Flexible capacity strategy with multiple market periods under demand uncertainty and investment constraint

Liu Yang\textsuperscript{a}, C.T. Ng\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a} Business School, University of International Business and Economics, Chaoyang District, Beijing 100029, China
\textsuperscript{b} Department of Logistics and Maritime Studies, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong Special Administrative Region, China

1. Introduction

Nowadays firms are facing incredible challenges ever, particularly due to volatile demand, fierce competition and critical financial environment. Worldwide financial crises make firms more cautious about their capacity investments. Meanwhile, the customer demand is getting more unpredictable and uncertain. To deal with these challenges, firms are doing their best to adopt different methods. On the one hand, firms are trying to get more flexibility in adjusting their productions to deal with demand uncertainty. On the other hand, firms need to keep and increase their market shares under competition and, therefore, they will keep at the lowest production level at least. Due to the competition pressure and fear of being forgotten by the customers, firms are not likely to stop their productions even when there is little demand. For example, manufacturers still produce ice cream even in winter when there is little demand. As a result, an appropriate ability in adjusting production is needed for a firm to survive, while keeping a suitable level of operation. Furthermore, a firm needs financial support when it builds up its capacity at the beginning of a planning horizon over multiple market periods. This financial support is so critical to the success of a capacity strategy. The financial budget often determines the affordability of a firm's capacity investment, and, therefore, determines a firm's capacity upper bound.

This paper investigates the flexible capacity strategy in terms of the firms’ ability in adjusting its production level under demand uncertainty. Particularly, we explore a firm’s flexibility throughout two stages: (1) production adjustment stage and (2) optimal sales stage. Furthermore, we integrate the financial budget into the model to formulate a firm’s financial consideration at the beginning of the planning horizon, which is over multiple market periods. To do so, we propose a general model which is able to present a firm’s flexibility to adjust the production, with full consideration of the financial investment constraint and the planning horizon of the business.

In the proposed model, the planning horizon includes \( n \) production/market periods. At the beginning of the planning horizon, the firm makes its capacity investment decision, i.e., determining its optimal capacity, aiming at maximizing the total profit over the \( n \) periods under the financial budget constraint. In each production period, the firm makes a safety production decision before the demand information is known. This safety production decision is due to some reasons, such as: (i) sustaining its market share in a competitive market; (ii) keeping its customers to remember the products; (iii) promoting its products to some potential markets;
and (iv) dealing with some unexpected and urgent demand. After knowing the real demand in each market, the firm decides whether to make additional production to meet the unsatisfied demand and by how much, under the capacity constraint. To capture the interaction among a firm’s supply, market demand and market price, we adopt the responsive pricing. It means that the product price in the market is determined by the real demand and the total supply in the market. The responsive pricing has been adopted by a few previous studies, such as Van Mieghem and Dada (1999), Goyal and Netessine (2007), Anand and Girotra (2007).

As the capacity strategy is usually a relatively long-term decision, it often involves a great amount of capital and multiple cycles of consumption (demand). Some capacity investment, such as purchasing new equipment and recruiting new staff, usually covers at least a few market periods before the next capacity investment. Furthermore, the multiple period model is a generalization of the single period model. Therefore, results of the multiple period model can be applied to the single period model, but the converse is not true.

Differing from most studies in the literature which assumed firms to put all produced products into the market, we explore a firm’s flexibility in two stages: determining the optimal sales and making additional production decision. We solve for the optimal safety production, the optimal additional production and the optimal sales of each period under four different situations. Second, we solve for the optimal capacity over the entire planning horizon under the financial budget constraint. We show that there are two thresholds of the unit capacity cost. When the unit capacity cost is very low, the firm builds up the capacity to its upper bound which is determined by its financial budget; when the unit capacity cost is very high, the firm only builds the capacity to maintain at the lowest production level; and when the unit capacity cost is in between of the two thresholds, we obtain the analytical solution for the optimal capacity. Third, we analyze the impact of the planning horizon on the optimal capacity of the firm. We find that the firm benefits from the investment only when the designed planning horizon is longer than a threshold. Furthermore, a numerical example demonstrates that the optimal capacity is concavely increasing in the number of market periods. Finally, we explore the endogenous flexibility by investigating the safety production level in each market period. Interestingly, we find that higher flexibility is not always beneficial to the firm. Depending on the situations, there are different optimal safety production levels to maximize the firm’s expected profit.

The rest of the paper is organized as follows: In Section 2, we review the recent research in the literature and find out a research gap. In Section 3, we introduce a styled multiple-period capacity strategy model. In Section 4, we formulate the production decisions in each period under demand uncertainty. In Section 5, we investigate the endogenous safety production level that leads to the maximum profit over the entire planning horizon. In Section 6, we conduct some discussion with respect to our partial flexibility. Finally, we draw conclusions, explore management insights, and provide suggestions for further research in Section 7.

2. Literature review

Flexible capacity strategies have been investigated extensively in the past two decades. Comprehensive reviews include Sethi and Sethi (1990), De Groot (1994), Van Hoek (2001), Van Mieghem (2003) and Volling, Matzke, Grunewald, and Spengler (2013). Flexible capacity strategy has been regarded as an effective and efficient method to improve a firm’s ability to hedge against demand uncertainty and increase the firm’s competitiveness by many studies, such as, Fine and Freund (1990), Röller and Tombak (1993), Lee and Tang (1997), Van Mieghem (1998), Aviv and Federgruen (2001), Buxey (2005), Malhotra and Mackelprang (2012) and Georgiadis and Athanasiou (2013). Gerwin (1993) established a research agenda with respect to the uncertainty type and the adaptive method. His study classified the capacity flexibility into seven categories, which are different but highly relevant to each other.

Focusing on the timing of postponement, Van Mieghem and Dada (1999) investigated six alternative strategies. They considered that the firm could postpone a few decisions including capacity, production, sales and pricing. Based on the comparison of the six alternatives, they concluded that production and pricing postponement strategy is the most effective strategy to maximize a firm’s expected profit. Assuming market clearance, Anupindi and Jiang (2008) extended the production and pricing postponement strategy in Van Mieghem and Dada (1999) to a duopoly model. They showed the equivalence of the equilibrium under quantity competition and price competition when two firms adopt the flexible strategies. Recently, Choi, Narasimhan, and Kim (2012) applied the postponement strategy in a global supply chain. Their results suggested the timing of postponement has a significant effect on the overall cost efficiency. However, these studies only investigated the problem in a single period. They did not consider the impact of the planning horizon which embraces multiple market periods.

Considering a firm’s ability in switching between different products to deal with the demand uncertainty, Bish, Muriel, and Biller (2005) studied a capacity allocation mechanism in a two-plant two-product manufacturing setting under a make-to-order environment. Chod and Rudi (2005) addressed a similar issue under responsive pricing. Other similar studies include Bish and Wang (2004) and Goyal and Netessine (2007). Under the setting of these studies, a firm makes its production decisions as long as they have received the demand information. In other words, there was an ideal make-to-order environment that firms could keep the production as zero if the demand is too low to obtain positive profits. Contrary to these studies, this paper investigates a bounded flexibility in which a firm has to make a safety production level, meanwhile the firm is also constrained by its financial budget over the planning horizon. Our model is more general and realistic than these studies as it also covers the possibility of zero safety production level. Furthermore, our model can be treated as a partial flexibility model as the total production level in each period cannot be less than the safety production level, Manganini, Van Oyen, and Sims (2005) analyzed a type of operational flexibility, namely the structural flexibility, which can be created by using multipurpose resources such as cross-trained labors, flexible machines or flexible factories. Anand and Girotra (2007) studied early/delay differentiaization as a strategic decision of firms in competition. They showed that, under plausible conditions, the benefit associated with delay differentiation can be significantly diminished. Some research incorporated other business factors into the investigation of flexible capacity. Yang, Ng, and Cheng (2011) studied the impact of technology investment under flexible capacity strategy; Li and Womber (2012) considered scheduling in optimizing the supply chain configuration, Moon, Yi, and Ngai (2012) proposed an instrument for measuring supply chain flexibility for the textile and clothing companies through an empirical study. Köber and Heinecke (2012) conducted a case study at the agricultural machinery of a manufacturer with volatile and seasonal demand. Their results showed the combination of make-to-order and make-to-stock is an appealing production strategy. Other relating studies include Hallgren and Olhager (2009), Vickery, Dröge, and Markland (1997) and Patel, Terjesen, and Li (2012). To the best of our knowledge, there has been little research to investigate the partial flexible capacity strategy over multiple market periods. This research is perhaps to fill part of
the research gap in studying the benefit of flexible capacity strategy over multiple marketing periods.

3. System features

We consider that the planning horizon of a firm's capacity investment consists of \( n \) market periods, as shown in Fig. 1. The firm makes its capacity investment at the beginning of the planning horizon followed by \( n \) market periods. Here it is noted that: (i) capacity investment is a long-term decision; (ii) a firm's capacity represents a firm's maximum production level during its designed planning horizon. Therefore, a firm makes its capacity investment decision very cautiously, with full considerations of two influential factors: (i) the demand uncertainty within the planning horizon and (ii) the financial affordability when the firm makes the capacity investment. For example, when a firm is going to build some capacity by purchasing some equipment, it needs to consider the designed life cycle of the equipment and the financial budget. Particularly, the length of the planning horizon is mainly determined by the designed life cycle of the equipment; while the financial budget limits the volume of the capacity investment. Let \( k \) represent a firm's capacity and \( C_0 > 0 \) be the unit capacity cost. Therefore, the total capacity cost is \( C_k \). Denote the total financial budget by \( C_B > 0 \), when a firm is going to build its capacity. The financial budget constraint can be formulated as \( C_T \geq k - C_b \).

After determining the capacity volume at the beginning of the planning horizon, a firm goes through four stages in each market period, as shown in Fig. 2. These four stages include safety production, additional production, responsive pricing and demand satisfaction. For example, we consider a logistics firm which invests in trucks available to use cannot exceed the trucks the firm invested in at the beginning of the planning horizon. Since the safety production \( q_s \) is not affected by the fluctuating demand, the additional production \( q_a \) is within the range \( q_0 \leq [0, k - q_s] \). Accordingly, the total production quantity \( q = q_s + q_a \) is within the range \( q \in [q_s, k] \). The maximum adjustment of production level is \( k - q_s \). This adjustment range reflects a firm's ability in making adjustment of its production.

Throughout this paper, we adopt the inverse demand function as \( p(x) = x - s \), where \( p \) is the product price in the market; \( s \) is the total product put into the market and \( x \) is the realization of the uncertain demand, which follows a general distribution with mean \( \mu \), cumulative distribution function \( F(x) \), and probability density function \( f(x) \). Define \( F(x) = 1 - F(x) \). In our model, if there is inventory left at the end of a market period, the inventory is not considered in the next market period. The safety production level is assumed to keep the same for all market periods. Considering that each market period is a short term and trying to keep the operation simple and standard, it is reasonable that the firm keeps a constant safety production level within the whole planning horizon of the capacity building. To facilitate the presentation, we define \( \tau \) as the safety production level in each period, i.e., \( q_s = \tau \), in each market period. It should be noted here that the safety production represents the lowest bound of the total production, i.e., \( \tau \leq q \). Also, we noted that the total production is restricted by the capacity of the firm, i.e., \( q \leq k \). Therefore, we have the relationship between the safety production level and the capacity of the firm, i.e., \( \tau \leq k \). \( \mu_0 \) and \( \beta_0 \) are defined as the unit production costs at the safety production stage and the additional production stage, respectively. To facilitate the discussion, we define \( L(x) = \int_{x}^{q_s} (x - x)^{f(x)}dx \), \( x \geq x \geq 0 \), which is proved to be a strictly decreasing and convex function in \( x \), and \( L(x) \in (0, \mu] \) for \( x \geq 0 \). Define \( X(C) \) as the inverse function of \( L(x) \) which is strictly decreasing in \( C \in (0, \mu] \). To enhance the readability of the paper, all proofs are provided in Appendix A.

4. Model formulation

4.1. Decisions in each period

With a given safety production level \( \tau \), the firm produces production \( \tau \) before observing the demand information. Then the firm holds the inventory \( \tau \) until it observes the demand information. After knowing the demand information, the firm uses its current inventory \( \tau \) to meet the demand first and determines the optimal sales, \( s \), (i.e., how many products are put into the market) to maximize its ex-post profit. The problem is formulated as:

\[
\text{Max } \pi(s|x) = s(x - s), \quad \text{s.t. } 0 \leq s \leq x. \tag{1}
\]

The constraint \( s \leq x \) reflects that the firm uses its current inventory to meet the demand.
When the inventory is enough to meet the demand, the firm will not produce any additional production, i.e., \( q_{II} = 0 \). When the inventory \( \tau \) is not enough to meet the demand, the firm considers making additional production at additional production stage II to meet the demand. Since there is no need to produce more than enough, all additional production will be sold in the market. The total sales is then \( s = \tau + q_{II} \). Since the capacity of the firm is \( k \) that represents the maximum of the total production, i.e., \( q \leq k \), the total sales is less or equal to the total production, i.e., \( s \leq k \). Together with the relationship \( s = \tau + q_{II} \), the firm makes its optimal sale decision under the constraints that \( \tau \leq s \leq k \). With any given demand realization \( s \), a firm aims to maximize its ex-post profit by determining the optimal sales, which is formulated as

\[
\max \; \pi(s|\alpha, k, \tau) = s(\alpha - s) - \beta_s(s - \tau), \quad \text{s.t.} \; \tau \leq s \leq k
\]

A firm’s optimal production is provided by Proposition 1 below.

**Proposition 1.** Given a safety production level \( \tau \), the capacity \( k \) and the demand realization \( \alpha \), the optimal sales of a firm in each market period is

\[
s = \begin{cases} 
S_a & 0 \leq \alpha < \alpha_A \\
\tau & \alpha_A \leq \alpha < \alpha_B \\
S_b & \alpha_B \leq \alpha < \alpha_C \\
k & \alpha_C \leq \alpha 
\end{cases}
\]

where \( \alpha_A = 2\tau \), \( \alpha_B = 2\tau + \beta_b \), \( \alpha_C = 2k + \beta_b \), \( S_a = \frac{1}{2} \alpha \) and \( S_b = \frac{1}{2}(\alpha - \beta_b) \). \( \square \)

Based on Proposition 1, we can obtain the following Proposition 2 and Proposition 3 to feature the optimal additional production and the optimal total production, respectively.

**Proposition 2.** Given a safety production level \( \tau \), the capacity \( k \) and the demand realization \( \alpha \), the optimal additional production at stage II in each market period is

\[
q_{II} = \begin{cases} 
0 & 0 \leq \alpha < \alpha_B \\
q_{IIb} & \alpha_B \leq \alpha < \alpha_C \\
k - \tau & \alpha_C \leq \alpha 
\end{cases}
\]

where \( \alpha_B = 2\tau + \beta_b \), \( \alpha_C = 2k + \beta_b \) and \( q_{IIb} = \frac{1}{2}(\alpha - \beta_b) - \tau \). \( \square \)

**Proposition 3.** Given a safety production level \( \tau \), the capacity \( k \) and the demand realization \( \alpha \), the optimal production of a firm in each market period is

\[
q = \begin{cases} 
\tau & 0 \leq \alpha < \alpha_B \\
q_b & \alpha_B \leq \alpha < \alpha_C \\
k & \alpha_C \leq \alpha 
\end{cases}
\]

where \( \alpha_B = 2\tau + \beta_b \), \( \alpha_C = 2k + \beta_b \) and \( q_b = \frac{1}{2}(\alpha - \beta_b) \). \( \square \)

On the basis of Propositions 1–3, we can derive the optimal ex-post profit of a firm under various situations, as stated in Proposition 4.

**Proposition 4.** Given a safety production level \( \tau \), the capacity \( k \) and the demand realization \( \alpha \), the optimal ex-post profit of a firm in each market period is

\[
\pi = \begin{cases} 
\pi_A & 0 \leq \alpha < \alpha_B \\
\pi_B & \alpha_B \leq \alpha < \alpha_C \\
\pi_C & \alpha_C \leq \alpha 
\end{cases}
\]

where \( \alpha_A = 2\tau \), \( \alpha_B = 2\tau + \beta_b \), \( \pi_A = \frac{1}{2} \alpha^2 \), \( \pi_B = \tau(\alpha - \tau) \), \( \pi_C = \frac{1}{2}(\alpha - \beta_b)^2 + \tau \beta_b \) and \( \beta = k(\alpha - k - \beta_b) + \tau \beta_b \). \( \square \)

To get an intuitive understanding, we plot Fig. 4 to represent Propositions 1–3 and to demonstrate: (1) how the operational decisions including additional production, total production and targeting sales are affected by the marketing environment and (2) the inter-relationship between these operational decisions.

Fig. 4 shows that as the demand realization \( \alpha \) increases, we have four different situations. Here, with a given \( \alpha \), the optimal decisions include the optimal sales, the optimal additional production and the optimal total production. It is noted that since the firm makes a safety production level \( \tau \) before knowing the demand information, it will use the safety production level to respond to the demand first. We summarize the four situations below:

- **Situation I:** When the demand is very low within \( \alpha \in [0, 2\tau] \), then the firm uses the safety production to meet the demand. With a given demand realization \( \alpha \), the optimal sales is \( s = \frac{1}{2} \alpha \) and the

![Fig. 3. Consequential events in each market period.](image)

![Fig. 4. Optimal productions and sales in a monopoly model.](image)
optimal ex-post profit is $\frac{1}{2}x^2$. Under this situation, the additional production is zero, i.e., $q_{II} = 0$, and the total production is the safety production $\tau$.

- **Situation II:** When demand is relatively low, i.e., $\alpha \in (2\tau, 2\tau + \beta\delta)$, then the firm calculates that the optimal sales are $(\tau + \frac{1}{2}\beta\delta)$ if the production cost is not considered. Under this situation, the safety production level is not enough to obtain the optimal revenue. It is noted here that the production cost of the safety production occurred before the firm knows the actual demand. Therefore, when the firm uses the safety production to meet the demand, the firm does not need to consider the production cost. However, if the firm decides to make the additional production after observing the actual demand, it needs to balance between the revenue by selling more products and the production cost of the addition products. It turns out that the net profit decreases if the firm makes any additional product under this situation. As a result, the firm will not produce any additional products and it will put all products which it already has into the market. Therefore, under this situation, the optimal additional production is zero; the total sales is $\tau$; the optimal total production is $\tau$ and the optimal ex-post profit is $\pi(\tau - \tau)$.

- **Situation III:** When the demand is relatively high, i.e., $\alpha \in (2\tau + \beta\delta, 2\tau + \beta\delta)$, the firm considers making additional production. With full considerations of the competitive price and the balance between the revenue and the production cost, the firm’s optimal additional production amount is $q_{II} = \frac{1}{2}(x - \beta\delta) - \tau$. It should be noted that there is no necessary for the firm to produce more after knowing the demand information, the firm will put its entire inventory and the additional production produced into the market. Therefore, the best response sales to different demand realization $\alpha \in (2\tau + \beta\delta, 2\tau + \beta\delta)$ is the sum of the safety production and the additional production, i.e., $s_b = \tau + q_{II} = \frac{1}{2}(x - \beta\delta)$. The optimal total production equals the sales, i.e., $q = s_b = \frac{1}{2}(x - \beta\delta)$. Accordingly, the optimal ex-post profit is $\pi = (x - s_b)s - q_{II}b = \frac{1}{2}(x - \beta\delta)^2 + \tau\beta\delta$. We summarize the four situations in Table 1.

4.2. Capacity decision stage

At capacity-decision stage, a firm determines its capacity to maximize its expected profit of the whole planning horizon embracing $n$ market periods. The capacity is also the maximum of a firm’s production ability. To facilitate the presentation, we define $\eta = C_1/C_0$, $C_0 > 0$. Then the financial budget constraint $C_1/k \leq C_0$ is equivalent to $k \leq \eta$, which is presented in terms of $\eta$. The capacity decision can be formulated as

$$\text{Max } II(k) = n\int_0^\infty (x - s)f(x)dx - \int_0^\infty q_{II}bf(x)dx - \tau\beta\delta - C_1k,$$

s.t. $\tau \leq k \leq n\eta$. \hfill (3)

In (3), $s$ and $q_{II}$ are the optimal solutions in the sales stage and production stage II, respectively. $I_{C_0}^\alpha$ is $s = x - \theta(x)$ if $d(x)$ is the expected total revenue in each market period. The term $\int_0^\infty q_{II}bf(x)dx + \tau\beta\delta k$ is the expected total production cost in each market period. $C_1k$ is the total capacity cost of the entire planning horizon. The optimal capacity and production quantity of a firm with a given $\tau$ is provided by Proposition 5 below. To facilitate the presentation, we set $A_0 = n\int_0^\infty \frac{1}{2}x^2f(x)dx + \int_0^\infty (x - \tau)\theta(x)dx + \tau\beta\delta f(x_\delta) - \tau\beta\delta$.

**Proposition 5.** In a monopoly model with given the safety production level $\tau$, we have:

(i) If $C_1 \leq C_0$, then $k = \tau$. $II = A_0 + n\int_0^x (x - \tau - \beta\delta)\theta(x)dx - C_1k$;

(ii) If $C_1 > C_0 < C_2$, then $k = \frac{1}{2}X(C_1/n - \beta\delta)$, the expected profit is $II = A_0 + n\int_0^x \frac{1}{2}(x - \beta\delta)^2\theta(x)dx + \int_0^x \frac{1}{2}(x - \beta\delta)^2\theta(x)dx$;

(iii) If $C_0 \leq C_1 \leq C_2$, then $k = \eta$, the expected profit is $II = A_0 + n\int_0^x \frac{1}{2}(x - \beta\delta)^2\theta(x)dx + \int_0^x \frac{1}{2}(x - \beta\delta)^2\theta(x)dx - C_1\eta$, where $C_1 = nl(2\tau + \beta\delta)$ and $C_2 = nl(2\eta + \beta\delta)$.

Proposition 5 provides the optimal capacity which gains the maximum of the expected profit over the planning horizon under different costing environments. There are three situations as stated in Proposition 5:

(i) When the capacity cost over the horizon is larger than an upper threshold, i.e., $C_1 < C_2$, $C_1 = nl(2\eta + \beta\delta)$, then the optimal capacity is the lower bound of the capacity, i.e., $k = \tau$. Under this situation, a firm will not invest in any excess capacity due to the high capacity cost.

(ii) When the capacity cost over the horizon is lower than another lower threshold, i.e., $C_0 < C_1 < C_2 = nl(2\eta + \beta\delta)$, then the optimal capacity is the upper bound of the capacity, i.e., $k = \eta$. Under this situation, since the capacity cost is relatively very low, the firm will build its capacity as much as possible. Therefore, the firm will build in the maximum of the capacity, which is determined by its financial budget constraint.

(iii) When the capacity cost is in the middle of the two thresholds, i.e., $C_0 < C_1 < C_2$, then the optimal capacity is $k = \frac{1}{2}X(C_1/n - \beta\delta)$, which is between the lower and upper bounds of the capacity, i.e., $\tau < k < \eta$.

It is interesting to note that the expected profits under three situations all include $A_0 = n\int_0^\infty \frac{1}{2}x^2f(x)dx + \int_0^\infty \frac{1}{2}(x - \tau)\theta(x)dx + \tau\beta\delta f(x_\delta) - \tau\beta\delta$. This indicates that $A_0$ occurs in every market period and it is not related to the capacity $k$ and capacity cost. Interestingly and surprisingly, the optimal capacity ($k$) is

<table>
<thead>
<tr>
<th>Range of $\alpha$</th>
<th>$\alpha \in (0, 2\tau)$</th>
<th>$\alpha \in (2\tau, 2\tau + \beta\delta)$</th>
<th>$\alpha \in (2\tau + \beta\delta, 2k + \beta\delta)$</th>
<th>$\alpha \in (2k + \beta\delta, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales $s$</td>
<td>$\frac{1}{2}x$</td>
<td>$\tau$</td>
<td>$\frac{1}{2}(x - \beta\delta)$</td>
<td>$k$</td>
</tr>
<tr>
<td>Safety production</td>
<td>$\tau$</td>
<td>$\tau$</td>
<td>$\tau$</td>
<td>$\tau$</td>
</tr>
<tr>
<td>Additional production</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2}(x - \beta\delta) - \tau$</td>
<td>$k - \tau$</td>
</tr>
<tr>
<td>Total production</td>
<td>$\tau$</td>
<td>$\tau$</td>
<td>$\frac{1}{2}(x - \beta\delta)$</td>
<td>$k$</td>
</tr>
<tr>
<td>Expected profit in one period</td>
<td>$\frac{1}{2}x^2$</td>
<td>$\tau(x - \tau)$</td>
<td>$\frac{1}{2}(x - \beta\delta)^2 + \tau\beta\delta$</td>
<td>$k(x - k) + \tau\beta\delta$</td>
</tr>
</tbody>
</table>

Table 1: Sales, production and expected profit in one market period under various $\alpha$. 

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independent of the unit production cost at safety production stage \( (\beta_s) \) in all three situations. Since the safety production level \( \tau \) is given in the model, the total safety production cost is actually determined over the planning horizon as \( p_s^\tau \text{m} \). As the firm’s capacity \( k \) is always larger than or equal the safety production level \( \tau \), the total safety producing cost does not affect the optimal capacity. It is worth to note here that if the safety production level \( \tau \) is generated endogenously, then we can see from the later discussion that the optimal capacity will depend on \( \beta_s \).

With respect to the number of \( n \), we found that under the situations (i) and (iii) in Proposition 5, the optimal capacity \( k \) is independent of the number of markets \( n \). Under the situation (ii) in Proposition 5, the optimal capacity is affected by the number of markets \( n \). The following Proposition 6 characterizes the relationship between the optimal capacity \( k \) and the number of market periods \( n \).

**Proposition 6.** In a monopoly model with given \( \tau, \eta, \beta_s, \beta_o \) and \( C_s \), set \( C_1 = nL(2\tau + \beta_o) \) and \( C_2 = nL(2\eta + \beta_s) \) When \( C_1 < k < C_2 \), within the range \( k \in [\tau, \eta] \) we have:

(i) the optimal capacity \( k \) is increasing in the number of market periods \( n \), i.e., \( k^{(1)}(n) > 0 \);
(ii) the optimal capacity \( k \) is concave in the number of market periods \( n \), i.e., \( k^{(2)}(n) < 0 \).

To obtain an intuitive understanding of the effect of \( n \) on the optimal capacity \( k \), we conduct a numerical example with the exponential distribution \( f(x) = \lambda e^{-\lambda x}, x \in [0, \infty) \) for some \( \lambda > 0 \). Table 2 shows the basic parameters of the problem.

As we see from Fig. 5, the optimal capacity \( k \) is concavely increasing in the number of market periods \( n \). As the number of market periods \( n \) increases, the optimal capacity also increases, but with a smaller speed. This indicates that the marginal capacity for each market period becomes less as \( n \) increases.

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### 5. Determination of safety production level and planning horizon

In the previous discussion, both the length of planning horizon and the safety production level are exogenously given. To run a business, we need to determine their values so that the firm’s expected profit is maximized, or at least the firm has a positive expected profit. In this section, we will settle these important issues.

#### 5.1. Endogenous safety production level \( \tau \)

In this section, we assume that the planning horizon is given, and will determine the optimal safety production level \( \tau \). According to Section 4, over the planning horizon of \( n \) market periods, the optimal safety production level \( \tau \) can be obtained by solving the following mathematical programming problem:

\[
\max_{\tau} H(\tau) = n \left[ \int_{0}^{2\tau \beta_o} f(x) \, dx + \int_{2\tau \beta_o}^{3\tau \beta_o} (x - \tau)^2 f(x) \, dx + \int_{3\tau \beta_o}^{\infty} (x - \tau)^3 f(x) \, dx \right] + \frac{\beta_o}{2} \int_{0}^{2\tau \beta_o} f(x) \, dx + \int_{2\tau \beta_o}^{\infty} k(x - \beta_o) f(x) \, dx - C_s k \quad \text{s.t.} \quad \tau \leq k \leq \eta.
\]

In (4), \( k \) is the optimal capacity at the capacity stage.

<table>
<thead>
<tr>
<th>( \beta_s )</th>
<th>Unit capacity cost ( C_s )</th>
<th>Distribution ( f(x) )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>4</td>
<td>( f(x) = \lambda e^{-\lambda x} )</td>
<td>0.1</td>
</tr>
</tbody>
</table>

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**Based on Proposition 5**, there are three situations of the optimal capacity. In the first situation when \( nL(2\tau + \beta_o) \leq C_s \), we have \( k = \tau \). Therefore, we need to substitute \( k = \tau \) in to the expected profit to find out the optimal \( \tau \) to maximize the \( H(\tau) \). In the second situation in Proposition 5, when \( nL(2\eta + \beta_s) < C_s < nL(2\tau + \beta_o) \), we have \( k = \frac{1}{2} X(C_s/n - \beta_s) \), which is independent of \( \tau \). In the third situation when \( C_s \leq nL(2\eta + \beta_s) \), we have \( k = \eta \), which is independent of \( \tau \). Therefore, when \( C_s < nL(2\tau + \beta_o) \), the optimal capacity \( k \) is independent of \( \tau \) and \( k > \tau \). As a result, the optimal \( \tau \) over the whole planning horizon is the same as the optimal \( \tau \) in each market period. After considering these three situations, we combine the discussion and derive the optimal endogenous \( \tau \) as stated in Proposition 7 below. To facilitate the discussion, in the following, we let \( \tau_2 = \frac{1}{2} \left[ X\left(\frac{C_s}{n} - \beta_s\right) \right] \) if \( \mu > \frac{\beta_o}{2} \); and \( \tau_3 = \frac{1}{2} X\left(\frac{C_s}{n} + \beta_s\right) \) if \( \mu > \frac{\beta_o}{2} + \beta_o \).

**Proposition 7.** In a monopoly model with given \( \beta_o, \beta_s, n, C_s \) and \( \eta \), a firm’s optimal endogenous safety production level \( \tau \) is as follows:

Case 1: When \( n < C_s/L(\beta_o) \), we have two sub-situations:

(i) If \( \mu < \frac{\beta_o}{2} + \beta_o \), then the optimal \( \tau = 0 \);
(ii) If \( \mu > \frac{\beta_o}{2} + \beta_o \), then the optimal \( \tau = \min(\tau_2, \eta) \);

Case 2: When \( n \geq C_s/L(\beta_o) \), we have three sub-situations:

(i) If \( \mu - L(\beta_s) - \beta_o > 0 \), then the optimal \( \tau = 0 \);
(ii) If \( \mu - L(\beta_s) - \beta_o < 0 \) and \( \tau_2 < \tau_3 \), then the optimal \( \tau = \min(\tau_2, \eta) \);
(iii) If \( \mu - L(\beta_s) - \beta_o < 0 \) and \( \tau_3 < \tau_2 \), then the optimal \( \tau = \min(\tau_2, \tau_3) \), where \( \tau_2 \) is the unique solution of the equation \( L(2\tau_2) - L(2\tau_3 + \beta_o) = \beta_o \).
high, which satisfy the relationship that \(\mu - L(b_0) - \beta_a > 0\), then the optimal safety production level \(\tau\) is determined by the comparison between \(\tau_a\) and \(\tau_b\). Under both situations \(\tau_a < \tau_b\) and \(\tau_a \geq \tau_b\), the optimal safety production level \(\tau\) is larger than zero. This means that the firm balances the benefit from exploring the flexibility and the higher production cost at the additional production stage. Therefore, the optimal safety production level is at a moderate level, rather than zero.

5.2. Number of market periods

Based on previous analyses, it is found that a firm’s optimal safety production level \(\tau\) depends on the number of market periods \(n\). Consequently, the firm’s operational decisions in each period are functions of the number of market periods \(n\). The length of planning horizon is critical to a firm’s operational decisions and profits. When a firm makes its capacity investment covering \(n\) market periods, it has to take into account all the following operational decisions to design the length of planning horizon. Then the number of market periods becomes an endogenous variable of the capacity planning. The following Proposition 8 provides the property of endogenous number of market periods \(n\).

**Proposition 8.** In a monopoly model with given \(\beta_a, \beta_0, C_a\), a firm gets increasing positive expected profit when \(n > n^*\), where \(n^* = \max(\frac{h}{L}, \frac{g}{L})\); and zero profit, otherwise, where \(n\) is the firm’s endogenous number of market periods in the planning horizon. \(\square\)

Proposition 8 reveals that the unique threshold of the number of market periods exists under a given costing environment. The threshold is the minimum number of market periods the planning horizon should include, so that the firm gets a positive expected profit. The expression shows that the threshold \(n^*\) is determined by four parameters: the capacity cost, the safety production cost, the additional production cost and the mean of distribution of \(x\). Generally speaking, the threshold \(n^*\) is inversely related to the costing parameters. As the capacity cost increases, the planning horizon needs to be designed longer to embrace more market periods, so that the firm is able to get a positive profit. With an increasing production cost, the firm will make less profit in each period, and therefore, the designed planning horizon needs to be longer too. Only when the capacity investment covers more than \(n\) market periods, the firm has incentive to make the investment. In other words, the firm benefits from the investment only when the planning horizon is longer than the threshold, as the firm needs to balance between the capacity cost and its revenue over the entire planning horizon. If the planning horizon is too short, the revenue earned cannot cover the total cost. This is one of the critical concerns of making capacity investment for a firm. In addition, the life time of the capacity investment, e.g., the life of the equipment bought at the beginning of the business, should at least be \(n\) market periods so that it is worth to make the capacity investment.

**Proposition 9.** In a monopoly model with given \(\beta_a, \beta_0, C_a\), the endogenous designed planning horizon \(n\) and the endogenous safety production level \(\tau\) are:

\[(i) \text{ When } \mu - L(b_0) - \beta_a > 0, \text{ the designed planning horizon should be longer than } C_t/(\mu - \beta_a) \text{ years; further, the endogenous safety production level depends on the designed number of market periods:} \]

\[(a) \text{ when } C_t/(\mu - \beta_a) < n < C_t/L(b_0), \text{ the optimal safety production } \tau \text{ is } \min(\tau_a, \tau_b); \]

\[(b) \text{ when } C_t/L(b_0) \leq n, \text{ we have: when } \tau_a < \tau_b, \text{ the optimal safety production level } \tau = \min(\tau_a, \tau_b); \text{ when } \tau_a \geq \tau_b, \text{ the optimal safety production level } \tau = 0. \]

\[(ii) \text{ When } \mu - L(b_0) - \beta_a < 0, \text{ the designed planning horizon should be longer than } C_t/L(b_0) \text{ years; the optimal safety production level } \tau = 0. \]

**Proposition 9** gives explicit results of the final stage to solve the problem. Based on this proposition, a firm can determine its designed planning horizon and the degree of the firm’s flexibility, which is reflected by the safety production level of the firm. The results highlight the importance of evaluating the production process before designing the planning horizon. When the safety production cost is relatively low or the additional production cost is high (i.e., \(\mu - L(b_0) - \beta_a > 0\)), the safety production is the crucial factor determining the designed planning horizon (i.e., the designed number of market periods \(n > C_t/(\mu - \beta_a)\)). The firm can benefit from producing products at the safety production stage and using the inventory to meet the demands. Therefore, a certain level of the safety production is better-off for the firm. Under the other situation where the safety production cost is high (or the additional production cost is relatively low), the full flexibility (i.e., \(\tau = 0\)) is beneficial to the firm. As a result, the additional production is the key to the designed planning horizon (i.e., the designed number of market periods \(n=C_t/L(b_0)\)).

6. Discussions

As discussed in Section 2, our paper can be regarded as a partial flexibility model. With suitable setting of the parameters, the model can be reduced to the case of purely make-to-order and the case of purely make-to-stock with holdback, respectively. The model can be considered as a generalization of the two extremes.

6.1. Make-to-order modeling setting

When the safety production level \(\tau = 0\), the production stage setting is an ideal make-to-order situation. It means, with a given capacity \(k\), the firm makes its production until it receives the actual demand information. If we consider a single-period problem, i.e., \(n = 1\), then the resulting production decision is the same as the price and production postponement strategy in the study of Van Mieghem and Dada (1999). Such make-to-order model setting is also adopted by other relevant research, combining with other specific considerations, such as demand correlation (Bish et al., 2005), competition effect (Goyal & Netessine, 2007; Anupindi & Jiang, 2008), and effect of technology level (Yang et al., 2011).

6.2. Make-to-stock with holdback

When the safety production level \(\tau = k\), the production stage setting becomes a make-to-stock with holdback. With an installed capacity \(k\), the firm transfers all of its capacity into production before observing the demand information. After knowing the demand information, the firm decides the optimal sales to maximize its profit at the production stage. When \(n = 1\), this production strategy is the same as the price postponement with holdback strategy in Van Mieghem and Dada (1999). The model setting is also investigated by Yang et al. (2011) with the consideration of technology level.

7. Conclusions

In this paper, we consider the flexible capacity strategy over multiple market periods under demand uncertainty. We formulate a firm’s flexible capacity strategy with full considerations of a few issues: (i) to deal with fluctuating demand, the firm is trying to build flexibility into its operation process; (ii) to keep the market competitive.
share and its customers to remember its products, the firm will have a lowest production level; and (iii) to consider the financial affordability, the firm will build its capacity restraint to financial budget. In the model, a firm makes its capacity decision at the beginning of the planning horizon of n market periods. In each market period, a firm goes through safety production stage, additional production stage and optimal sales stage.

First, we present a general flexible capacity model which is able to present a firm’s ability in adjusting its production in response to the demand uncertainty. We consider a firm has a safety production level and an additional production level. The analytical results show that as the demand increases, the firm’s operational decisions in each period consist of four pieces. (i) When demand is very low, the firm will put part of inventory into the market and will not make any additional production. (ii) When demand is relatively low, the firm will put all products produced at the safety production stage into the market, but will not make any additional production. (iii) When demand is relatively high, the firm will make an optimal additional production and put all products produced into the market. (iv) When demand is very high, the firm will use all its capacity to explore the maximum production.

Second, we analytically find the optimal capacity over multiple market periods under demand uncertainty, with consideration of its financial constraints. There are two thresholds of the unit capacity cost. When the capacity unit cost is lower than the lower threshold $C_L$, the firm will build the capacity as high as possible under the financial budget constraint. When the capacity unit cost is higher than the upper threshold $C_H$, the firm will build the capacity at its lowest operation level, and use all capacity to maintain a safety production level in each market period. Then, we investigate the impact of the number of market periods on the optimal capacity. It is demonstrated that as the number of market periods $n$ increases, the optimal capacity also increases, but with a smaller slope.

Third, we explore the endogenous safety production level $\tau$ in the model. Interestingly, we find that a higher flexibility is not always better-off for the firm. When the safety production cost is relatively high, a firm will benefit from keeping the safety production level as low as possible to get more flexibility. When the safety production cost is relatively low but the unit cost of additional production is relatively high, there exists an optimal flexibility (which is reflected by the optimal safety production level $\tau$), to which the firm could maximize its expected profit over the planning horizon.

Forth, we verified a threshold which is the minimum number of market periods the planning horizon should embrace to get a positive expected profit. Only when the designed planning horizon is longer than the threshold, the firm could benefit from the capacity investment. Furthermore, the threshold is determined by capacity and production costs. When the capacity cost increases, the firm needs to design a longer planning horizon to make up for a higher cost. When the production costs increase, the expected profit of each period will decrease, and therefore, the firm needs to lengthen the planning horizon. Thereby, the threshold value suggests the firm to ensure a sufficiently long planning horizon.

In this paper, we only consider the monopoly situation. However, one of the biggest challengers in a competitive market comes from the competitors. A firm has to consider not only its own decisions, but also decisions of its rivals. Such consideration will affect a firm’s decisions over the entire planning horizon. The competition is an interesting and challenging direction for future research. Another direction deserved attention is the financial constraint structure. In this paper, we have simplified the financial constraints as the budget of the total capacity cost. It should be noticed that the financial constraints are much complex than a budget. It is interesting to explore more influential perspectives of financial constraints in a firm’s capacity planning. The third direction is the inventory structure. In this paper, the inventory is not delivered into the next market period when there is inventory left. However, in reality, for some products with a relatively longer life cycle, the firm may benefit from holding the inventory till the next market period. Such problem would be a very interesting and practical issue to investigate.

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Appendix A

Proof of Proposition 1. We discuss two situations.

Situation 1, when $0 < s \leq \tau$: By (1), we can derive the optimal sales $s = \left\{ \begin{array}{ll} \alpha/2 & 0 < \alpha < 2\tau \quad \tau > 2 \tau \leq \alpha \\ \end{array} \right.$ . Situation 2, when $\tau < s \leq k$: By (2), the first- and second-order derivatives are $\pi^{(1)}(s) = -2s - \beta_s$, $\pi^{(2)}(s) = -2$. So, $\pi(s)$ is convex in $s$ and its unconstrained optimal solution is $s_b = \frac{1}{2}(\alpha - \beta_s)$. Note that $\tau < s \leq k$. Hence, the optimal production is $s = \left\{ \begin{array}{ll} \tau & 0 < \tau < s_b \\ s_b & s_b \leq \tau < s_c , \text{ where } s_b = 2\tau + \beta_s, \\ \tau & \tau < s_c \leq \alpha_c = 2\tau + \beta_s; \text{ Combining situations 1 and 2 and } \\ & \text{and comparing the corresponding profits, the result follows.} \right.$

Proof of Proposition 5. By Propositions 1 and 2, we substitute $s$ and $d_s$ into (3). Then, the expected profit can be represented as:

$$\Pi(k) = A_0 + n \left[ f_{\alpha_k}^C (x - \beta_s) f(x) dx + \int_{\alpha_k}^\infty k(x - k - \beta_s)f(x)dx \right] - C_k,$$

where $A_0 = n \left[ f_{\alpha_k}^\infty \frac{1}{2} x^2f(x) dx + \int_{\alpha_k}^\infty (x - \tau)f(x)dx + \tau \beta_s F(\alpha_k) - \tau \beta_s \right]$ is independent of $k$. With respect to $k$, the first- and second-order derivatives are $\Pi^{(1)}(k) = n \cdot L(2\tau + \beta_s) - C_k$; $\Pi^{(2)}(k) = -2nL(2\tau + \beta_s) < 0$. Thus, $\Pi(k)$ is concave in $k$. As $k \rightarrow \infty$, $\Pi^{(1)}(k) \rightarrow -C_k \rightarrow 0$. Therefore, the unconstrained optimal capacity satisfies $\Pi^{(1)}(k) = 0$, i.e., $k = \frac{1}{2}[X(C_i/n) - \beta_s]$. With the constraint $\tau < k < \eta$, we have three cases:

Case I: If $X(C_i/n) \leq 2\tau + \beta_s$, i.e., $nl(2\tau + \beta_s) < C_i$.

then $k = \tau$, the expected profit is $\Pi = A_0 + n \int_{\alpha_k}^\infty (x - \tau - \beta_s)f(x)dx - C_k$.

Case II: If $2\tau + \beta_s < X(C_i/n) < 2\eta + \beta_s$, i.e., $nl(2\eta + \beta_s) < C_i$.

then $k = \frac{1}{2}[X(C_i/n) - \beta_s]$, the expected profit is $\Pi = A_0 + n \int_{\alpha_k}^\infty \left( \frac{1}{2}(x - \beta_s) f(x) dx + \int_{\alpha_k}^{\alpha_c} \frac{1}{2}[X(C_i/n) - \beta_s] f(x) dx \right)$.

Case III: If $2\eta + \beta_s < X(C_i/n)$, i.e., $\eta > \infty$.

then $k = \eta$, the expected profit is $\Pi = A_0 + n \int_{\alpha_k}^\infty \left( \frac{1}{2}(x - \beta_s) f(x) dx + \int_{\alpha_k}^{\alpha_c} \eta (x - \eta - \beta_s) f(x) dx \right) - C_i\eta$.

Then the results follow.

Proof of Proposition 6. From Proposition 5, we have the optimal capacity $k = \frac{1}{2}[X(C_i/n) - \beta_s]$ when $C_i < C_s < \infty$. With respect to $n$, we can derive that $k^{(1)}(n) = \frac{C_i}{n} > 0$ and $k^{(2)}(n) = -\frac{C_i}{n^2} > 0$. Then the results follow.

Proof of Proposition 7. Based on Proposition 5, there are three situations.
Situation 1: If $C_k \leq C_a$, then $k = \tau$, $\Pi = A_0 + n \int_{s+b_1}^{\infty} f(x)dx - C_0$, where $C_k = n(2\tau + b_0)$. By substituting $k = \tau$ into $\Pi(\tau)$, we have $\Pi(\tau) = \left[ \int_{2\tau}^{\infty} f(x)dx + \int_{0}^{2\tau+b_1} f(x)dx \right] \times \tau$ and $\Pi(1) = n(\tau + b_0) - C_0$. Therefore, $\Pi(\tau) = \left( n(\tau + b_0) - C_0 \right) - \left( n(\tau + b_0) - C_0 \right) = 0$. Since $\Pi(\tau) = 0$, $\Pi(1) = 0$, and $\Pi(2) = 0$, the condition $C_k \leq C_a$ is equivalent to $\tau = \tau_a$. Therefore, we need $\eta \geq \tau_a$ in this case. Note that $\Pi(\tau) = n(\tau + b_0) - C_0$ is 0 equivalent to $\tau = \tau_a$. Therefore, the optimal $\tau = \text{median}(\tau_a, \tau_b, \eta)$.

Situation 2 and Situation 3: In situation 2, when $nL(2\eta + b_0) < C_k \leq nL(2\tau + b_0)$, i.e., $\eta > \tau_a > \tau$, we have $k = \frac{1}{2}X(C_k/n - b_0) - \tau_a$, which is independent of $\tau$. In situation 3, when $C_k \leq nL(\eta + b_0)$, we have $k = \eta$, which is also independent of $\tau$. Therefore, when $C_k \leq nL(\eta + b_0)$, the optimal capacity $k$ is independent of $\tau$ and $k > \tau$. It is noted that in situation 3, $k = \eta > \tau_a$, and in situation 3, $k = \eta < \tau_a$. Therefore, $k = \min(\tau_a, \eta)$. As a result, the optimal $\tau$ over the whole planning horizon is equivalent to that in each market period. For each market period, the optimal $\tau$ is formulated as below:

$$\max \pi(\tau) = \int_{\frac{1}{4}}^{\frac{1}{4}} x^2 f(x)dx + \int_{\frac{1}{2}}^{\frac{1}{2}} + b_0 \tau - \tau_a + \int_{\frac{1}{2}}^{\frac{1}{2}} f(x)dx + \int_{\frac{1}{2}}^{\frac{1}{2}} \Delta(x)dx \leq \tau_a$$

with respect to $\tau$. $\pi(\tau)$ is concave in $\tau$. By condition $C_k \leq nL(\eta + b_0)$, we have $0 < \tau < \min(\tau_a, \eta)$, implying $\tau_a > 0$. As $\tau \to 0^-$, $\pi(\tau) \to \min(\tau_a, \eta)$. Therefore, $\tau^* = \min(\tau_a, \eta)$. With respect to $\tau$, $\pi(\tau)$ is concave in $\tau$. By condition $C_k \leq nL(\eta + b_0)$, we have $0 < \tau < \min(\tau_a, \eta)$, implying $\tau_a > 0$. As $\tau \to 0^-$, $\pi(\tau) \to \min(\tau_a, \eta)$. Therefore, $\tau^* = \min(\tau_a, \eta)$.

Proof of Proposition 8. Based on Propositions 5 and 7, we have the following situations to discuss, respectively. Case 1: When $n \leq C_k/L(b_0)$, then based on Proposition 7, we have two sub-cases:

(i) If $\mu > \frac{\Delta}{\Delta + \beta}$, i.e., $n < C_k / \max(\beta, L(b_0))$, then the optimal $\tau = 0$. Based on proof of Proposition 8, situation 1 occurs. So, we have $k = \tau = 0$. Therefore, $\Pi = 0$.

(ii) If $\mu > \frac{\Delta}{\Delta + \beta}$, i.e., $0 < \frac{\Delta}{\Delta + \beta} < n < C_k / \max(\beta, L(b_0))$, then the optimal $\tau = \min(\tau_a, \eta)$. Based on proof of Proposition 7, situation 1 occurs, $\tau^* = \min(\tau_a, \eta)$. (a) If $\eta \leq \tau_a$, i.e., $n(\eta - \beta_a) > C_k$, then $k = \tau_a$. $\Pi = \int_{\eta}^{\infty} f(x)dx + \int_{\eta}^{\infty} f(x)dx - \eta \beta_a - \frac{C_k}{\eta - \beta_a} = 0$. Therefore, $\Pi(\tau) = 0$, $\Pi(1) = 0$, and $\Pi(2) = 0$. Since $\Pi(\tau) = 0$, $\Pi(1) = 0$, and $\Pi(2) = 0$, the condition $C_k \leq C_a$ is equivalent to $\tau = \tau_a$. Therefore, we need $\eta \geq \tau_a$ in this case. Note that $\Pi(\tau) = n(\tau + b_0) - C_0$ is 0 equivalent to $\tau = \tau_a$. Therefore, the optimal $\tau = \text{median}(\tau_a, \tau_b, \eta)$.

Based on the condition $0 < \tau < \eta$, the results follow. □
Therefore, \( \Pi \) is an increasing function of \( n \) when \( n(L(2\eta) - \beta_0) \geq C_k(b) \), by Proposition 8, the designed plan.

Therefore, \( \Pi = n \left[ \int_0^{s_{2g} \eta} \frac{1}{2} f(z)dx + \int_{s_{2g} \eta}^{s_{2g} \eta} \frac{1}{2} f(z)dx \right] + \frac{x_{2g} \eta}{2} f(x) dx + \frac{x_{2g} \eta}{2} f(x) dx + \frac{x_{2g} \eta}{2} f(x) dx > 0. \) Therefore, \( \Pi \) is an increasing function of \( n \) for all \( C_k(b) \). Combine cases (a) and (b), we have that \( \Pi \) is an increasing function of \( n \) for \( 0 < \frac{C_k(b)}{4} < n < \frac{C_k(b)}{4} \).

Case sub-cases (i) and (ii), we have

Case 1-i: If \( \mu - L(b_0) > \beta_0, \) then the sub-case (ii) does not exist. The optimal \( \tau = k = \tau_0 \) and \( \Pi = 0 \) for \( n \leq C_k(b_0) \). Case 1-ii: If \( \mu - L(b_0) < \beta_0, \) then \( \tau = k = \tau_0 \) and \( \Pi = 0 \) for \( n \leq C_k(b_0) \). Situation 3 occurs. For both situations 2 and 3, we have

(a) If \( \mu - L(b_0) < \beta_0, \) then \( \tau = k = \tau_0 \) and \( \Pi = 0 \) for \( n \leq C_k(b_0) \). Situation 3 occurs. For both situations 2 and 3, we have

(b) If \( \mu - L(b_0) > \beta_0, \) then when \( n \leq \frac{C_k(b)}{4} \), the optimal \( \tau = k = \tau_0 \) and \( \Pi = 0 \) for \( n \leq C_k(b_0) \). Situation 3 occurs. For both situations 2 and 3, we have

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