Total colorings of planar graphs without chordal 6-cycles

Bing Wang\textsuperscript{a,b}, Jian-Liang Wu\textsuperscript{b,\ast}, Hui-Juan Wang\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Zaozhuang University, Shandong, 277160, China
\textsuperscript{b} School of Mathematics, Shandong University, Jinan, 250100, China

\begin{abstract}
A total k-coloring of a graph G is a coloring of \( V(G) \cup E(G) \) using k colors such that no two adjacent or incident elements receive the same color. The total chromatic number of G is the smallest integer k such that G has a total k-coloring. In this paper, it is proved that if G is a planar graph with maximum degree \( \Delta \geq 7 \) and without chordal 6-cycles, then the total chromatic number of G is \( \Delta + 1 \).
\end{abstract}

\section{Introduction}

All graphs considered in this paper are simple, finite and undirected, and we follow [2] for the terminologies and notations not defined here. Let G be a graph. We use \( V(G) \), \( E(G) \), \( \Delta(G) \) and \( \delta(G) \) (or simply \( V \), \( E \), \( \Delta \) and \( \delta \)) to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively.

A total k-coloring of a graph G is a coloring of \( V \cup E \) using k colors such that no two adjacent or incident elements receive the same color. The total chromatic number \( \chi''(G) \) of G is the smallest integer k such that G has a total k-coloring. Clearly, \( \chi''(G) \geq \Delta + 1 \). Behzad [1] and Vizing [14] posed independently the following famous conjecture, which is known as the total coloring conjecture (TCC).

\textbf{Conjecture A.} For any graph G, \( \Delta + 1 \leq \chi''(G) \leq \Delta + 2 \).

This conjecture was confirmed for general graphs with \( \Delta \leq 5 \). For its history, readers can see [19]. For planar graphs, the only open case is \( \Delta = 6 \) (see [8,11]). Interestingly, planar graphs with high maximum degree allow a stronger assertion, that is, every planar graph with high maximum degree \( \Delta \) has a total (\( \Delta + 1 \))-coloring. This result was first established in [3] for \( \Delta \geq 14 \), which was extended to \( \Delta \geq 12 \) [4], \( \Delta \geq 10 \) [15], and finally to \( \Delta \geq 9 \) [9]. Recently, Shen and Wang [12] proved that if G is a planar graph with \( \Delta = 8 \) and G contains no chordal 5-cycles or no chordal 6-cycles, then \( \chi''(G) = \Delta + 1 \). Wang and Wu [17] proved that if G is a planar graph with \( \Delta \geq 7 \) and every vertex is incident with at most one triangle, then \( \chi''(G) = \Delta + 1 \). Wang and Wu [18] proved that if G is a planar graph with \( \Delta \geq 7 \) and without 4-cycles, then \( \chi''(G) = \Delta + 1 \) (later, it is extended to \( \Delta \geq 6 \) by Shen and Wang [13]). Wang et al. [16] proved that if G is a planar graph with \( \Delta \geq 7 \) and without chordal 5-cycles, then \( \chi''(G) = \Delta + 1 \). In this paper, we obtain that if G is a planar graph with \( \Delta \geq 7 \) and without chordal 6-cycles, then \( \chi''(G) = \Delta + 1 \). To prove the result, we first establish various structural properties of G. Relying on these properties, we use the discharging method in the detailed proof to obtain a contradiction.

\textsuperscript{\ast} This work is supported by the National Natural Foundation of China (No. 11271006) and the Natural Science Foundation of Shandong Province (ZR2012AL08).

\textsuperscript{\ast} Corresponding author. Tel.: +86 53187906969; fax: +86 053188364654.

E-mail addresses: jlwu65@sina.com, jlwu@sdu.edu.cn (J.-L. Wu).

http://dx.doi.org/10.1016/j.dam.2014.02.004

0166-218X/© 2014 Elsevier B.V. All rights reserved.
2. Main result and its proof

We will introduce some more notations and definitions here for convenience. Let \( G = (V, E, F) \) be a plane graph, where \( F \) is the face set of \( G \). For a vertex \( v \in V \), let \( N(v) \) denote the set of vertices adjacent to \( v \), and let \( d(v) = |N(v)| \) denote the degree of \( v \); and for a face \( f \), the degree of a face \( f \), denoted by \( d(f) \), is the number of edges incident with it, where each cut-edge is counted twice. A \( k \)-vertex, a \( k^+ \)-vertex or a \( k^- \)-vertex is a vertex of degree \( k \), at least \( k \) or at most \( k \), respectively. Similarly, a \( k \)-face or a \( k^+ \)-face or a \( k^- \)-face is a face of degree \( k \) or at least \( k \), respectively. Let \( n_t(v) \) be the number of \( t \)-vertices adjacent to a vertex \( v \), and \( f_k(v) \) the number of \( k \)-faces incident with \( v \). Especially, let \( f_3(v) = t \). Let \( v_1, v_2, \ldots, v_d \) be neighbors of \( v \) in an anticlockwise order. Let \( f_i \) be face incident with \( v, v_i \) and \( v_{i+1} \), for all \( i \) such that \( i \in \{1, 2, \ldots, d\} \). Note that all the subscripts in the paper are taken modulo \( d \). For convenience, \( (d_1, d_2, \ldots, d_d) \) denotes a cycle (or a face) whose boundary vertices are of degree \( d_1, d_2, \ldots, d_d \) in the anticlockwise order. Specially, \( (i, j^+, k^-) \)-face is a \( 3 \)-face \( uvw \) such that \( d(u) = i \leq j \leq d(v) \leq k \leq d(w) \).

**Theorem 1.** Let \( G \) be a planar graph without chordal 6-cycles. If \( \Delta \geq 7 \), then \( \chi''(G) = \Delta + 1 \).

**Proof.** In [12], Theorem 1 was established for \( \Delta = 8 \). So we assume that \( \Delta = 7 \). Let \( G \) be a minimal counterexample to Theorem 1 in terms of the number of vertices and edges, respectively. Then every proper subgraph of \( G \) has a total-8-coloring, but \( G \) is not. We first show some known properties on \( G \).

(a) \( G \) is 2-connected and the boundary of each face in \( G \) is exactly a cycle (see [5]);
(b) The subgraph \( G_{27} \) of \( G \) induced by all edges joining 2-vertices to 7-vertices is a forest (see [3,5]);
(c) \( G \) contains no edge \( uw \) with \( \min\{d(u), d(v)\} \leq \lfloor \frac{\Delta}{2} \rfloor \) and \( d(u) + d(v) \leq \Delta + 1 \) (see [5]);
(d) \( G \) contains no 3-face incident with more than one 4-vertex (see [10]);
(e) If \( v \) is a 7-vertex of \( G \) with \( n_2(v) \geq 1 \), then \( n_4(v) \geq 1 \) (see [6]).

**Lemma 2.** \( G \) contains no configurations depicted in Fig. 1, where the vertices marked by \( \bullet \) have no other neighbors in \( G \).

**Proof.** The proof that \( G \) contains no configurations depicted in Fig. 1(1,2,4,5) can be found in [7]. The proof that \( G \) contains no configuration depicted in Fig. 1(3) and (6) can be found in [5,16], respectively.

**Lemma 3.** \( G \) contains no configurations depicted in Fig. 2, where the vertices marked by \( \bullet \) have no other neighbors in \( G \).

**Proof.** Suppose that \( G \) contains a configuration depicted in Fig. 2(1). Then \( G' = G - vv_2 \) has a total-8-coloring \( \psi \) with the color set \( C = \{1, 2, \ldots, 8\} \) by the minimality of \( G \). Erase the color on \( v_2 \). For a vertex \( x \in V(G) \), let \( C(x) = \{\psi(xy) : y \in N(x)\} \). First, we consider \( v_2 v_3 \) as follows. If \( |C(v_2) \cup C(v_3)| < 7 \), then we can color \( v_2 v_3 \) with a color in \( C \setminus (\{\psi(v)\} \cup C(v_2) \cup C(v_3)) \). Otherwise, without loss of generality (WLOG), we assume that \( \{\psi(vv_1), \psi(vv_2), \psi(vv_3), \psi(vv_4), \psi(vv_5), \psi(vv_6), \psi(vv_7)\} = \{1, 2, 3, 4, 5, 6, 7, 8\} \). Then we obtain a total-8-coloring of \( G \) by coloring \( vv_2 \) with a color in \( \{5, 6, 7\} \setminus C(v_2) \), and coloring \( vv_2 \) with 3. Otherwise, if \( \psi(vv_4) = 1 \), then we exchange the colors of edges \( v_2 v_3 \) and \( vv_4 \), color \( vv_2 \) with 2. Otherwise, we exchange the colors of edges \( v_1 v_2 \) and \( vv_4 \), color \( vv_2 \) with 1 and color \( vv_3 \) with 3. Hence we obtain a total-8-coloring \( \psi' \) of \( G \) in which \( v_2 \) is uncolored.

Now we begin to recolor \( v_2 \). Let \( \alpha \) be the color on \( v_2 v_3 \) and \( D = C(v_2) \cup \{\alpha, 8\} \cup \{\psi(x) : x \in N(v_2)\} \). If \( |D| < 8 \), then we obtain a total-8-coloring of \( G \) by coloring \( v_2 \) with a color in \( C \setminus D \), a contradiction. Otherwise \( C = D \). WLOG, we assume that \( v_2 y, v_2 v_1, v_2 v_3, v_1, v, v_3 \) is colored with 1, 2, 3, 4, 5, 6, 7, 8. First, we have 5, 6, 8 \( \in C(v) \), for otherwise, we recolor \( vv_2 \) with a color in \( \{5, 6, 8\} \setminus C(v) \), and \( v_2 \) with 3, a contradiction. Since \( d(v) = 5 \) and \( \{3, 5, 6, 8\} \subset C(v) \), color 2 or 4 does not appear at \( v \), WLOG, 4 \( \not\in C(v) \). If \( \psi(vv_2) \in \{5, 6\} \), then we exchange the colors of edges \( v_2 v_3 \) and \( vv_3 \), recolor \( v_2 \) with 4. Otherwise, \( \psi(vv_2) \in \{1, 2\} \). If \( \psi(vv_2) = 1 \), then we exchange the colors of edges \( vv_3 \) and \( v_2 v_3 \), color \( v_2 \) with 2. Otherwise, we exchange the colors of edges \( v_2 v_1 \) and \( vv_1 \), recolor \( v_2 \) with 4, a contradiction, too.

Suppose that \( G \) contains a configuration depicted in Fig. 2(2), where \( d(v) = 7 \). Then \( G' = G - vv_7 \) has a total-8-coloring \( \psi \). Erase the colors on all black 3\(^{-}\)-vertices. For a vertex \( x \in V(G) \), let \( C(x) = \{\psi(xy) : y \in N(x)\} \). If \( \psi(v_3 v_7) \in C(v) \cup \{\psi(v)\} \),
then the forbidden colors for $vv_j$ is at most 7, so $vv_j$ can be properly colored. By recoloring the uncolored vertices, we obtain a total-8-coloring of $G$, a contradiction. So $\psi(vv_j) \not\in C(v)$, that is, $|C(v) \cup \{\psi(vv_j), \psi(vv_j)\}| = 8$. WLOG, assume that $\psi(v) = 8$, $\psi(vv_7) = 7$, and $\psi(vv_j) = j$ for $j \in \{1, 2, 5, 6\}$. If $\psi(vv_1v_2) \neq 7$, then we can recolor $vv_1$ with 7, and color $vv_7$ with 1 to obtain a total-8-coloring of $G$, a contradiction. So $\psi(vv_1v_2) = 7$. Similarly, we have $7 \in C(vv_6)$ and $\psi(vv_1v_2) = 2$. If $2 \not\in C(vv_6)$, then we exchange the colors of edges $vv_7v_7$ and $x_7v_1v_1v_2v_2$ and $vv_7v_7$ with 2, and color $vv_6$ with 6 to obtain a total-8-coloring of $G$, a contradiction. Hence $C(vv_6) = \{2, 6, 7\}$. If $\psi(vv_5v_6) = 7$, then we just exchange the colors of the edges $vv_5v_6$ and $vv_5$, and color $vv_7$ with 5. Otherwise, we exchange the colors of the edges $vv_7v_7$ and $x_7v_1v_1v_2v_2$ and $vv_7v_5$ and $vv_5$ and color $vv_7$ with 5. By recoloring the uncolored vertices, we obtain a total-8-coloring of $G$, a contradiction.

Since $G$ is a planar graph, by Euler's formula, we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -12 < 0.$$ 

Now we define the initial charge function $ch(x)$ of $x \in V \cup F$ to be $ch(v) = 2d(v) - 6$ if $v \in V$ and $ch(f) = d(f) - 6$ if $f \in F$. It follows that $\sum_{x \in V \cup F} ch(x) < 0$. Now we design appropriate discharging rules and redistribute weights accordingly.

Note that any discharging procedure preserves the total charge of $G$. If we can define suitable discharging rules to change the initial charge function $ch$ to the final charge function $ch'$ on $V \cup F$, such that $ch'(x) \geq 0$ for all $x \in V \cup F$, then we get an obvious contradiction.

Our discharging rules are defined as follows.

R1 Let $v$ be a 2-vertex. If $v$ is incident with a 3-face, then it receives 1 from each of its neighbors. Otherwise, $v$ receives $\frac{1}{2}$ from its child and $\frac{1}{4}$ from its parent.

R2 Let $f$ be a 3-face. If $f$ is incident with a 3-vertex, then it receives $\frac{3}{2}$ from each incident 6-vertex. If $f$ is incident with a 4-vertex, then it receives $\frac{1}{2}$ from the 4-vertex and receives $\frac{3}{2}$ from each incident 5-vertex. If $f$ is not incident with any 4-vertex, then it receives 1 from each incident 5-vertex.

R3 Let $f$ be a 4-face. If $f$ is incident with two 3-vertices, then it receives 1 from each incident 6-vertex. If $f$ is incident with the unique 3-vertex $u$, then it receives $\frac{1}{2}$ from each incident 6-vertex adjacent to $u$ and receives $\frac{1}{4}$ from its incident 4-vertex opposite to $u$. If $f$ is incident with no 3-vertices, then it receives $\frac{1}{2}$ from each incident vertex.

R4 Let $f$ be a 5-face. If $f$ is incident with two 3-vertices, then it receives $\frac{1}{2}$ from each incident 4-vertex. If $f$ is incident with one 3-vertex, then it receives $\frac{1}{4}$ from each incident 4-vertex. If $f$ is not incident with any 3-vertex, then it receives $\frac{1}{3}$ from each incident vertex.

The rest of this paper is to check $ch'(x) \geq 0$ for all $x \in V \cup F$. Firstly note that our discharging rules are just designed such that $ch'(f) \geq 0$ for all $f \in F$ and $ch'(v) \geq 0$ for all 2-vertices $v \in V$. So we only check that $ch'(v) \geq 0$ for all 3-vertices of $G$.

Let $v$ be a vertex of $G$. If $d(v) = 3$, then $ch'(v) = ch(v) = 0$. If $d(v) = 4$, then $v$ sends at most $\frac{1}{2}$ to each incident face by R2 and R3, and it follows that $ch'(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$.

Suppose $d(v) = 5$. Then $d(v_i) \geq 4$ for all $i \in \{1, 2, \ldots, 5\}$ by (c) and $v$ is incident with at most three 3-faces by the choice of $G$, that is, $f_1(v) = t \leq 3$. Assume that $t = 3$. If $f_2(v) \geq 2$, then $ch'(v) = ch(v) - \frac{3}{4} \times 3 > 0$ by R2. Otherwise, $v$ is incident with at least one $(5^+, 5^+, 5^+, 5^+)$-face by Lemma 3. Then $v$ sends at most $\frac{1}{2}$ to a $(5^+, 5^+, 5^+, 5^+)$-face by R2, at most $\frac{1}{4}$ to a $4^+$-face by R3 and at most $\frac{1}{2}$ to a $5^+$-face by R4. Hence $ch'(v) \geq ch(v) - \frac{3}{4} \times 2 - \frac{1}{4} = \frac{1}{2} > 0$. Assume that $t = 2$. Then $f_3(v) \geq 1$. Hence $ch'(v) \geq ch(v) - \frac{3}{4} \times 2 - \frac{1}{4} \times 2 > 0$. Assume that $t \leq 1$. Then $f_3(v) \leq 2$. Hence $ch'(v) \geq ch(v) - \frac{3}{4} \times t - \frac{1}{2} \times (5 - 2 - t) = \frac{12 - 7t}{4} > 0$.

Suppose $d(v) = 6$. Then $t \leq 4$ and $d(v_i) \geq 3$ for all $i \in \{1, 2, \ldots, 6\}$. If $3 \leq t \leq 4$, then $f_2(v) \geq 2$, and it follows that $ch'(v) \geq ch(v) - \frac{3}{4} \times t - (6 - 2 - t) = \frac{4t - 12}{4} \geq 0$. If $t = 2$, then $f_4(v) \leq 2$, and it follows that $ch'(v) \geq ch(v) - \frac{3}{4} \times 2 - \frac{1}{4} \times \frac{1}{2} = \frac{1}{2} > 0$. If $0 \leq t \leq 1$, then $f_4(v) \leq 3$, and it follows that $ch'(v) \geq ch(v) - \frac{3}{4} \times t - \frac{1}{4} \times (6 - 3 - t) = \frac{12 - 7t}{8} > 0$.

![Fig. 2. Reducible configurations.](image-url)
Suppose $d(v) = 7$. Then $\chi^*(v) = 2 \times 7 - 6 = 8$. If $n_2(v) \geq 2$, then any 2-vertex adjacent to $v$ is not incident with a 3-face by Lemma 2(1), so $v$ sends at most $\frac{8^2+2}{2} = 2$ to all its adjacent 2-vertices by R1. Moreover, $v$ sends at most $\frac{1}{2}$ to a 3-face by R2, at most 1 to a 4-face by R3 and at most $\frac{1}{2}$ to a 5-face by R4.

By the choice of $C$, we have the following lemma.

**Lemma 4.** Suppose that $d(v_i) = d(v_j) = 2$ and $d(v_k) \geq 3$ for all $i = 1, \ldots, k - 1$. If $f_i, f_{i+1}, \ldots, f_{k-1}$ are 4*-faces, then $v$ sends at most $\left[\frac{k-1}{2}\right] + \left[\frac{k-2}{2}\right] \times \frac{1}{2}$ (in total) to $f_i, f_{i+1}, \ldots, f_{k-1}$.

**Proof.** By the choice of $C$, $v$ is incident with at most $\left[\frac{k-1}{2}\right]$ 4*-faces and $\left[\frac{k-2}{2}\right]$ 5*-faces. Moreover, $v$ sends at most 1 to each incident 4*-face by R3 and at most $\frac{1}{2}$ to each incident 5*-face by R4. Then $v$ sends at most $\left[\frac{k-1}{2}\right] + \left[\frac{k-2}{2}\right] \times \frac{1}{2}$ to $f_i, f_{i+1}, \ldots, f_{k-1}$. $lacksquare$

We consider the following cases.

Case 1. $n_2(v) \geq 6$. Then $f_{0+}(v) \geq 5$ by Lemma 2(4) and $t = 0$ by Lemma 2(1). So $\chi'(v) \geq 8 - \frac{6+2}{2} = 6 > 0$.

Case 2. $n_2(v) = 5$. Then $f_{0+}(v) \geq 3$. So $\chi'(v) \geq 8 - \frac{5+2}{2} - \max\{4 \times 1, \frac{1}{2} + 2 \times 1\} > 0$.

Case 3. $n_2(v) = 4$. All 2-vertices adjacent to $v$ are located as shown in Fig. 3, where the vertices marked by $\cdot$ are 2-vertices.

For Fig. 3(a), $f_4, f_5, f_6$ are 6*-faces by Lemma 2(4). By Lemma 2(1), $f_5, f_6$ are 4*-faces, thus $f_1, f_2, f_3$ can be 3-faces, that is, $f_1(v) = t \leq 2$. So $\chi'(v) \geq 8 - \frac{4+2}{2} - \frac{1}{2} \times t - (7 - 3 - t) = \frac{2}{t} + \frac{1}{2} > 0$.

For Fig. 3(b) (the case Fig. 3(c) can be settled similarly), $t \leq 1$ and $f_{0+}(v) \geq 2$. By Lemma 2(4), $f_5, f_6$ are 6*-faces, if $t = 1$, then $f_1$ is a 3-face, $f_2, f_3, f_4, f_5$ are 4*-faces. So $v$ sends at most $\frac{3}{2}$ to $f_1$ by R2, at most $2 \times 1$ to $f_2$ and $f_3$ by R3 and at most $(1 + \frac{1}{3})$ to $f_4$ and $f_5$ by Lemma 4. Hence $\chi'(v) \geq 8 - \frac{4+2}{2} - \frac{1}{2} - (1 + \frac{1}{2}) = 0$. Otherwise, we have that $\chi'(v) \geq 8 - \frac{4+2}{2} - 5 \times 1 = 0$.

For Fig. 3(d), we have $t = 0$ and $f_{0+}(v) \geq 1$. Then $\chi'(v) \geq 8 - \frac{4+2}{2} - 3 \times (1 + \frac{1}{3}) > 0$ by Lemma 4.

Case 4. $n_2(v) = 3$. All 2-vertices adjacent to $v$ are located as shown in Fig. 4, where the vertices marked by $\cdot$ are 2-vertices.

For Fig. 4(a), $t \leq 3$ and $f_{0+}(v) \geq 2$. If $0 \leq t \leq 1$, then $\chi'(v) \geq 8 - \frac{\frac{1+2}{2}}{2} - \frac{1}{2} \times t - (7 - 2 - t) = \frac{t+1}{2} > 0$. Otherwise, $f_{0+}(v) \geq 3$, and it follows that $\chi'(v) \geq 8 - \frac{\frac{3+2}{2}}{2} - \frac{1}{2} \times t - (7 - 3 - t) = \frac{4+1}{2} > 0$.

For Fig. 4(b), $t \leq 2$ and $f_{0+}(v) \geq 1$. Then $v$ sends at most $(1 + \frac{1}{2})$ to $f_4$ and $f_5$ by Lemma 4. If $t = 2$, then $f_{0+}(v) \geq 2$ and it follows that $\chi'(v) \geq 8 - \frac{\frac{1+2}{2}}{2} - \frac{1}{2} \times 2 - 1 - (1 + \frac{1}{2}) = \frac{1}{2} > 0$. If $t \leq 1$, then $f_4(v) \leq 3$ and it follows that $\chi'(v) \geq 8 - \frac{\frac{3+2}{2}}{2} - \frac{1}{2} \times t - 3 \times 1 - \frac{1}{2} \times (7 - 3 - t) > 0$.

For Fig. 4(c), $t \leq 2$ and $f_{0+}(v) \geq 1$. Suppose $t = 2$. Then $f_{0+}(v) \geq 2$. If $f_{0+}(v) \geq 3$, then $\chi'(v) \geq 8 - \frac{\frac{3+2}{2}}{2} - \frac{1}{2} - 2 - 2 > 0$. Otherwise, by Lemma 2(5) and (d), $v$ is incident with at least one $(4, 5^{+}, 6^{+})$-face and at least two 4-1 faces each of which incident with at most one $3^-$-vertex. Moreover, $v$ sends at most $\frac{1}{2}$ to a $(4, 5^{+}, 6^{+})$-face by R2, at most $\frac{1}{2}$ to a 4*-face incident with at most one $3^-$-vertex by R3. Hence $\chi'(v) \geq 8 - \frac{\frac{3+2}{2}}{2} - \frac{1}{2} - \frac{1}{2} \times 2 - 2 = 0$. Suppose $t = 1$. Then $f_4(v) \leq 3$. If $f_4(v) \leq 3$, then $\chi'(v) \geq 8 - \frac{\frac{3+2}{2}}{2} - \frac{1}{2} - 3 - \frac{1}{2} \times 2 > 0$. Otherwise, $f_4(v) = 4$. Then $v$ is incident with at least one $(4, 5^{+}, 6^{+})$-face and at least two 4-1 faces each of which incident with at most one $3^-$-vertex by Lemma 2(5), and it follows
that $c'(v) \geq 8 - \frac{3+2}{2} - \frac{5}{4} - \frac{7}{4} \times 2 - (2 + \frac{1}{2}) > 0$ by Lemma 4. For $t = 0$, then $c'(v) \geq 8 - \frac{3+2}{2} - (2 + \frac{1}{2}) \times 2 > 0$ by Lemma 4.

For Case 4(d), $t \leq 1$. Suppose $t = 0$. Then $c'(v) \geq 8 - \frac{3+2}{2} - (1 + \frac{1}{2}) > 0$ by Lemma 4. Suppose $t = 1$. Then $f_3(v) \leq 4$. If $f_3(v) \leq 3$, then $c'(v) \geq 8 - \frac{3+2}{2} - 3 \times 1 - \frac{1}{2} \times 3 = 0$. Otherwise, $v$ is incident with at least one (4, 5, 6)-face and at least two 4-faces each of which incident with at most one 3-face. Then $c'(v) \geq 8 - \frac{3+2}{2} - \frac{5}{4} - 2 \times (1 + \frac{1}{2}) > 0$ by Lemma 4.

Case 5. $n_2(v) = 2$. All 2-vertices adjacent to $v$ are located as shown in Fig. 5, where the vertices marked by * are 2-vertices. For Fig. 5(a), $t \leq 3$ and $d(f_3) \geq 5$ by (b). Thus $v$ sends at most $\frac{1}{2}$ to $f_3$ by R4. Suppose $t = 3$. Assume that 3-faces incident with $v$ are adjacent mutually, without loss of generality, say $f_1, f_2, f_3$ are 3-faces. Moreover, $d(v^3) \geq 4$ and $d(v^3) \geq 4$ by Lemma 2(6), then $v$ is incident with at least one (4, 5, 6)-face. If $f_3(v) \geq 2$, then $c'(v) \geq 8 - \frac{3+2}{2} - \frac{5}{4} - 1 - \frac{1}{2} \times 2 > 0$. Otherwise, $f_3(v) \geq 2$. Then $c'(v) \geq 8 - \frac{3+2}{2} - \frac{7}{4} \times 2 - 3 \times 1 - \frac{1}{2} \times 2 > 0$. Assume that 3-faces incident with $v$ are not adjacent mutually. That is $f_1, f_2, f_3$ are 3-faces, then $f_3, f_3$ are 6-faces by the choice of $G$, and it follows that $c'(v) \geq 8 - \frac{3+2}{2} - 3 \times 1 - \frac{1}{2} \times (7 - 3 - t) = \frac{10-7t}{6} > 0$.

For Fig. 5(b), $t \leq 3$. Note that $v$ sends at most (1 + $\frac{1}{2}$) to $f_3$ and $f_3$ by Lemma 4. Suppose $t = 3$. That is, $f_1, f_2, f_3$ are 3-faces, then $f_4, f_5$ are 6-faces. It follows that $c'(v) \geq 8 - \frac{3+2}{2} - \frac{5}{4} - 3 \times 1 - \frac{1}{2} \times 2 > 0$. Suppose $t \geq 2$. Then $f_3(v) \geq 1$. Assume that $v$ is incident with two adjacent 3-faces. Then $f_3(v) \geq 2$ by the choice of $G$, and it follows that $c'(v) \geq 8 - \frac{3+2}{2} - \frac{5}{4} - 2 \times 3 - 1 - \frac{1}{2} \times 2 > 0$. Assume that the two 3-faces incident with $v$ are not adjacent. That is, $f_1, f_2$ are 3-faces, then $d(f_3) \geq 6$ by the choice of $G$. If $d(v) \geq 5$, then $c'(v) \geq 8 - \frac{3+2}{2} - \frac{5}{4} - 2 - 1 - \frac{1}{2} (\frac{1}{2} + 1) = \frac{1}{6} > 0$.

Otherwise, $f_3(v) \geq 2$. Then $c'(v) \geq 8 - \frac{3+2}{2} - \frac{7}{4} \times 2 - 3 \times 1 - \frac{1}{2} \times 2 > 0$. Otherwise, $f_3(v) \geq 4$, then $v$ is incident with at least one 4-face incident with at most one 3-face, and it follows that $c'(v) \geq 8 - \frac{3+2}{2} - \frac{7}{4} \times 2 - 3 \times 1 - \frac{1}{2} \times 2 > 0$. Suppose $t = 0$. Then $f_3(v) \geq 3$. We have $c'(v) \geq 8 - \frac{3+2}{2} - \frac{7}{4} \times 2 - \frac{1}{2} \times 3 > 0$.

For Fig. 5(c), $t \leq 3$. Suppose $t = 3$. That is, $f_1, f_2, f_3$ are 3-faces, then $f_4, f_5$ are 6-faces. Moreover, $f_3(v) \geq 4$ by (c) and then $f_4, f_5$ are 6-faces each of which incident with at most one 3-face by Lemma 2(5) and (6), and it follows that $c'(v) \geq 8 - \frac{3+2}{2} - \frac{7}{4} \times 2 - \frac{1}{2} \times 2 > 0$. Suppose $t = 2$. Then $f_3(v) \geq 1$. If $f_3(v) \geq 2$, then $c'(v) \geq 8 - \frac{3+2}{2} - \frac{7}{4} \times 2 - \frac{1}{2} \times 2 > 0$. Suppose $t = 1$. Then $f_3(v) \geq 1$. If $f_3(v) \geq 2$, then $c'(v) \geq 8 - \frac{3+2}{2} - \frac{7}{4} \times 2 - \frac{1}{2} \times 2 > 0$. Otherwise, $v$ is incident with at least one 4-face incident with at most one 3-face, and it follows that $c'(v) \geq 8 - \frac{3+2}{2} - \frac{7}{4} \times 2 - \frac{1}{2} \times 2 > 0$.

Case 6. $n_2(v) = 1$. Without loss of generality, assume that $d(v) = 2$. By (e), we have $n_4(v) \geq 1$.

Case 6.1. $v_2$ is incident with a 3-face, that is, $f_6$ is a (2, 7, 7)-face. Then $v$ sends at most 1 to $v_2$ by R1. Note that $t \leq 5$ and all other 3-faces except $f_6$ incident with $v$ are (4, 5, 6)-face by Lemma 2(2) and (3). If $4 \leq t \leq 5$, then $f_6(v) \geq 2$. So $c'(v) \geq 8 - 1 - \frac{1}{2} - \frac{1}{2} \times (t - 1) - (7 - 2 - t) = \frac{10-7t}{6} > 0$. If $2 \leq t \leq 3$, then $f_6(v) \geq 1$, and it follows that $c'(v) \geq 8 - \frac{3+2}{2} - \frac{5}{4} - 2 \times 3 - 1 - \frac{1}{2} \times 2 > 0$. If $t = 1$, then $f_6(v) \geq 1$, and it follows that $c'(v) \geq 8 - \frac{3+2}{2} - 3 \times 1 > 0$.

Case 6.2. $v_7$ is not incident with any 3-face. Then $v$ sends at most $\frac{1}{2}$ to $v_7$. By R1. Moreover, we obtain that $t \leq 4$. Suppose $0 \leq t \leq 1$. Then $f_6(v) \geq 2$. Thus $c'(v) \geq 8 - \frac{3+2}{2} - \frac{5}{4} - 2 \times (7 - 2 - t) = \frac{10-7t}{6} > 0$. Suppose $t = 2$. Then $f_6(v) \geq 1$. If $f_6(v) \geq 2$, then $c'(v) \geq 8 - \frac{3+2}{2} - 2 \times 3 > 0$.
Suppose \( t = 3 \). Then \( f_6^+(v) \geq 2 \), and it follows that \( ch'(v) \geq 8 - \frac{3}{2} - 3 \times \frac{3}{2} - 2 \times 1 = 0 \). Suppose \( t = 4 \). Then \( f_6^+(v) \geq 2 \).

By Lemma 2(6), \( v \) is incident with at least one \((4,5^+,6^+)\)-face. If \( v \) is incident with at least two \((4,5^+,6^+)\)-faces, then \( ch'(v) \geq 8 - \frac{3}{2} - 2 \times \frac{3}{2} - 2 \times \frac{3}{2} - 1 = 0 \). Otherwise, \( v \) is incident with one 4-face incident with at most one 3'-vertex by Lemma 3. Thus \( ch'(v) \geq 8 - \frac{3}{2} - 3 \times \frac{3}{2} - 2 \times 2 = 0 \).

Case 7. \( n_2(v) = 0 \). Note that \( t \leq 5 \). If \( 4 \leq t \leq 5 \), then \( f_6^+(v) \geq 2 \), and it follows that \( ch'(v) \geq 8 - \frac{3}{2} \times t - (7 - 2 - t) = \frac{6 - t}{2} > 0 \).

If \( t = 3 \), then \( f_6^+(v) \geq 1 \), and it follows that \( ch'(v) \geq 8 - \frac{3}{2} - 2 \times \frac{3}{2} - 3 - 3 > 0 \). If \( 0 \leq t \leq 2 \), then \( ch'(v) \geq 8 - \frac{3}{2} \times t - (7 - t) = \frac{2 - t}{2} \geq 0 \).

Hence we complete the proof of the theorem. □

References