Approximate homotopy symmetry and infinite series solutions to the perturbed mKdV equation

Xiaoyu Jiao a,*, Ying Zheng b, Bo Wu a

a School of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing 210046, China
b School of Science, Nanjing University of Posts and Telecommunications, Nanjing 210046, China

ABSTRACT

We devise a homotopy model for the perturbed mKdV equation with weak fourth order dispersion and weak dissipation, and demonstrate that the related approximate equations lead to similarity reduction equations and similarity reduction solutions of different orders which are formally coincident, respectively. Series reduction solutions for the perturbed mKdV equation are thus derived. Painlevé II type equations or hyperbolic secant function solutions are obtained for zero-order similarity reduction equations. Higher order similarity reduction equations are linear variable coefficients ordinary differential equations.

1. Introduction

Nonlinear partial differential equations arise from various fields of natural science and engineering, and are generally difficult to solve explicitly. Lie group theory [1–3], a sophisticated technique, concerning the invariance of partial differential equations under Lie group transformation, can be employed to construct group-invariant solutions. Although finding exact analytic solutions is significant, many exact analytic solutions can not be used to describe phenomena in reality. So the research of approximate analytic solutions is irreplaceable. Perturbation skills [4–6] are efficacious to construct approximate analytic solutions, which are often more valuable.

Approximate symmetry method is based on perturbation theory and Lie group theory. If we consider the perturbation form of Lie group generator and the related approximate invariance of partial differential equations, we are using the approximate symmetry method by Baikov et al. [7,8]. If we first decompose an equation into a series of equations by the perturbation form of the dependent variable, and consider the exact symmetry of the resulted equations, we have the approximate symmetry method by Fushchich and Shtelen [9]. The second method is superior to the first one owing to the comparison in Refs. [10,11] and is further extended to construct series reduction solutions to perturbed partial differential equations in Refs. [12–16].

The combination of perturbation theory and homotopy theory brings in the homotopy analysis method by Liao [17–23] which can be used to decompose nonlinear equations into infinite linear equations. This method is superior to perturbation theory, since it is applicable to non-perturbed problems, and can recover the series solutions by perturbation method, artificial parameter method [24], Delta-perturbation expansion method [25] and Adomian decomposition method [26].

The approximate homotopy symmetry method by Lou [27] obtained from the combination of homotopy theory and the approximate symmetry differs from the homotopy analysis method in that it decomposes nonlinear equations into infinite nonlinear equations. The solutions obtained by approximate symmetry method can also be retrieved by this method.

* Corresponding author.
E-mail address: jiaoxxy@yahoo.com.cn (X. Jiao).

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The first and critical step of the approximate homotopy symmetry method is the construction of homotopy model. For a nonlinear partial differential equation

\[ A(u) = A(x, t, u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0, \]  

(1)

where \( A \) is a nonlinear operator, \( u = u(x,t) \) is an undetermined function, and \( \{x,t\} \) are independent variables, we introduce a homotopy model

\[ H(u, q) = 0, \]  

(2)

with \( q \in [0,1] \) an embedding homotopy parameter. The above homotopy model has the property

\[ H(u, 0) = \mathcal{H}_0(u) \quad H(u, 1) = A(u), \]  

(3)

where \( \mathcal{H}_0(u) = 0 \) is a differential equation of which the solutions can be easily obtained.

We make an ansatz that the homotopy model (2) has the homotopy series solution

\[ u = \sum_{i=0}^{\infty} u_i q^i, \]  

(4)

where \( u_i \) solves the system

\[ O(q^0) : \mathcal{H}_0(u_0) = 0, \]  

(5a)

\[ O(q^1) : \mathcal{H}_0(u_0)u_1 + F_1(u_0) = 0, \]  

(5b)

\[ O(q^2) : \mathcal{H}_0(u_0)u_2 + F_2(u_0, u_1) = 0, \]  

(5c)

\[ \ldots \]  

\[ O(q^i) : \mathcal{H}_0(u_0)u_i + F_i(u_0, u_1, \ldots, u_{i-1}) = 0, \]  

(5d)

\[ \ldots \]

in which the operator \( \mathcal{H}_0(u_0) \) is defined as

\[ \mathcal{H}_0(u_0) f = \frac{\partial}{\partial q} \mathcal{H}_0(u_0 + \varepsilon f)|_{\varepsilon=0}, \]  

(6)

for arbitrary function \( f \), and all \( F_i \equiv F_i(u_0, u_1, \ldots, u_{i-1}) \) satisfy

\[ F_i = \frac{1}{i!} \frac{\partial^i}{\partial q^i} \mathcal{H}_0 \left( \sum_{k \neq i} u_k q^k, q \right)|_{q=0}. \]  

(7)

Then, the solutions of the original nonlinear system (1) read

\[ u = \sum_{i=0}^{\infty} u_i. \]  

(8)

For convenience, we often take the homotopy model as

\[ (1 - q)\mathcal{H}_0(u) + q\lambda A(u) = 0, \]  

(9)

with \( \lambda \neq 0 \) an auxiliary parameter. It is easily seen that Eq. (9) varies asymptotically from \( \mathcal{H}_0(u) = 0 \) to Eq. (1) as \( q \) goes gradually from 0 to 1. When \( \mathcal{H}_0 \) is fixed as a linear operator, the homotopy model (2) is just the usual one applied in Refs. [17–23].

2. The approximate homotopy symmetry reduction of perturbed mKdV equation

The perturbed mKdV equation

\[ u_t + 6\alpha u^2 u_x + u_{xxx} = \epsilon (u_{xx} + u_{xxxx}) \quad (a = \pm 1), \]  

(10)

arise in a number of places, such as models of shallow water on tilted planes [28]. Soliton perturbation property of the mKdV equation was analyzed in Refs. [29–31].

We consider two special forms of the above equation with weak fourth order dispersion and weak dissipation

\[ u_t + 6\alpha u^2 u_x + u_{xxx} = \epsilon u_{x^{(4+\delta)}}, \]  

(11)

where the subscript \( x^{(n)} \) means the differentiation with respect to \( x \) in \( n \)-order. \( u_{x^{(n)}} \) represents fourth order dispersion for \( \delta = 1 \) and dissipation for \( \delta = -1 \).

To apply the approximate homotopy symmetry method to Eq. (11), we take

\[ \mathcal{H}_0(u) = u_t + 6\alpha u^2 u_x + u_{xxx}, \]  

(12)

and replace \( 1 - \theta \) by \( \varepsilon \), the homotopy model can be reified as

\[ (1 - q\varepsilon)(u_t + 6\alpha u^2 u_x + u_{xxx}) - q\epsilon(1 - \theta)u_{x^{(4+\delta)}} = 0. \]  

(13)
It is easily seen that the homotopy model becomes the mKdV equation when \( q = 0 \).
Substituting the series solution ansatz (4) into the homotopy model (13) and collecting the coefficients of different powers of \( \epsilon \), we get a series of equations (approximate equations of different orders)

\[
u_{k+1} + 6\alpha \sum_{j=0}^{k} \sum_{i=0}^{j} u_{j-i} u_{i} + u_{k,xxx} - \epsilon(1-\theta) \sum_{i=0}^{k-1} (k-1-i)u_{i,x}^{i+1} = 0, \quad (k = 0, 1, \ldots),
\]

where \( u_{-1} = 0 \). All the quantities with negative subscripts are taken to zero in the following text.

To investigate similarity reduction of Eq. (14), we first write out the related linearized equations

\[
s_{k+1} + 6\alpha \sum_{j=0}^{k} \sum_{i=0}^{j} s_{j-i}s_{i} + s_{k,xxx} - \epsilon(1-\theta) \sum_{i=0}^{k-1} (k-1-i)s_{i,x}^{i+1} = 0, \quad (k = 0, 1, \ldots),
\]

where \( s_{k} \) are functions of \( x \) and \( t \). Eq. (15) means that Eq. (14) is invariant under the transformation \( u_{k} \to u_{k} + \epsilon \sigma_{k} \ (k = 0, 1, \ldots) \) with \( \epsilon \) an infinitesimal parameter.

It should be noted that \( \sigma_{k} \) in the linearized equations (15) can be specified as

\[
s_{k} = Xu_{k,x} + Tu_{k,t} - U_{k}, \quad (k = 0, 1, \ldots),
\]

where the functions \( X, T \) and \( U_{k} \) with respect to \( x, t \) and \( u_{k} \) \((k = 0, 1, \ldots)\) can be determined by finite equations in Eqs. (14)–(16).

When \( \theta = 1 \) (fourth order dispersion in Eq. (11)), confining the maximum of \( k \) to 2 in Eqs. (14)–(16), we see that the independent variables of \( X, T, U_{0}, U_{1} \) and \( U_{2} \) are accordingly restricted to \( x, t, u_{0}, u_{1} \) and \( u_{2} \). More than 700 determining equations are obtained by inserting Eq. (16) into Eq. (15), eliminating \( u_{0,xx}, u_{1,t} \) and \( u_{2,t} \) in terms of Eq. (14) and vanishing coefficients of different partial derivatives of \( u_{0}, u_{1} \) and \( u_{2} \).

To solve the determining equations, we first extract the simplest equations for \( T \)

\[
T_{x} = T_{u_{0}} = T_{u_{1}} = T_{u_{2}} = 0,
\]

from which we have \( T = T(t) \). Simplifying the determining equations by this condition, we get the simplest equations for \( X \)

\[
X_{u_{0}} = X_{u_{1}} = X_{u_{2}} = 0,
\]

with the solution \( X = X(x,t) \). Further simplifying the determining equations by this condition, we get the simplest equations for \( U_{0} \) and the solution

\[
U_{0,xx} = U_{0,x} = 0, \quad U_{0,u_{1}} = U_{0,u_{2}} = U_{0,u_{0},u_{1}} = U_{0,u_{0},u_{2}} = 0, \quad U_{0} = F_{1}(x,t)u_{0} + F_{2}(x,t) + F_{3}(t)u_{1}.
\]

Considering this condition, we select the simplest equations for \( U_{1} \) and the solution

\[
U_{1,xx} = U_{1,x} = U_{1,uu_{1}} = U_{1,uu_{2}} = U_{1,uu_{0},u_{1}} = 0, \quad U_{1} = F_{4}(x,t)u_{0} + F_{5}(x,t)u_{1} + F_{6}(x,t).
\]

In this case, we get the simplest equations for \( U_{2} \)

\[
U_{2,xx} = U_{2,w_{1}} = U_{2,w_{2}} = U_{2,uu_{1}} = U_{2,uu_{2}} = U_{2,uu_{0},u_{1}} = 0,
\]

which imply

\[
U_{2} = F_{7}(x,t)u_{0} + F_{8}(x,t)u_{1} + F_{9}(x,t)u_{2} + F_{10}(x,t).
\]

The determination for unknown functions in \( U_{1}, U_{2}, U_{3} \) are also step by step. The simplest equations in the simplified determining equations are

\[
F_{2} = F_{3} = F_{4} = F_{6} = F_{7} = F_{10} = F_{1,x} = 0.
\]

Considering this condition, we select the simplest equations again

\[
X_{xx} = X_{1,x} = X_{5,x} = X_{8,x} = X_{9,x} = 0.
\]

In this case, the determining equations are simplified to

\[
X_{1} = X_{5,t} = X_{8,t} = X_{9,t} = 0, \quad T_{1} = 3X_{1}, \quad T_{1} = 4X_{x} - F_{1} + F_{5}, \quad T_{1} = X_{1} - 2F_{1},
\]

\[
\theta(4X_{x} - T_{1} = F_{1} + F_{9}) + F_{8} = 0, \quad T_{1} = 4X_{x} - F_{1} + 2F_{5} + F_{9}, \quad T_{1} = 4X_{x} - F_{1} + F_{9},
\]

from which we get

\[
X = C_{1}x + C_{3}, \quad T = 3C_{1}t + C_{2}, \quad U_{0} = -C_{1}u_{0}, \quad U_{1} = -2C_{1}u_{1}, \quad U_{2} = C_{1}(u_{0} - 3u_{2}),
\]

where \( C_{1}, C_{2} \) and \( C_{3} \) are arbitrary constants.

In the same way, limiting the maximum of \( k \) to 3 in Eqs. (14)–(16), we execute similar computation and obtain

\[
X = C_{1}x + C_{3}, \quad T = 3C_{1}t + C_{2}, \quad U_{0} = -C_{1}u_{0}, \quad U_{1} = -2C_{1}u_{1}, \quad U_{2} = C_{1}(u_{0} - 3u_{2}), \quad U_{3} = 2C_{1}(u_{0} - 2u_{1}),
\]

where \( C_{1}, C_{2} \) and \( C_{3} \) are arbitrary constants.
Enlarge the domain of \( k \) by degrees and repeat similar procedures, we discover the formal coherence of \( X, T \) and \( U_k \) \((k = 0, 1, \ldots)\), i.e.,
\[
X = C_1 x + C_3, \quad T = 3C_1 t + C_2, \quad U_k = C_1 [(k - 1) \partial u_{k-1} - (k + 1) u_k] \quad (k = 0, 1, \ldots),
\]
(19)
where \( C_1 \) and \( C_3 \) are arbitrary constants.

When \( \delta = -1 \) (second order dissipation in Eq. (11)), the general formulas for \( X, T \) and \( U_k \) \((k = 0, 1, \ldots)\) can also be obtained. The solutions to the determining equations for \( \delta = \pm 1 \) can be merged into
\[
X = C_1 x + C_1, \quad T = 3C_1 t + C_2, \quad U_k = C_1 \delta [(k - 1) \partial u_{k-1} - (k + \delta) u_k] \quad (k = 0, 1, \ldots).
\]
(20)

Similarity solutions to \( k \)-order approximate equation (14) are dependent upon the symmetry transformation (16) for \( \sigma_k = 0 \), which are equivalent to solving the characteristic equations
\[
\frac{dx}{X} = \frac{dt}{T} = \frac{du_0}{U_0} = \frac{du_1}{U_1} = \frac{du_k}{U_k} = \ldots = \frac{dt}{T}.
\]
(21)

Two subcases are distinguished in the following text.

2.1. Homotopy symmetry reduction of Painlevé II type

When \( C_1 \neq 0 \), without loss of generality, we rewrite the constants \( C_2 \) and \( C_3 \) as \( 3C_1 C_2 \) and \( C_1 C_3 \), and change Eq. (20) to
\[
X = C_1 (x + C_3), \quad T = 3C_1 (t + C_2), \quad U_k = C_1 \delta [(k - 1) \partial u_{k-1} - (k + \delta) u_k] \quad (k = 0, 1, \ldots).
\]
(22)

From the first two equations in Eq. (21), we get the invariants
\[
I(x, t) = \xi = (x + C_3)(t + C_2)^{-1},
\]
(23)
\[
I_0(x, t, u_0) = P_0 = u_0(t + C_2)^{1/3}.
\]
(24)

Viewing \( P_0 \) as a function of \( \xi \), we have
\[
u_0 = P_0(\xi)(t + C_2)^{-1/3}.
\]
(25)

Similarly, we get other similarity solutions
\[
u_1 = P_1(\xi)(t + C_2)^{-1/3}, \quad \nu_2 = \partial P_1(\xi)(t + C_2)^{-1/3} + P_2(\xi)(t + C_2)^{-2/3}, \quad \nu_3 = \partial^2 P_1(\xi)(t + C_2)^{-1/3} + 2\partial P_2(\xi)(t + C_2)^{-2/3} + P_3(\xi)(t + C_2)^{-3/3},
\]
(26)
(27)
(28)

which conform to the general expression
\[
u_k = \delta_{k0} P_0(\xi)(t + C_2)^{-1/3} + \sum_{i=0}^{k-1} \binom{k - 1}{i} \partial^{k-i}(t + C_2)^{-i/3} P_{i+1}(\xi), \quad (k = 0, 1, \ldots),
\]
(29)

with the similarity variable
\[
\xi = (x + C_3)(t + C_2)^{-1/3}.
\]
(30)
The notation \( \delta_{k0} \) satisfies \( \delta_{00} = 1 \) and \( \delta_{k0} = 0 \) \((k \neq 0)\).

From Eqs. (29) and (4), we get the series solution to homotopy model (13)
\[
u = \sum_{k=0}^{\infty} P_k(\xi) \left( \frac{q}{1 - q \theta} \right)^k (t + C_2)^{-k/3},
\]
(31)
and when further setting \( q = 1 \), we have
\[
u = \sum_{k=0}^{\infty} P_k(\xi)(1 - \theta)^{-k}(t + C_2)^{-k/3},
\]
(32)
which is a homotopy series solution to the perturbed mKdV equation.

The determination of similarity reduction equations depends on finite equations in Eqs. (14) and (29). It should be emphasized that all the previous similarity reduction equations should be considered when we eliminate \( u_k \) in the \( k \)-order approximate equation (14) in terms of the similarity solutions (29). We sum up the general formula for the similarity reduction equations
\[
3P_{k+2} + \xi P_{k+1} - \xi P_{k+2} - (1 + \delta k) P_k + 18d \sum_{j=0}^{k} \sum_{i=0}^{j} P_{k-j} P_{j-i} \partial P_{i+1} + 3\epsilon(\theta - 1) P_{k-1} \partial P_{i+1} = 0. \quad (k = 0, 1, \ldots).
\]
(33)
When \( k = 0 \), Eq. (33) is equivalent to the Painlevé II type equation. Specific forms of Eq. (33) depend on the solutions \( P_0, P_1, \ldots, P_{k-1} \) and we can rearrange the terms in Eq. (33) as

\[
3P_{k+2} - \varepsilon P_{k+1} - (1 + \delta_k)P_k + 18a[2(1 - \delta_k)P_0P_{0,k} + P_0^2P_{k,0}] = f_k(\xi) \quad (k = 0, 1, \ldots),
\]

where all \( f_k(\xi) \) are functions with respect to \( \{P_0, P_1, \ldots, P_{k-1}\} \)

\[
f_k(\xi) = 3\varepsilon(1 - \theta)P_{k-1,2} - 18a\left(\sum_{j=1}^{k-1} \sum_{i=0}^{j} P_{k-j,1}P_{i,1} + \sum_{i=1}^{k-1} P_0P_{k-i,1}\right).
\]

Eq. (34) is actually a third order linear variable coefficients ordinary differential equation of \( P_k \) provided \( P_0, P_1, \ldots, P_{k-1} \) are known. Theoretically, Eq. (34) can be solved one by one from zero-order.

### 2.2. Homotopy symmetry reduction of the traveling wave form

When \( \xi = 0 \), from the characteristic Eq. (21), we get the similarity solutions

\[
u_k = P_k(\xi) \quad (k = 0, 1, \ldots),
\]

with \( \xi = x - \frac{ct}{c_k^2} \). We take an equivalent travelling wave form \( \xi = x + ct \) for this similarity variable with \( c \) an arbitrary velocity constant.

From Eqs. (4) and (8), we obtain a series solution to the homotopy model (13)

\[
u = \sum_{k=0}^{\infty} q_k P_k(\xi),
\]

and a homotopy series solution to the perturbed mKdV equation

\[
u = \sum_{k=0}^{\infty} P_k(\xi).
\]

Substituting the similarity solutions (36) to approximate equation (14), we get the similarity reduction equations

\[
P_{k,2} - \varepsilon P_{k+1} + 2a \sum_{j=0}^{k} \sum_{i=0}^{j} P_{k-j,1}P_{i,1} + \varepsilon(\theta - 1) \sum_{i=0}^{k-1} \theta^{k-1-i}P_{i,2} + A_k = 0, \quad (k = 0, 1, \ldots),
\]

where all \( A_k \) are arbitrary integral constants.

For zero-order similarity reduction equation, taking \( a > 0 \) and \( A_0 = 0 \), we get the hyperbolic secant function solution

\[
P_0 = \sqrt{\frac{c}{a}} \text{sech}(\sqrt{c}\xi).
\]

Other similarity reduction equations are equivalent to

\[
P_{k,2} - \varepsilon P_{k+1} + 2a(3 - 2\delta_{k,1})P_0P_0^2 = g_k(\xi) \quad (k = 0, 1, \ldots),
\]

where

\[
g_k(\xi) = -2a\left(\sum_{j=0}^{k-1} \sum_{i=0}^{j} P_{k-j,1}P_{i,1} + P_o\sum_{i=1}^{k-1} P_{k-i,1}\right) + \varepsilon(1 - \theta) \sum_{i=0}^{k-1} \theta^{k-1-i}P_{i,2} + A_k \quad (k = 0, 1, \ldots).
\]

It is easily seen that Eq. (41) is a second order linear variable coefficients ordinary differential equation with the solution

\[
P_k = P_0 \left[ e_k + \int P_0^{-2} \left( d_k + \int P_0 g_k d\xi \right) d\xi \right] \quad (k = 1, 2, \ldots).
\]

where \( d_k \) and \( e_k \) are arbitrary integral constants.

Therefore, given a solution to zero-order similarity reduction equation, we can obtain solutions to other similarity reduction equations one by one. The equations grow more and more complicate as the order increases. It is suggested that the series solutions be truncated in order that the computation involved does not grow too much.

### 3. Conclusion and discussion

For the perturbed mKdV equation, we investigate the approximate homotopy symmetry reduction with weak fourth order dispersion and weak dissipation. The general formulas for similarity reduction equations and the related similarity


