Globally Optimal Kalman Filtering with Correlated Noises, Random One-Step Sensor Delay and Multiple Packet Dropouts

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Abstract: In this paper, we investigate the globally optimal Kalman filtering problem for uncertain stochastic systems with one-step autocorrelated process noises, cross-correlated noises, random one-step sensor delay and multiple packet dropouts. The multiplicative noises are used to characterize the random disturbances existing in systems. Random one-step sensor delay and multiple packet dropouts are characterized by employing two Bernoulli distributed random variables with known conditional probabilities. By separating the random variables from the non-random terms in the transmission and measurement matrices of the addressed dynamical systems, the process noises and measurement noises in the augmented systems depend on the state and the stochastic uncertain perturbations. The process noises are one-step autocorrelated and cross-correlated with the measurement noises. For this complicated systems, a globally optimal Kalman filtering algorithm is developed in the minimum mean square error (MMSE) sense. Finally, we provide a simulation example to illustrate the performance of the proposed filtering approach.

Key Words: Globally optimal Kalman filtering, Stochastic uncertain systems, Different sources noises, Random one-step sensor delay, Multiple packet dropouts.

1 Introduction

Due to the strong advantages and easy-to-implement of the Kalman filtering algorithm, it has been widely applied in various research areas such as control engineering, guidance, communication and signal processing. As pointed out in [1], we know that the Kalman filtering is a globally optimal linear estimation in the minimum mean square error (MMSE) sense. Recently, many important results have been published based on this optimal principle. In [2, 3, 4], the Kalman filters have been designed for systems with state-delays and/or sensor delays.

For one research frontier in the recent years, the uncertainties should be addressed in the system modelling because the accurate information of the system model usually cannot be obtained in practice. Different description of modeling uncertainties includes the norm-bounded uncertainties, polytopic uncertainties, stochastic uncertainties, etc. The state-dependent multiplicative noises have been used in [5, 6] to describe the stochastic uncertainties in systems due to its wide existence in target tracking, guidance systems and image processing [7, 8]. Accordingly, the optimal filtering problems have been studied in [5, 6] for systems with multiplicative noises. On the other hand, with the rapid developments of the network technology, the design problems of the Kalman filter for networked control systems (NCSs) have been extensively studied to deal with the effects from the network-induced phenomena, see e.g. [9, 10, 11, 12]. To mention a few, a new model has been proposed in [4] to characterize the random one-step delay and multiple packet dropouts in a unified framework and then thoroughly employed for optimal filtering problems in [13, 14, 15] in an elegant way. In order to reflect the complexity of the network-induced phenomena, a new model has been introduced in [12] to describe the random sensor delays, multiple packet dropouts and uncertain observation and an unbiased linear optimal filtering approach has been proposed. Based on the method in [12], the Kalman filter has been designed in [3] for systems with one-step random delay and consecutive packet dropouts. Nevertheless, the proposed Kalman-type filter in [5] is local optimal. Thus, the globally optimal Kalman filter has been designed in [6] instead of the Kalman-type filter in [5] for systems with one-step random delay and consecutive packet dropouts. As in [4], the phenomena of random one-step delay and multiple packet dropouts are characterized by employing two Bernoulli distributed random variables with known conditional probabilities. Based on a model characterizing the random sensor one-step delay and multiple packet dropouts in a unified framework, an augmented system is obtained by augmenting the system state. A globally optimal Kalman filtering algorithm is developed.
oped in terms of the MMSE principle.

2 Problem Formulation and Preliminaries

Consider the following uncertain systems with correlated noises, random one-step sensor delay and multiple packet dropouts:

\[
\begin{align*}
\dot{x}_{k+1} &= \left( \tilde{A}_k + \tilde{A}_k \xi_k \right) \tilde{x}_k + \tilde{B}_k \tilde{\omega}_k \\
\tilde{z}_k &= \tilde{C}_k \tilde{x}_k + \tilde{v}_k \\
y_k &= \gamma_k \tilde{z}_k + (1 - \gamma_k)(1 - \gamma_{k-1}) \lambda_k \tilde{z}_{k-1} + (1 - \gamma_k)(1 - \gamma_{k-1}) \lambda_k \tilde{y}_{k-1}
\end{align*}
\]

(1) and (3), we have zero mean process noise and one-step autocorrelated. The statistics of process noises, random one-step sensor delay and multiple packet dropouts include Bernoulli distribution and have the following statistical properties and is uncorrelated with other noise signals.

\[
\mathbb{E}\{\tilde{\omega}_k \tilde{v}_k^T\} = \tilde{S}_k.
\]

The uncorrelated random variables \( \gamma_k \) and \( \lambda_k \) obey the Bernoulli distribution and have the following statistical properties:

\[
\begin{align*}
\mathbb{P}\{\gamma_k = 1\} &= \mathbb{E}\{\gamma_k\} = \alpha, \quad \mathbb{P}\{\gamma_k = 0\} = 1 - \alpha \\
\mathbb{P}\{\lambda_k = 1\} &= \mathbb{E}\{\lambda_k\} = \beta, \quad \mathbb{P}\{\lambda_k = 0\} = 1 - \beta
\end{align*}
\]

where \( \alpha \in [0, 1] \) and \( \beta \in [0, 1] \) are known positive scalars. Assume that \( \gamma_k, \lambda_k, \tilde{\omega}_k, \tilde{v}_k \) and \( \xi_k \) are uncorrelated.

**Assumption 1:** The initial value \( \tilde{x}_0 \) has the following statistical property:

\[
\mathbb{E}\{\tilde{x}_0 \} = \bar{x}_0, \quad \mathbb{E}\{(\tilde{x}_0 - \bar{x}_0)(\tilde{x}_0 - \bar{x}_0)^T\} = \bar{P}_0
\]

and is uncorrelated with other noise signals.

**Remark 1:** As in [4], the model (3) can describe the random one-step sensor delay and multiple packet dropouts in a unified framework by using random variables \( \gamma_k \) and \( \lambda_k \) for data transmission through networks. It is easy to see that the probability of the sensor receives the data at time instant \( k \) is \( \mathbb{P}\{\gamma_k = 1\} = \alpha \), the probability of the sensor occurs one-step time delay is \( \mathbb{P}\{\gamma_k = 0, \lambda_k = 0\} = (1 - \lambda_k)^2 \beta \), and the probability of the sensor occurs packet dropouts are \( \mathbb{P}\{\gamma_k = 0, \lambda_k = 1\} + \mathbb{P}\{\gamma_k = 0, \lambda_k = 0\} = (1 - \alpha)\lambda_k + (1 - \alpha)^2(1 - \beta) \). In addition, we can see that \( \alpha + (1 - \alpha)^2\beta + (1 - \alpha)\lambda_k + (1 - \alpha)^2(1 - \beta) = 1 \).

Based on the approach in [4], denote \( \tilde{Z}_k = (1 - \gamma_k)\tilde{z}_k \) and \( \tilde{Y}_k = \gamma_k\tilde{y}_k \). Noting \( \gamma_k^2 = \gamma_k \) and \( \gamma_k(1 - \gamma_k) = 0 \), from (2) and (3), we have

\[
\begin{align*}
Z_k &= (1 - \gamma_k)\tilde{C}_k \tilde{x}_k + (1 - \gamma_k)\tilde{v}_k, \\
Y_k &= \gamma_k \tilde{C}_k \tilde{x}_k + \gamma_k \tilde{v}_k.
\end{align*}
\]

Then, the systems (1)-(3) can be rewritten as the following compact form:

\[
\begin{align*}
x_{k+1} &= \hat{A}_k x_k + \hat{B}_k \omega_k, \\
y_k &= \hat{C}_k x_k + \gamma_k \nu_k
\end{align*}
\]

(6) and (7) where

\[
\begin{align*}
x_{k+1} &= \begin{bmatrix} \tilde{x}_{k+1}^T \\ \tilde{z}_{k+1}^T \end{bmatrix}, \quad \omega_k = \begin{bmatrix} \tilde{z}_k^T \\ \omega_k \end{bmatrix}, \quad \nu_k = \tilde{v}_k, \\
\hat{A}_k &= \begin{bmatrix} \hat{A}_k + \tilde{A}_k, \xi_k & 0 \\ 1 - \gamma_k & \gamma_k \hat{C}_k \end{bmatrix}, \quad \hat{B}_k = \begin{bmatrix} \hat{B}_k & 0 \\ 0 & \gamma_k I_m \end{bmatrix}, \\
\hat{C}_k &= \begin{bmatrix} \gamma_k \hat{C}_k & (1 - \gamma_k) I_m \end{bmatrix}
\end{align*}
\]

and the statistical properties of process noises \( \omega \) and measurement noises \( \nu \) have the following statistical properties:

\[
\begin{align*}
\mathbb{E}\{\omega_{j, k} \omega_{k}^T\} &= Q_k \delta_{k-j} + Q_j, k \delta_{k-j-1} + Q_{j, k} \delta_{k-j+1}, \\
\mathbb{E}\{\omega_{j, k} \nu_{k}^T\} &= S_{k} \delta_{k-j}, \\
\mathbb{E}\{\omega_{j, k} \nu_{k}^T\} &= S^T_{k} \bar{R}_k, \\
Q_k &= \begin{bmatrix} \tilde{Q}_k & \tilde{S}_k \\ \tilde{S}_k^T & \tilde{R}_k \end{bmatrix}, \quad S_{k} = \begin{bmatrix} \tilde{S}_k \\ \tilde{R}_k \end{bmatrix}, \quad Q_{j, k} = \text{diag}\{\tilde{Q}_{j, k}, 0\}.
\end{align*}
\]

To facilitate the subsequent developments, we introduce the following notations:

\[
\begin{align*}
\hat{A}_k &= A_{1, k} + \gamma_k A_{2, k} + (1 - \gamma_k)\lambda_k A_{3, k} + \xi_k A_{4, k}, \\
\hat{B}_k &= \mathbb{E}\{\hat{A}_k\} = A_{1, k} + \alpha A_{2, k} + (1 - \alpha)\beta A_{3, k}, \\
\Delta A_k &= \hat{A}_k - A_k = (\gamma_k - \alpha) A_{2, k} + [(1 - \gamma_k) \lambda_k - (1 - \alpha)\beta] A_{3, k} + \xi_k A_{4, k}, \\
A_{1, k} &= \begin{bmatrix} \hat{A}_k & 0 & 0 & 0 \\ 0 & \hat{C}_k & 0 & 0 \\ 0 & 0 & \tilde{C}_k & 0 \\ 0 & 0 & 0 & \tilde{C}_k \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
A_{2, k} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{C}_k & 0 \\ 0 & 0 & 0 & \tilde{C}_k \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
A_{3, k} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
A_{4, k} &= \text{diag}\{\tilde{A}_{s, k}, 0, 0, 0\}
\end{align*}
\]
\[ \bar{B}_k = B_{1,k} + \gamma_k B_{2,k}, \quad \bar{B}_k = E(\bar{B}_k) = B_{1,k} + \alpha B_{2,k}, \]
\[ \Delta B_k = \bar{B}_k - B_k = (\gamma_k - \alpha) B_{2,k}, \]
\[ B_{1,k} = \begin{bmatrix} \bar{B}_k & 0 \\ 0 & I_m \end{bmatrix}, \quad B_{2,k} = \begin{bmatrix} 0 & 0 \\ 0 & -I_m \end{bmatrix}, \]
\[ \bar{C}_k = C_{1,k} + \gamma_k C_{2,k} + (1 - \gamma_k) \lambda_k C_{3,k}, \]
\[ C_k = E(C_k) = C_{1,k} + C_{2,k} + (1 - \gamma_k) \lambda_k C_{3,k}, \]
\[ \Delta C_k = \bar{C}_k - C_k = (\gamma_k - \alpha) C_{3,k}, \]
\[ C_{1,k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_{2,k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]
\[ C_{3,k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

It can easily be verified that \( E(\Delta A_k) = 0, \) \( E(\Delta B_k) = 0 \) and \( E(\Delta C_k) = 0. \)

Subsequently, the systems (6)-(7) can be equivalently rewritten as follows:
\[ x_{k+1} = A_k x_k + \bar{B}_k \omega_k, \quad y_k = C_k x_k + \nu_k, \]
where the process noise \( \omega_k \) and measurement noises \( \nu_k \) are described by
\[ \omega_k = \Delta A_k x_k + \bar{B}_k \omega_k, \]
\[ \nu_k = \Delta C_k x_k + \gamma_k \nu_k. \]

The purpose of this paper is to design a globally optimal Kalman filter \( \bar{x}_{k+1|k+1} \) of \( \bar{x}_{k+1|k+1} \) in the sense of minimum variance for original systems (1)-(3) based on the observation sequence \( \{y_1, y_2, \ldots, y_{k+1}\} \). By noting the relationship between the original systems and the augmented systems, we have \( \bar{x}_{k+1|k+1} = \begin{bmatrix} I_n & 0 & 0 \end{bmatrix} \bar{x}_{k+1|k+1}. \)

3 Main Results

To begin, some lemmas and definitions are given which are useful for further derivations. As in [5], letting \( \psi_k = E(x_k \omega_k^T) \), we have
\[ \psi_k = B_{k-1} Q_{k-1, k}. \]

By using the similar methods mentioned in [6], the following results can be obtained:

**Lemma 1** Setting \( \Xi_{k+1} = E(x_{k+1} x_{k+1}^T) \), one has
\[ \Xi_{k+1} = A_k \Xi_k A_k^T + \alpha (1 - \alpha) A_{2,k} \Xi_k A_{2,k} + \beta A_k \Xi_k A_k^T \]
\[ + (1 - \alpha) \beta A_{3,k} \Xi_k A_{3,k} + A_{2,k} \Xi_k A_{2,k} + A_{3,k} \Xi_k A_{3,k} \]
\[ + A_{2,k} \Xi_k A_{2,k} + A_{3,k} \Xi_k A_{3,k} \]
\[ + (1 - \alpha) A_{3,k} \Xi_k A_{3,k} + B_k Q_k B_k^T \]
\[ + (1 - \alpha) B_k Q_k B_k^T + \tilde{A}_k + \tilde{A}_k^T \]
\[ A_k = \tilde{A}_k \psi_k B_k^T + \alpha (1 - \alpha) A_{2,k} \psi_k B_k^T + \alpha (1 - \alpha) A_{3,k} \psi_k B_k^T \]
\[ + (1 - \alpha) B_k Q_k B_k^T + \tilde{A}_k + \tilde{A}_k^T. \]

**Proof:** The proof is omitted for brevity.

**Lemma 2** The autocorrelation and cross-correlation matrices of the process noises \( \omega_k \) and the measurement noises \( \nu_k \) in the systems (10)-(11) are given as follows:
\[ E(\omega_j \omega_k^T) = Q_k \delta_{j-k} + Q_{j,k} \delta_{k-j-1} + Q_{j,k} \delta_{k-j+1}, \]
\[ E(\nu_j \nu_k^T) = \tilde{R}_k \delta_{j-k}. \]

where
\[ Q_k = \alpha (1 - \alpha) A_{2,k} \Xi_k A_{2,k} + (1 - \alpha) \beta (1 - 1 - \alpha) \beta A_{3,k} \Xi_k A_{3,k} + A_{2,k} \Xi_k A_{2,k} + B_k \]
\[ \times Q_k B_k^T + \alpha (1 - \alpha) B_k Q_k B_k^T + B_k + B_k^T, \]
\[ \tilde{R}_k = \alpha (1 - \alpha) A_{2,k} \psi_k B_k^T + \alpha (1 - \alpha) B_k Q_k B_k^T + B_k + B_k^T. \]

Proof: From the expression of \( \bar{\omega}_k \) given by (12), when \( j = k \), setting \( \tilde{Q}_k = E(\bar{\omega}_k \bar{\omega}_k^T) \), the following formula can be obtained
\[ \tilde{Q}_k = E(\Delta A_k x_k + \bar{B}_k \omega_k)^T (\Delta A_k x_k + \bar{B}_k \omega_k) \]
\[ = \alpha (1 - \alpha) A_{2,k} \Xi_k A_{2,k} + \alpha (1 - \alpha) \beta (1 - 1 - \alpha) \beta A_{3,k} \Xi_k A_{3,k} + A_{2,k} \Xi_k A_{2,k} + B_k \]
\[ \times Q_k B_k^T + \alpha (1 - \alpha) B_k Q_k B_k^T + B_k + B_k^T, \]
\[ \tilde{R}_k = E(\Delta C_k x_k + \bar{B}_k \omega_k)^T (\Delta C_k x_k + \bar{B}_k \omega_k) \]
\[ = \alpha (1 - \alpha) A_{2,k} \psi_k B_k^T + \alpha (1 - \alpha) B_k Q_k B_k^T + B_k + B_k^T, \]
\[ \tilde{Q}_{j,k} = E((\Delta A_j x_j + \bar{B}_j \omega_j)^T (\Delta A_k x_k + \bar{B}_k \omega_k)) \]
\[ = B_j Q_{j,k} B_k^T. \]

For other \( j \leq k - 2 \) or \( j \geq k + 2 \), we have \( E(\bar{\omega}_k \bar{\omega}_k^T) = 0 \). Thus, we obtain (17).

Similarly, set \( \tilde{R}_k = E(\bar{\nu}_k \bar{\nu}_k^T) \). Note that \( E(x_k \nu_k^T) = 0, \) \( E(\Delta C_k) = 0 \), we have
\[ \tilde{R}_k = E((\Delta C_k x_k + \gamma_k \nu_k)^T (\Delta C_k x_k + \gamma_k \nu_k)) \]
\[ = \alpha (1 - \alpha) B_k Q_k B_k^T + \alpha (1 - \alpha) \beta A_{3,k} \Xi_k A_{3,k} + A_{2,k} \Xi_k A_{2,k} + B_k \]
\[ \times Q_k B_k^T + \alpha (1 - \alpha) B_k Q_k B_k^T + B_k + B_k^T, \]
\[ \tilde{R}_{j,k} = E((\Delta C_j x_j + \gamma_j \nu_j)^T (\Delta C_k x_k + \gamma_k \nu_k)) \]
\[ = \alpha (1 - \alpha) B_k Q_k B_k^T + \alpha (1 - \alpha) \beta A_{3,k} \Xi_k A_{3,k} + A_{2,k} \Xi_k A_{2,k} + B_k \]
\[ \times Q_k B_k^T + \alpha (1 - \alpha) B_k Q_k B_k^T + B_k + B_k^T. \]

For other \( j \leq k - 1 \) or \( j \geq k + 1 \), note that \( \gamma_k \) is uncorrelated with \( \gamma_j \) and \( E(\Delta C_{j, k}) = 0 \), \( E(\bar{\nu}_j \bar{\nu}_k^T) = 0 \) can be obtained, and we have the following results:
\[ E(\bar{\nu}_j \bar{\nu}_k^T) = \tilde{R}_k \delta_{j-k}. \]
At last, for $E\{\dot{\omega}_t \dot{\nu}_T^T\}$, when $j = k$, set $S_k = E\{\dot{\omega}_k \dot{\nu}_T^T\}$. By using $E\{x_t \dot{\nu}_T^T\} = 0$ and $\dot{B}_k = \dot{B}_k + \Delta \dot{B}_k$, one has

$\tilde{S}_k = E\{(\Delta \dot{A}_k x_k + \tilde{B}_k \dot{w}_k)(\Delta C_k x_k + \gamma_k \dot{v}_k)\} = \alpha(1 - \alpha)A_2 \bar{x}_k C_T^{2,k} + \alpha(\alpha - 1) \beta A_{3,k}
\times \bar{x}_k C_T^{2,k} + \alpha(\alpha - 1) \beta A_{2,k} \bar{x}_k C_T^{3,k} + (1 - \alpha)
\times \beta[(1 - 1 - \alpha) \bar{x}_k C_T^{5,k} + \alpha \times \beta(1 - 1 - \alpha) \bar{x}_k C_T^{5,k} + \alpha
\times \bar{x}_k C_T^{5,k} + \alpha(1 - \alpha) \beta B_{2,k} \bar{x}_k S_k.$

For other $j \leq k - 1$ or $j \geq k + 1$, notice that $\gamma_k$ is uncorrelated with $\gamma_j$, $E\{\Delta C_k\} = 0$, $E\{\Delta A_j\} = 0$ and $E\{\omega_j \dot{\nu}_T^T\} = 0$. Then, we have $E\{\omega_j \dot{\nu}_T^T\} = 0$. So far, the proof of this lemma is complete.

By using the above lemmas, the covariance matrices between the process noises $\dot{\omega}_k$ and the state $x_k$ observation $y_k$ can be calculated.

**Lemma 3** Letting $\Psi_k = Cov(\dot{\omega}_k, x_k)$, $\Phi_k = Cov(\dot{\omega}_k, y_k)$, the following results can be obtained:

$\Psi_k = Q_k^{T,k-1,k}$, $\Phi_k = \Psi_k C_T^{k} + \tilde{S}_k$. \hspace{1cm} (20)

Proof: The proof is omitted for brevity.

In order to facilitate the subsequent developments, we introduce the following definition.

**Definition 1** It is well known that the optimal linear estimator of $x_{k+1}$ by using $Y_j = L(y_1, \cdots, y_j)$ is given as follows:

$\hat{x}_{k+1|j} = E\{x_{k+1}\} + Cov(x_{k+1}, Y_j)(Var Y_j)^{-1}
\times (Y_j - E\{Y_j\}).$ \hspace{1cm} (22)

Similarly, we can obtain the expressions of $\hat{y}_{k+1|j}$ and $\hat{\omega}_{k+1|j}$.

By (22), the following parameters are:

$\Delta y_{k+1} = y_{k+1} - E\{y_{k+1}\} - Cov(y_{k+1}, Y_k)
\times (Var Y_k)^{-1}(Y_k - E\{Y_k\})$, \hspace{1cm} (23)

$P_{k+1|k} = Var(x_{k+1} - Cov(x_{k+1}, Y_k)
\times (Var Y_k)^{-1}Cov(Y_k, x_{k+1}),$, \hspace{1cm} (24)

$U_{k+1} = Cov(x_{k+1}, y_{k+1}) - Cov(x_{k+1}, Y_k)
\times (Var Y_k)^{-1}Cov(Y_k, y_{k+1}),$, \hspace{1cm} (25)

$V_{k+1} = Var(y_{k+1} - Cov(y_{k+1}, Y_k)
\times (Var Y_k)^{-1}Cov(Y_k, y_{k+1}),$, \hspace{1cm} (26)

$D_k = Cov(\dot{\omega}_k, y_k) - Cov(\dot{\omega}_k, Y_k)(Var Y_k)^{-1}
\times Cov(Y_k, \dot{\omega}_k + \dot{\omega}_k),$, \hspace{1cm} (27)

$N_{k+1} = Cov(x_{k+1}, y_{k+1}) - Cov(x_{k+1}, Y_k)
\times (Var Y_k)^{-1}Cov(Y_k, y_{k+1}),$, \hspace{1cm} (28)

$F_{k+1} = Var(x_{k+1} - Cov(x_{k+1}, Y_k)
\times (Var Y_k)^{-1}Cov(Y_k, y_{k+1}).$ \hspace{1cm} (29)

**Lemma 4** For the following block-partitioned matrices with appropriate dimensions,

$M = \begin{bmatrix} M_{11} & M_{12} \\ \end{bmatrix}$

$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$,

$H = \begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix}$

if $G$ and $G_{11}$ are both invertible, then we have $\Theta = G_{22} - G_{21}G_{11}^{-1}G_{12}$ is invertible, and $MG - 1H = M_{11}G_{11}^{-1}H_{11} + (M_{12} - M_{11}G_{11}^{-1}G_{12})^{-1}(H_{21} - G_{21}G_{11}^{-1}H_{11}).$ 

**Lemma 5** $D_k$, $N_{k+1}$ and $F_{k+1}$ satisfy the following equations:

$D_k = \Phi_k,$ \hspace{1cm} (30)

$N_{k+1} = D_k + \bar{A}_k U_k,$ \hspace{1cm} (31)

$F_{k+1} = \bar{A}_k P_{k|k-1} \bar{A}_k + \bar{Q}_k + \bar{A}_k \Psi_k^T + \Psi_k \bar{A}_k^T.$ \hspace{1cm} (32)

Proof: It follows from the definition of $D_k$ and $Cov(\dot{\omega}_k, Y_{k-1}) = 0$, we have

$D_k = Cov(\dot{\omega}_k, y_k) - Cov(\dot{\omega}_k, Y_{k-1})(Var Y_{k-1})^{-1}
\times Cov(Y_{k-1}, y_k + \dot{A}_k Cov(x_k, y_k) - Cov(x_k, Y_{k-1})(Var Y_{k-1})^{-1}Cov(Y_{k-1}, y_k)]
= D_k + \bar{A}_k U_k.$

Substituting (10) into the definition of $N_{k+1}$, the following formula can be derived

$N_{k+1} = Cov(\dot{\omega}_k, y_k) - Cov(\dot{\omega}_k, Y_{k-1})(Var Y_{k-1})^{-1}
\times Cov(Y_{k-1}, y_k + \dot{A}_k Cov(x_k, y_k) - Cov(x_k, \dot{\omega}_k, Y_{k-1})(Var Y_{k-1})^{-1}Cov(Y_{k-1}, y_k)]
= D_k + \bar{A}_k U_k.$

Substituting (10) into the definition of $F_{k+1}$ and noting $Cov(\dot{\omega}_k, Y_{k-1}) = 0$, $Cov(x_k, \dot{\omega}_k) = \Psi_k$, $Cov(\dot{\omega}_k, \dot{\omega}_k) = E\{\omega_k \dot{\nu}_T^T\}$, we have

$F_{k+1} = Cov(\dot{A}_k x_k + \dot{\omega}_k, \dot{A}_k x_k + \dot{\omega}_k)
- Cov(\dot{A}_k x_k + \dot{\omega}_k, Y_{k-1})(Var Y_{k-1})^{-1}
\times Cov(Y_{k-1}, \dot{A}_k x_k + \dot{\omega}_k)
= \dot{A}_k P_{k|k-1} \bar{A}_k^T + \bar{Q}_k + \dot{A}_k \Psi_k^T + \Psi_k \bar{A}_k^T.$

The proof of this lemma is complete.

In the following parts, based on the above definitions and lemmas, we can design the globally optimal Kalman filter $\hat{x}_{k+1|k+1}$ of $x_{k+1}$ in the sense of minimum variance for augmented systems (10)-(11).

**Theorem 1** For the systems (10)-(11), the globally optimal Kalman filter can be given as follows:

$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + U_{k+1} V_{k+1|k}^{-1} \Delta y_{k+1},$ \hspace{1cm} (33)

$\hat{x}_{k+1|k} = \dot{A}_k \hat{x}_{k|k} + D_k V_{k-1|k}^{-1} \Delta y_{k},$ \hspace{1cm} (34)

$\Delta y_{k+1} = y_{k+1} - \dot{A}_k \hat{x}_{k|k} + D_k V_{k-1|k}^{-1} \Delta y_{k},$ \hspace{1cm} (35)

$P_{k+1|k+1} = P_{k+1|k} - U_{k+1} V_{k+1|k}^{-1} U_{k+1}^T,$ \hspace{1cm} (36)

$P_{k+1|k+1} = F_{k+1} - N_{k+1} V_{k+1|k}^T,$ \hspace{1cm} (37)

$U_{k+1} = \dot{C}_{k+1} P_{k+1|k} \bar{C}_{k+1}^T + \bar{R}_{k+1},$ \hspace{1cm} (38)

$V_{k+1} = \dot{C}_{k+1} P_{k+1|k} \bar{C}_{k+1}^T + \bar{R}_{k+1}.$ \hspace{1cm} (39)

Proof: From definition 1 and lemma 4, one has

$\hat{x}_{k+1|k+1} = E\{x_{k+1}\} + Cov(x_{k+1}, Y_{k+1})
\times (Var Y_{k+1})^{-1} (Y_{k+1} - E\{Y_{k+1}\})
= \hat{x}_{k+1|k} + U_{k+1} V_{k+1|k}^{-1} \Delta y_{k+1}.$

Then, (33) is true.
Subsequently, we aim to derive $\hat{x}_{k+1|k}$. Firstly, let us derive $\hat{\omega}_{k|k}$. It follows from $E\{\omega_k\} = 0$, definition 1 and lemma 4 that
\[
\hat{\omega}_{k|k} = E\{\hat{\omega}_k\} + Cov(\omega_k, Y_k)(Var Y_k)^{-1}(Y_k - E\{Y_k\})
= D_kV_k^{-1}\Delta y_k.
\] (40)

When deriving (40), we have used the fact that $Cov(\omega_k, Y_k|k) = 0$. Taking projection on both sides of (10) onto the linear space spanned by $Y_k$, we have
\[
\hat{x}_{k+1|k} = \hat{A}_k\hat{x}_{k|k} + \hat{\omega}_{k|k}.
\] (41)

Substituting (40) into (41) yields (34).

Thirdly, it is easy to see $\hat{\nu}_{k+1|k} = 0$, then
\[
\Delta y_{k+1} = y_{k+1} - \hat{y}_{k+1|k} = y_{k+1} - \hat{C}_k\hat{x}_{k+1|k} + \hat{\nu}_{k+1|k}.
\]

Thus, (35) can be obtained.

Next, by using definition 1 and lemma 4, the recursion of $P_{k+1|k+1}$ can be deduced:
\[
P_{k+1|k+1} = E\left\{\left[\hat{x}_{k+1} - \hat{x}_{k+1|k+1}\right]\left[\hat{x}_{k+1} - \hat{x}_{k+1|k+1}\right]^T\right\}
= Var(\hat{x}_{k+1} - \hat{C}_k\hat{x}_{k+1|k})
\times (Var Y_k)^{-1}Cov(Y_{k+1}, x_{k+1})
= P_{k+1|k} - \hat{C}_k\hat{P}_{k+1|k}^{T}\hat{C}_k^{T}.
\]

Now, (36) can be obtained. Similarly, the recursion of $P_{k+1|k}$ can be calculated.

Finally, let us derive the recursions of $U_{k+1}$ and $V_{k+1}$. Since $Cov(x_{k+1}, \hat{v}_{k+1}) = 0$, $Cov(Y_{k+1}, \hat{v}_{k+1}) = 0$ and $y_{k+1} = \hat{C}_k\hat{x}_{k+1|k} + \hat{\nu}_{k+1|k}$, we have
\[
U_{k+1} = Cov(x_{k+1}, y_{k+1}) - Cov(x_{k+1}, Y_k)
\times (Var Y_k)^{-1}Cov(Y_k, y_{k+1})
= [Var x_{k+1} - Cov(x_{k+1}, Y_k)(Var Y_k)^{-1}
\times Cov(Y_k, x_{k+1})]\hat{C}_k^{T}
= P_{k+1|k}\hat{C}_k^{T},
\]
and
\[
V_{k+1} = Var y_{k+1} - Cov(y_{k+1}, Y_k)
\times (Var Y_k)^{-1}Cov(Y_k, y_{k+1})
= \hat{C}_k\hat{P}_{k+1|k}\hat{C}_k^{T} + \hat{C}_k\hat{P}_{k+1|k}\hat{C}_k^{T} + \hat{R}_{k+1}.
\]

Up to now, the proof of the theorem is complete.

**Remark 2.** Note that the process noises of systems (10)-(11) are one-step autocorrelated, the process noises and measurement noises at time $k$ are cross-correlated. Consequently, it is necessary to derive the optimal filtering of systems (10)-(11). For this case, the traditional Kalman filtering is one special case of the proposed Kalman filtering. Also, the proposed results are the extension of [6].

Now, let us summarize the calculation process of the globally optimal Kalman filtering scheme in Theorem 1.

**Algorithm 1: Kalman filtering with different sources noises, random one-step sensor delay and multiple packet dropouts**

1. **Step 1.** Set the initial values $x_0$, $\Delta y_0$, $P_{0|0}$.
2. **Step 2.** Substitute $Q_{k-1,k}$ into (20). Substituting $S_k$ and $\Psi_k$ into (21) leads to $D_k$.
3. **Step 3.** Calculate $\hat{x}_{k+1|k}$ by using $\hat{x}_{k|k}$, $V_k$, $\Delta y_k$ and $D_k$.
4. **Step 4.** Calculate $N_{k+1|k}$ by using $D_k$ and $U_k$.
5. **Step 5.** Obtain $\hat{Q}_k$ according to $\bar{Z}_k$ and $\psi_k$.
6. **Step 6.** Calculate $F_{k+1|k}$ by substituting $P_{k|k-1}$, $\hat{Q}_k$, $\Psi_k$ into (32).
7. **Step 7.** Compute $P_{k+1|k}$ by substituting $F_{k+1|k}$, $N_{k+1|k}$ and $V_k$ at the latest time into (37).
8. **Step 8.** Obtain $\bar{U}_{k+1}$ by substituting $F_{k+1|k}$ into (38).
9. **Step 9.** According to $\bar{Z}_k$, $\bar{v}_k$ and (15), we have $\bar{X}_{k+1}$.
10. **Step 10.** Obtain $\bar{R}_{k+1}$ by $\bar{Z}_{k+1}$. Substituting $P_{k+1|k}$ and $\bar{R}_{k+1}$ into (39) yields $V_{k+1}$. Using $y_{k+1}$ and $\hat{x}_{k+1|k}$, we have $\Delta y_{k+1}$.
11. **Step 11.** Substituting $\hat{x}_{k+1|k}$, $U_{k+1}$, $V_{k+1}$ and $\Delta y_{k+1}$ into (33), $\hat{x}_{k+1|k+1}$ can be computed.
12. **Step 12.** According to (36) and using $P_{k+1|k}$, $U_{k+1}$, $V_{k+1}$, we obtain $P_{k+1|k+1}$.

**4 An Illustrative Example**

In this section, we present a simulation to illustrate the effectiveness of the proposed filtering algorithm.

Consider the discrete stochastic system (1)-(3) with
\[
A_k = \begin{bmatrix}
0.2 & -0.15 \\
0 & 0.15
\end{bmatrix}, \quad \bar{A}_{s,k} = \begin{bmatrix}
0.01 & 0 \\
0 & 0.01
\end{bmatrix}, \quad \bar{B}_k = \begin{bmatrix}
2 \\
2.5
\end{bmatrix}, \quad \bar{C}_k = \begin{bmatrix}
1.5 & 1
\end{bmatrix},
\]
\[
\bar{a}_k = a_0\epsilon_{k} + a_1\epsilon_{k-1}, \quad \bar{v}_k = b_0\epsilon_{k-1},
\]
and $\bar{x}_k = [\bar{x}_{k,1} \bar{x}_{k,2}]^T$, $\epsilon_k, \epsilon_{k-1}$ are zero-mean Gaussian white noises with unity covariances. Letting $a_0 = 0.05$, $a_1 = 0.1$, $b_0 = 0.15$, then the process noises are one-step autocorrelated and cross-correlated with measurement noises, $\hat{Q}_k = 0.0125$, $\hat{Q}_{k-1,k} = 0.005$, $\hat{Q}_{k+1,k} = 0.005$, $\bar{S}_k = a_1b_0 = 0.015$.

Set $x_0 = [-0.1 \quad 0.1 \quad 0.3 \quad 0.4 \quad 0.2]^T$, $\Delta y_0 = 0$, $\alpha = 0.95$, $\beta = 0.9$ and $P_{0|0} = \text{diag}[1,1,1,1,1]$. MSEi denotes the mean-square error for the filtering of $\bar{x}_{k,i}$, i.e.,
\[
(1/M)\sum_{i=1}^{M}(\bar{x}_{k,i} - \bar{x}_{k,i|k})^2 \quad (i = 1, 2),
\]
where $M$ is the number of simulation steps. According to Theorem 1, the globally optimal Kalman filter can be constructed recursively. From Figs. 1-4, we can see that our globally optimal Kalman filtering algorithm has a good performance.

**5 Conclusions**

The problem of the globally optimal Kalman filtering has been investigated for systems with different sources noises, random one-step sensor delay and multiple packet dropouts. A new globally optimal Kalman filter has been designed. Finally, we have given a simulation example to demonstrate the effectiveness of the proposed filtering method.

**References**


Fig. 1: The trajectories of $x_{k,1}$ and $\hat{x}_{k|1}$

Fig. 2: The trajectories of $x_{k,2}$ and $\hat{x}_{k|2}$

Fig. 3: log(MSE1)

Fig. 4: log(MSE2)


