepsilon_j w_{j,k} > x, A_1, Z_t > D\right). \quad (45)
$$

For the first term in (45), because $\varepsilon_0$ is a nonnegative constant, we can get from Lemma 4 and relation (30) that, uniformly for all $\bar{t}_{0,k} \in [0, T]^{k+1}$,

$$
P\left(\sum_{j=0}^{k} \varepsilon_j w_{j,k} > x + \varepsilon_0 \psi_0 + c_0 D, A_1\right) \geq P\left(\sum_{j=1}^{k} \varepsilon_j w_{j,k} > x + \varepsilon_0 \psi_0 - \varepsilon_0 b + c_0 D, A_1\right)
$$

$$
\sim \sum_{j=1}^{k} P\left(\varepsilon_j w_{j,k} > x + \varepsilon_0 \psi_0 - \varepsilon_0 b + c_0 D, A_1\right) \sim \sum_{j=1}^{k} P\left(\varepsilon_j w_{j,k} > x, A_1\right)
$$

$$
\geq \sum_{j=1}^{k} P\left(\varepsilon_j w_{j,k} > x\right) - \sum_{m=2}^{3} \sum_{j=1}^{k} P\left(\varepsilon_j w_{j,k} > x, A_m\right). \quad (46)
$$
Similar to the derivations in (41)–(43), we can obtain

\[
\limsup_{B \to \infty} \limsup_{b \to 0} \limsup_{x \to \infty} \sup_{\tilde{t}_{0,k} \in [0,T]^{k+1}} \sum_{m=2}^{3} \sum_{j=1}^{k} \frac{\mathbb{P}(\varepsilon_j w_{j,k} > x, A_m)}{\mathbb{F}(x)} = 0. \tag{47}
\]

For the second term in (45), because \(\varepsilon_0\) is a nonnegative constant, from (13) and the definition of \(\mathcal{L}\), we can get that, for all \(\tilde{t}_{0,k} \in [0,T]^{k+1}\), all \(t \in [0,T]\), and all large \(x\),

\[
\mathbb{P}\left( \sum_{j=0}^{k} \varepsilon_j w_{j,k} > x, A_1, Z_t > D \right) \leq \mathbb{P}\left( \sum_{j=1}^{k} \varepsilon_j B > x - \varepsilon_0 B \right) \mathbb{P}(Z_t > D) \\
\leq k \mathbb{P}\left( \varepsilon_1 B > \frac{x - \varepsilon_0 B}{k} \right) \mathbb{P}(Z_t > D) \leq C \mathbb{F}(x) \mathbb{P}(Z_T > D).
\]

Noting that \(\mathbb{E}Z_T = \mathbb{E} \int_0^T e^{-L_\theta(s)} \, ds = \int_0^T e^{t \psi_0(1)} \, ds < \infty\), we have

\[
\limsup_{D \to \infty} \limsup_{x \to \infty} \sup_{\tilde{t}_{0,k} \in [0,T]^{k+1}, t \in [0,T]} \frac{\mathbb{P}\left( \sum_{j=0}^{k} \varepsilon_j w_{j,k} > x, A_1, Z_t > D \right)}{\mathbb{F}(x)} = 0. \tag{48}
\]

Combining (45)–(48) with relation (31) gives that, uniformly for all \(\tilde{t}_{0,k} \in [0,T]^{k+1}\) and all \(t \in [0,T]\),

\[
\mathbb{P}\left( \sum_{j=0}^{k} \varepsilon_j w_{j,k} - c_0 Z_t > x + \varepsilon_0 \psi_0 \right) \geq \sum_{j=1}^{k} \mathbb{P}\left( \varepsilon_j w_{j,k} > x \right). \tag{49}
\]

From (44) and (49), we can conclude the proof. \(\square\)

Now we are ready to establish the following four crucial lemmas for the proofs of the main results. Among them, Lemmas 8–10 will be used in the proofs of Theorems 1 and 2, and Lemma 11 will be used in the proof of Theorem 2.

**Lemma 8**

Under the conditions of Theorem 1 with \(Z_t\) and \(\{W_j(t), j \geq 0\}\) defined in (23) and (24), for arbitrarily fixed \(T > 0\), \(c_0 \geq 0\) and integer \(M \geq 1\), the relation

\[
\sum_{k=1}^{M} \mathbb{P}\left( \sum_{j=0}^{\infty} \varepsilon_j W_j(t) - c_0 Z_t > x, N_t = k \right) \sim \sum_{k=1}^{M} \sum_{j=1}^{k} \mathbb{P}(\varepsilon_j W_j(t) > x, N_t = k) \tag{50}
\]

holds uniformly for all \(t \in [0,T]\). Specifically, if \(F \in \mathcal{R}_\alpha\) for some \(0 < \alpha < \infty\), then it holds uniformly for all \(t \in [0,T]\) that

\[
\sum_{k=1}^{M} \mathbb{P}\left( \sum_{j=0}^{\infty} \varepsilon_j W_j(t) - c_0 Z_t > x, N_t = k \right) \sim \mathbb{F}(x) \sum_{k=1}^{M} \sum_{j=1}^{k} \mathbb{E}\left[ W_j(0)^\alpha \mathbb{I}_{[N_t = k]} \right]. \tag{51}
\]

**Proof**

Notice that given \(N_t = k\) and \(\tau_0 = 0\), \(\{W_j(t), j \geq 0\}\) in (24) is

\[
W_0(t) = \sum_{n=0}^{k} \varphi_n e^{-L_\theta(\tau_n)} - \varphi_0; \quad W_j(t) = \sum_{n=j}^{k} \varphi_{n-j} e^{-L_\theta(\tau_n)}, \quad 1 \leq j < k; \quad W_j(t) = 0, \quad j > k.
\]
Denote the distribution function of the random vector \( \tilde{\tau}_{t,n} = (\tau_1, \ldots, \tau_n) \) by \( G_{\tilde{\tau}_{t,n}} \) for any \( n \geq 1 \), and let \( E_{t,k+1} = \{(t_0, \ldots, t_{k+1}) : 0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t, t_{k+1} > t\} \). By Lemma 7, it holds uniformly for all \( t \in [0, T] \) that

\[
\sum_{k=1}^{M} \mathbb{P} \left( \sum_{j=0}^{\infty} \varepsilon_j W_j(t) - c_0 Z_t > x, N_t = k \right) = \sum_{k=1}^{M} \mathbb{P} \left( \sum_{j=0}^{k} \varphi_{n-j} e^{-L_0(t_n)} - c_0 Z_t > x + \varepsilon_0 \varphi_0, N_t = k \right)
\]

\[
= \sum_{k=1}^{M} \int \cdots \int_{E_{t,k+1}} \mathbb{P} \left( \sum_{j=0}^{k} \varepsilon_j w_{j,k} - c_0 Z_t > x + \varepsilon_0 \varphi_0 \right) dG_{\tilde{\tau}_{t,k+1}}(t_1, \ldots, t_{k+1})
\]

\[
\approx \sum_{k=1}^{M} \sum_{j=1}^{k} \int \cdots \int_{E_{t,k+1}} \mathbb{P} (\varepsilon_j w_{j,k} > x) dG_{\tilde{\tau}_{t,k+1}}(t_1, \ldots, t_{k+1})
\]

\[
= \sum_{k=1}^{M} \sum_{j=1}^{k} \mathbb{P} \left( \varepsilon_j \sum_{n=j}^{k} \varphi_{n-j} e^{-L_0(t_n)} > x, N_t = k \right) = \sum_{k=1}^{M} \sum_{j=1}^{k} \mathbb{P} (\varepsilon_j W_j(t) > x, N_t = k).
\]

Relation (51) follows immediately by applying relation (32) to (52).

Lemma 9
Under the conditions of Theorem 1 with \( \{W_j(t), j \geq 0\} \) defined in (24), for arbitrarily fixed \( T > 0 \) and \( q \geq 0 \), we have

\[
\limsup_{M \to \infty} \limsup_{x \to \infty} \sup_{t \in [0, T]} \sum_{k=M+1}^{\infty} k^q \mathbb{P} \left( \sum_{j=0}^{\infty} \varepsilon_j W_j(t) > x, N_t = k \right) F(x) = 0.
\]

Proof
Recall that \( w_{t}^+ = \sup_{n \geq 0} \{\varphi_n\} e^{-\inf_{s \in [0,t]} L_0(s)} \). From the definitions of \( \{W_j(t), j \geq 0\} \) in (24), we can get that given \( N_t = k \) and \( r_0 = 0 \),

\[
W_j(t) \leq \sum_{n=0}^{k} \varphi_{n-j} e^{-L_0(t_n)} \leq (k+1)w_{t}^+, \quad 0 \leq j \leq k; \quad W_j(t) = 0, \quad j > k.
\]

Thus, we have

\[
\mathbb{P} \left( \sum_{j=0}^{\infty} \varepsilon_j W_j(t) > x, N_t = k \right) \leq \mathbb{P} \left( \sum_{j=0}^{k} \varepsilon_j (k+1)w_{t}^+ > x \right) \mathbb{P} (N_t = k).
\]

Take \( p > \frac{q}{F} \). Recall that the lemma of Willekens [27] and \( \mathbb{E} e^{-pL_0(T)} = e^{T \psi_0(p)} < \infty \) imply that \( \mathbb{E} (w_{t}^+)^p < \infty \) (see (34) for details). Because \( \varepsilon_0 \) is a nonnegative constant, we can get from Lemma 2, Chebyshev’s inequality, and (13) and (14) that, for all \( t \in [0, T] \) and all large \( x \),

\[
\mathbb{P} \left( \sum_{j=0}^{k} \varepsilon_j (k+1)w_{t}^+ > x \right) \leq \sum_{j=1}^{k} \mathbb{P} \left( \varepsilon_j w_{t}^+ > \frac{x}{(k+1)^2} \right) + \mathbb{P} \left( \varepsilon_0 w_{t}^+ > \frac{x}{(k+1)^2} \right)
\]

\[
\leq C k F \left( \frac{x}{(k+1)^2} \right) \mathbb{E} \left( (w_{t}^+)^p + I_{[w_{t}^+ < 1]} \right) + C(k+1)^2 \mathbb{E} (w_{t}^+)^p \leq C k^{2p+1} F(x).
\]

Thus, for all \( t \in [0, T] \) and all large \( x \),

\[
\sum_{k=M+1}^{\infty} k^q \mathbb{P} \left( \sum_{j=0}^{\infty} \varepsilon_j W_j(t) > x, N_t = k \right) \leq C F(x) \sum_{k=M+1}^{\infty} k^{2p+q+1} \mathbb{P} (N_t = k)
\]

\[
= C F(x) \mathbb{E} \left( N_t^{2p+q+1} I_{[N_t > M]} \right) \leq C F(x) \mathbb{E} \left( N_T^{2p+q+1} I_{[N_T > M]} \right).
\]
Because \( r_1 \) is not degenerate at zero, by Lemma 3.2 in [28], there exists some \( h > 0 \) such that \( \mathbb{E}e^{hn} < \infty \). It follows that the last expectation mentioned previously tends to 0 as \( M \to \infty \). Therefore, relation (53) holds.

\[ \tag{54} \]

Lemma 10

Consider the renewal risk model introduced in Section 1 with \( \{W_j(t), j \geq 1\} \) defined in (24). Then, for any \( t \geq 0 \),

\[
\sum_{j=1}^{\infty} \mathbb{P}(\varepsilon_j W_j(t) > x, \tau_j \leq t) = \int_0^t \mathbb{P}\left( \varepsilon_1 \sum_{k=0}^{\infty} \varphi_k e^{-\alpha(x+\tau_k)} \mathbb{I}_{[s+\tau_k \leq t]} > x \right) \, d\lambda_s,
\]

and for any \( \alpha > 0 \),

\[
\sum_{j=1}^{\infty} \mathbb{E}[W_j^\alpha(t)] = \sum_{j=1}^{\infty} \mathbb{E}W_j^\alpha(t) = \int_0^t \mathbb{E}\left( \sum_{k=0}^{\infty} \varphi_k e^{-\alpha(x+\tau_k)} \mathbb{I}_{[s+\tau_k \leq t]} \right)^\alpha \, d\lambda_s.
\]

Proof

By the definitions of \( \{W_j(t), j \geq 1\} \) in (24) and the strong Markov property of \( \{L\vartheta(t), t \geq 0\} \), we can derive that for any integer \( j \geq 1 \) and any \( t \geq 0 \),

\[
\mathbb{P}(\varepsilon_j W_j(t) > x, \tau_j \leq t) = \mathbb{P}\left( \varepsilon_1 \sum_{k=0}^{\infty} \varphi_k e^{-\alpha(x+\tau_k)} \mathbb{I}_{[s+\tau_k \leq t]} > x, \tau_j \leq t \right)
\]

\[
= \mathbb{P}\left( e^{-\alpha(x)} \varepsilon_1 \sum_{k=0}^{\infty} \varphi_k e^{-[\alpha(x+\tau_k)]-\alpha(\tau_j)} \mathbb{I}_{[s+\tau_k \leq t]} > x, \tau_j \leq t \right)
\]

\[
= \int_0^t \mathbb{P}\left( e^{-\alpha(x)} \varepsilon_1 \sum_{k=0}^{\infty} \varphi_k e^{-[\alpha(x+\tau_k)]-\alpha(\tau_j)} \mathbb{I}_{[s+\tau_k \leq t]} > x \right) \, d\tau_j
\]

\[
= \int_0^t \mathbb{P}\left( e^{-\alpha(x)} \varepsilon_1 \sum_{k=0}^{\infty} \varphi_k e^{-[\alpha(x+\tau_k)]-\alpha(s)} \mathbb{I}_{[s+\tau_k \leq t]} > x \right) \, d\tau_j
\]

\[
= \int_0^t \mathbb{P}\left( \varepsilon_1 \sum_{k=0}^{\infty} \varphi_k e^{-\alpha(x+\tau_k)} \mathbb{I}_{[s+\tau_k \leq t]} > x \right) \, d\tau_j.
\]

Hence, for any \( t \geq 0 \),

\[
\sum_{j=1}^{\infty} \mathbb{P}(\varepsilon_j W_j(t) > x, \tau_j \leq t) = \sum_{j=1}^{\infty} \int_0^t \mathbb{P}\left( \varepsilon_1 \sum_{k=0}^{\infty} \varphi_k e^{-\alpha(x+\tau_k)} \mathbb{I}_{[s+\tau_k \leq t]} > x \right) \, d\tau_j
\]

\[
= \int_0^t \mathbb{P}\left( \varepsilon_1 \sum_{k=0}^{\infty} \varphi_k e^{-\alpha(x+\tau_k)} \mathbb{I}_{[s+\tau_k \leq t]} > x \right) \, d\lambda_s.
\]

By the same approach as in (56) and (57), we can obtain (55).

Recall that for every fixed \( \vartheta \in (0, 1), \kappa_{\vartheta} > 0 \) is the unique value satisfying \( \psi_{\vartheta}(\kappa_{\vartheta}) = 0 \). For notational convenience, write \( \kappa_{\vartheta} = \infty \) when \( \vartheta = 0 \) in the rest of this paper.

Lemma 11

Consider \( \{W_j, j \geq 0\} \) defined in (25). For arbitrarily fixed \( q \geq 0 \) and \( 0 < p < \kappa_{\vartheta} \) with \( \vartheta \in [0, 1) \), we have

\[
\mathbb{E}W_0^p + \sum_{j=1}^{\infty} j^q \mathbb{E}W_j^p < \infty.
\]

Furthermore, under the conditions of Theorem 2, we have

\[
\mathbb{P}\left( \sum_{j=0}^{\infty} \varepsilon_j W_j > x \right) \leq \mathcal{F}(x) \sum_{j=1}^{\infty} \mathbb{E}W_j^q.
\]
Proof
We first prove (58). Denote $\vartheta_n = e^{-L_\theta(\tau_n)}$ for any $n \geq 1$. Because $\sup_{n \geq 0} \{ \varphi_n \} < \infty$, from the definitions of $\{W_j, j \geq 0\}$ in (25), we can get
\[
\mathbb{E}W_0^p + \sum_{j=1}^{\infty} j^q \mathbb{E}W_j^p \leq C \left( \sum_{n=1}^{\infty} \vartheta_n \right)^p + C \sum_{j=1}^{\infty} j^q \mathbb{E} \left( \sum_{n=j}^{\infty} \vartheta_n \right)^p \leq C \sum_{j=1}^{\infty} j^q \mathbb{E} \left( \sum_{n=j}^{\infty} \vartheta_n \right)^p . \quad (60)
\]
For the case $p > 1$, by Hölder’s inequality, we have
\[
\sum_{j=1}^{\infty} j^q \mathbb{E} \left( \sum_{n=j}^{\infty} \vartheta_n \right)^p \leq \sum_{j=1}^{\infty} j^q \left( \sum_{n=j}^{\infty} \frac{1}{n^2} \right)^{p-1} \sum_{n=j}^{\infty} n^{2p-2} \mathbb{E} \vartheta_n^p \leq C \sum_{j=1}^{\infty} j^q \sum_{n=j}^{\infty} n^{2p-2} \mathbb{E} \vartheta_n^p
\]
\[
= C \sum_{n=1}^{\infty} n^{2p-2} \mathbb{E} \vartheta_n^p \sum_{j=1}^{n} j^q \leq C \sum_{n=1}^{\infty} n^{2p+q-1} \mathbb{E} \vartheta_n^p . \quad (61)
\]
For the case $0 < p \leq 1$, by $C_r$ inequality, we have
\[
\sum_{j=1}^{\infty} j^q \mathbb{E} \left( \sum_{n=j}^{\infty} \vartheta_n \right)^p \leq \sum_{j=1}^{\infty} j^q \sum_{n=j}^{\infty} \mathbb{E} \vartheta_n^p \leq \sum_{n=1}^{\infty} n^{q+1} \mathbb{E} \vartheta_n^p . \quad (62)
\]
Thus, by (60)–(62), we can get that for any $p > 0$ and any $q \geq 0$,
\[
\mathbb{E}W_0^p + \sum_{j=1}^{\infty} j^q \mathbb{E}W_j^p \leq C \sum_{n=1}^{\infty} n^{2p+q+1} \mathbb{E} \vartheta_n^p = C \sum_{n=1}^{\infty} n^{2p+q+1} \left( \mathbb{E}e^{-pL_\theta(\tau_1)} \right)^n .
\]
For every fixed $\theta \in [0, 1)$, by the definition of $\kappa_\theta$, we obtain $\varphi_\theta(p) < 0$ for any $0 < p < \kappa_\theta$ (see (12) and $\psi_0(p) = -rp$). It follows that $\mathbb{E}e^{-pL_\theta(\tau_1)} = \mathbb{E}e^{rt_1\varphi_\theta(p)} < 1$. This, together with the aforementioned inequality, implies that (58) holds for any $q \geq 0$ and $0 < p < \kappa_\theta$.

We next prove relation (59). For arbitrarily fixed $0 < \delta < 1 < B < \infty$ and integer $M > 1$ satisfying $\sum_{j=M+1}^{\infty} j^{-2} < 1$, we have
\[
\mathbb{P} \left( \sum_{j=0}^{\infty} \varepsilon_j W_j > x \right) \leq \mathbb{P} \left( \sum_{j=0}^{M} \varepsilon_j W_j > \delta x, W_0 \leq B \right) + \mathbb{P} \left( \sum_{j=0}^{M} \varepsilon_j W_j > \delta x, W_0 > B \right)
\]
\[
+ \mathbb{P} \left( \sum_{j=M+1}^{\infty} \varepsilon_j W_j > x - \delta x \right) := K_1 + K_2 + K_3 . \quad (63)
\]
In view of (58), we can get from Lemma 5 and the definitions of $R_{-\alpha}$ and $\mathcal{L}$ that
\[
K_1 \leq \mathbb{P} \left( \sum_{j=1}^{M} \varepsilon_j W_j > \delta x - \varepsilon_0 B \right) \sim \mathcal{F}(\delta x - \varepsilon_0 B) \sum_{j=1}^{M} \mathbb{E}W_j^q \leq \delta^{-\alpha} \mathcal{F}(x) \sum_{j=1}^{\infty} \mathbb{E}W_j^q . \quad (64)
\]
Take $0 < p_1 < \alpha < p_2 < \kappa_\theta$. For the fixed $0 < \delta < 1$, by Lemma 3 and Chebyshev’s inequality, we can get that for all large $x$,
\[
K_2 \leq \sum_{j=1}^{M} \mathbb{P} \left( \varepsilon_j W_j > \delta x \frac{M+1}{M+1}, W_0 > B \right) + \mathbb{P} \left( \varepsilon_0 W_0 > \delta x \frac{M+1}{M+1} \right)
\]
\[
\leq C \mathcal{F}(x) \sum_{j=1}^{\infty} \mathbb{E} \left[ \mathbb{I}_{W_j > [0,B]} W_j^{p_1} + W_j^{p_2} \mathbb{I}_{W_0 > B} \right] + C x^{-p_2} \mathbb{E}W_0^p .
\]
Hence, from (58) and relation (14) we can get

$$\limsup_{B \to \infty} \limsup_{x \to \infty} \frac{K_3}{F(x)} = 0. \quad (65)$$

For $K_3$, by Lemma 3 and $\sum_{j=M+1}^{\infty} j^{-2} < 1$, we can get that for all large $x$,

$$K_3 \leq \mathbb{P}\left( \sum_{j=M+1}^{\infty} \mathbb{E}_j W_j > \sum_{j=M+1}^{\infty} \frac{x - \delta x}{j^2} \right) \leq \mathbb{P}\left( \mathbb{E}_j W_j > \frac{x - \delta x}{j^2} \right) \leq C \mathcal{F}(x) \sum_{j=M+1}^{\infty} \left[ j^{2p_1} \mathbb{E}_j W_j^{p_1} + j^{2p_2} \mathbb{E}_j W_j^{p_2} \right].$$

According to (58), the last sum mentioned previously tends to 0 as $M \to \infty$. Hence,

$$\limsup_{M \to \infty} \limsup_{x \to \infty} \frac{K_3}{F(x)} = 0. \quad (66)$$

By (63)–(66), we obtain relation (59) and conclude the proof.

\section*{4. Proofs}

Using the notations in (21)–(24), we can express $\Psi_\theta(x, t)$ and $\Psi_\theta(x)$ as

$$\Psi_\theta(x, t) = \mathbb{P}\left( \sup_{0 < s \leq t} V_\theta(s) > x \right) = \mathbb{P}\left( \sup_{0 < s \leq t} \left( \sum_{j=0}^{\infty} \mathbb{E}_j W_j(s) - c Z_s \right) > x \right) \quad (67)$$

and

$$\Psi_\theta(x) = \mathbb{P}\left( \sup_{0 < s < \infty} V_\theta(s) > x \right) = \mathbb{P}\left( \sup_{0 < s < \infty} \left( \sum_{j=0}^{\infty} \mathbb{E}_j W_j(s) - c Z_s \right) > x \right). \quad (68)$$

\textbf{Proof of Theorem 1}

We first prove the upper bound version of relation (15). By (67) and the definitions of $\{W_j(t), j \geq 0\}$ in (24) and $Z_t$ in (23), we can get that for every fixed $M \geq 1$,

$$\Psi_\theta(x, t) \leq \left( \sum_{k=1}^{M} + \sum_{k=M+1}^{\infty} \right) \mathbb{P}\left( \sum_{j=0}^{\infty} \mathbb{E}_j W_j(t) > x, N_t = k \right). \quad (69)$$

By relation (50) with $c_0 = 0$, it holds uniformly for all $t \in \Lambda_{T_0, T}$ that

$$\sum_{k=1}^{M} \mathbb{P}\left( \sum_{j=0}^{\infty} \mathbb{E}_j W_j(t) > x, N_t = k \right) \sim \sum_{k=1}^{M} \sum_{j=1}^{k} \mathbb{P}\left( \mathbb{E}_j W_j(t) > x, N_t = k \right) \leq \sum_{k=1}^{\infty} \sum_{j=1}^{k} \mathbb{P}\left( \mathbb{E}_j W_j(t) > x, N_t = k \right) = \sum_{j=1}^{\infty} \mathbb{P}\left( \mathbb{E}_j W_j(t) > x, \tau_j \leq t \right). \quad (70)$$

Because $\varphi_0 > 0$, we can take $0 < \delta < 1$ such that $\mathbb{P}\left( \varphi_0 e^{-\sup s \in [0, T]} L_\theta(s) > \delta \right) > 0$. Then, by the definition of $W_1(t)$ and inequality (13), we have, for all $t \in \Lambda_{T_0, T}$ and all large $x$,

$$\mathbb{P}\left( \mathbb{E}_1 W_1(t) > x, \tau_1 \leq t \right) \geq \mathbb{P}\left( \mathbb{E}_1 \varphi_0 e^{-L_\theta(\tau_1)} > x, \tau_1 \leq t \right) \geq \mathbb{P}\left( \mathbb{E}_1 \delta > x, \varphi_0 e^{-\sup s \in [0, t]} L_\theta(s) > \delta, \tau_1 \leq t \right) \geq \mathbb{P}\left( \mathbb{E}_1 \delta > x \right) \mathbb{P}\left( \varphi_0 e^{-\sup s \in [0, T_1]} L_\theta(s) > \delta \right) \mathbb{P}(\tau_1 \leq T_0) \geq C \mathcal{F}(x). \quad (71)$$
This, together with Lemma 9, implies that

$$\limsup_{M \to \infty} \limsup_{x \to \infty} \sup_{t \in \Lambda_{T_0, T}} \frac{\sum_{k=M+1}^{\infty} \mathbb{P}\left( \sum_{j=0}^{\infty} \varepsilon_j W_j(t) > x, N_t = k \right)}{\mathbb{P}(\varepsilon_1 W_1(t) > x, \tau_1 \leq t)} = 0. \quad (72)$$

Combining (70) and (72) with (69) gives that, uniformly for all $t \in \Lambda_{T_0, T}$,

$$\Psi_\theta(x, t) \leq \sum_{j=1}^{\infty} \mathbb{P}(\varepsilon_j W_j(t) > x, \tau_j \leq t). \quad (73)$$

By applying (54) to (73), we obtain the upper bound version of relation (15).

We next prove the lower bound version of relation (15). By expression (67) and relation (50) with $c_0 = c$, it holds uniformly for all $t \in \Lambda_{T_0, T}$ that

$$\Psi_\theta(x, t) \geq \sum_{k=1}^{M} \mathbb{P}\left( \sum_{j=0}^{\infty} \varepsilon_j W_j(t) - c Z_1 > x, N_t = k \right) - J_1(x, t, M),$$

where $M \geq 1$ is a temporarily fixed integer. Trivially, for any integer $k > M$,

$$\sum_{j=1}^{k} \mathbb{P}(\varepsilon_j W_j(t) > x, N_t = k) \leq k \mathbb{P}\left( \sum_{j=0}^{\infty} \varepsilon_j W_j(t) > x, N_t = k \right).$$

Then, from Lemma 9 and (71), we can get

$$\limsup_{M \to \infty} \limsup_{x \to \infty} \sup_{t \in \Lambda_{T_0, T}} \frac{J_1(x, t, M)}{\mathbb{P}(\varepsilon_1 W_1(t) > x, \tau_1 \leq t)} = 0.$$

Hence, it holds uniformly for all $t \in \Lambda_{T_0, T}$ that

$$\Psi_\theta(x, t) \geq \sum_{j=1}^{\infty} \mathbb{P}(\varepsilon_j W_j(t) > x, \tau_j \leq t). \quad (74)$$

By applying (54) to (74), we obtain the lower bound version of relation (15) and conclude the proof of Theorem 1. \qed

**Proof of Theorem 2**

First, we follow the proof of Theorem 1 to establish that, uniformly for all $t \in \Lambda_{T_0, T}$,

$$\Psi_\theta(x, t) \sim \mathcal{F}(x) \sum_{j=1}^{\infty} \mathbb{E}W_j^q(t). \quad (75)$$

From the definitions of $\{W_j(t), j \geq 1\}$ in (24), we can get

$$\sum_{k=1}^{\infty} \sum_{j=1}^{k} \mathbb{E}\left[ W_j^q(t) \mathbb{1}_{[N_t=k]} \right] = \sum_{j=1}^{\infty} \mathbb{E}\left[ W_j^q(t) \mathbb{1}_{[\tau_j \leq t]} \right] = \sum_{j=1}^{\infty} \mathbb{E}W_j^q(t). \quad (76)$$
For all \( t \in \Lambda_{T_0,T} \) and all \( x \geq 0 \), we have
\[
\mathcal{F}(x) \mathbb{E} W^\alpha(t) \geq \mathcal{F}(x) \mathbb{E} \left[ \varphi_0 e^{-L_\theta(t_1)} \varphi^\alpha \right] = \mathcal{F}(x) \varphi^\alpha_0 \int_{0^-}^t \mathbb{E} e^{-\alpha L_\theta(s)} \mathbb{P}(\tau_1 \in ds) \\
\geq \mathcal{F}(x) \varphi^\alpha_0 \min \{1, e^{\tau \varphi_0(a)}\} \mathbb{P}(\tau_1 \leq T_0) \geq C \mathcal{F}(x).
\]

(77)

Now following the derivations in (69)–(74) but using relations (51) and (77) instead of relations (50) and (71), respectively, we can get from (76) that relation (75) holds uniformly for all \( t \in \Lambda_{T_0,T} \).

Next, we extend the uniformity of relation (75) to the whole interval \( \Lambda_{T_0} \). For arbitrarily fixed \( \varepsilon > 0 \), because \( W_j(t) \) increases to \( W_j \) almost surely as \( t \to \infty \) for any \( j \geq 1 \) and (58) holds for any \( 0 < p < \kappa_\theta \) and \( q = 0 \), we can take \( T_1 > T_0 \) such that
\[
\sum_{j=1}^\infty \mathbb{E} W^\alpha_j \leq (1 + \varepsilon) \mathbb{E} W^\alpha_j(T_1).
\]

(78)

On the one hand, by expression (68), it holds uniformly for all \( t \in \Lambda \cap (T_1, \infty] \) that
\[
\Psi_\theta(x,t) \leq \Psi_\theta(x) \leq \mathbb{P} \left( \sum_{j=0}^{\infty} \varepsilon_j W_j > x \right) \leq \mathcal{F}(x) \sum_{j=1}^\infty \mathbb{E} W^\alpha_j \\
\leq \mathcal{F}(x) \sum_{j=1}^\infty \left[ \mathbb{E} W^\alpha_j(t) + \mathbb{E} W^\alpha_j(t) - \mathbb{E} W^\alpha_j(T_1) \right] \leq (1 + \varepsilon) \mathcal{F}(x) \sum_{j=1}^\infty \mathbb{E} W^\alpha_j(t),
\]

where we used relation (59) in the third step and relation (78) in the last step. On the other hand, it holds uniformly for all \( t \in \Lambda \cap (T_1, \infty] \) that
\[
\Psi_\theta(x,t) \geq \Psi_\theta(x, T_1) \sim \mathcal{F}(x) \sum_{j=1}^\infty \mathbb{E} W^\alpha_j(T_1) \\
\geq \mathcal{F}(x) \sum_{j=1}^\infty \left[ \mathbb{E} W^\alpha_j(t) - \mathbb{E} W^\alpha_j(t) + \mathbb{E} W^\alpha_j(T_1) \right] \geq (1 - \varepsilon) \mathcal{F}(x) \sum_{j=1}^\infty \mathbb{E} W^\alpha_j(t),
\]

where in the second step we used relation (75) with \( t \) replaced by \( T_1 \), while in the last step we used (78). Hence, for all \( t \in \Lambda \cap (T_1, \infty] \) and all large \( x \), say, \( x > x_1 > 0 \),
\[
(1 - \varepsilon) \mathcal{F}(x) \sum_{j=1}^\infty \mathbb{E} W^\alpha_j(t) \leq \Psi_\theta(x,t) \leq (1 + \varepsilon) \mathcal{F}(x) \sum_{j=1}^\infty \mathbb{E} W^\alpha_j(t).
\]

(79)

By relation (75), (79) still holds for all \( t \in \Lambda \cap [T_0, T_1] \) and all large \( x \), say \( x > x_2 > 0 \). Hence, (79) holds for all \( t \in \Lambda_{T_0} \) and all \( x \geq \max\{x_1, x_2\} \). By applying equality (55) to (79) and taking into account the arbitrariness of \( \varepsilon > 0 \), we obtain that relation (16) holds uniformly for all \( t \in \Lambda_{T_0} \).

\( \square \)

**Proof of Corollary 1**

From relation (16) and the strong Markov property of \( \{L_\theta(t), t \geq 0\} \), we can derive that
\[
\Psi_\theta(x) \sim \mathcal{F}(x) \int_{0^-}^\infty \mathbb{E} \left( \sum_{k=0}^\infty \varphi_k e^{-L_\theta(s+\tau_k)} \right)^\alpha d\lambda_s \\
= \mathcal{F}(x) \int_{0^-}^\infty \mathbb{E} e^{-\alpha L_\theta(s)} \mathbb{E} \left( \sum_{k=0}^\infty \varphi_k e^{-[L_\theta(s+\tau_k)-L_\theta(s)]} \right)^\alpha d\lambda_s \\
= \mathcal{F}(x) \mathbb{E} \left( \sum_{k=0}^\infty \varphi_k e^{-L_\theta(\tau_k)} \right)^\alpha \int_{0^-}^\infty e^{\tau \varphi_0(a)} d\lambda_s \\
= \mathcal{F}(x) \mathbb{E} \left( \sum_{k=0}^\infty \varphi_k e^{-L_\theta(\tau_k)} \right)^\alpha \frac{\mathbb{E} e^{\tau \varphi_0(a)}}{1 - \mathbb{E} e^{\tau \varphi_0(a)}},
\]
where the last equality holds because
\[
\int_{0-}^{\infty} e^{\psi_\theta(s)} \, d\lambda_s = \sum_{k=1}^{\infty} \int_{0-}^{\infty} e^{\psi_\theta(s)} \, d\lambda_s = \sum_{k=1}^{\infty} \mathbb{E} e^{\tau_k \psi_\theta(\omega)} = \frac{\mathbb{E} e^{\tau_1 \psi_\theta(\omega)}}{1 - \mathbb{E} e^{\tau_1 \psi_\theta(\omega)}}.
\]
It ends the proof.

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