Numerical solution of the convection diffusion equations by the second kind Chebyshev wavelets

Fengying Zhou, Xiaoyong Xu
School of Science, East China Institute of Technology, Nanchang 330013, PR China

Abstract
In this paper, a numerical method for solving convection diffusion equations is presented. The method is based upon the second kind Chebyshev wavelets approximation. The second kind Chebyshev wavelets operational matrix of integration is derived and utilized to transform the equation to a system of algebraic equations by combining collocation method. Numerical examples show that the proposed method has good efficiency and precision.

1. Introduction

Convection diffusion equations are widely used for modeling and simulations of various complex phenomena in science and engineering, such as migration of contaminants in a stream, smoke plume in atmosphere, dispersion of chemicals in reactors, tracer dispersion in a porous medium, etc. Since it is impossible to solve convection diffusion equations analytically for most application problems, efficient numerical algorithms are becoming increasingly important to numerical simulations involving convection diffusion equations. For this model, some authors have studied the numerical techniques such as the finite difference method [1–3], the Bessel collocation method [4], the finite element method [5,6], the wavelet-Galerkin method [7,8], the Crank–Nicholson method [9], the piecewise-analytical method [10] and ADI method [11]. Among these methods, the wavelet method is more attractive. In recent years, wavelets have received considerable attention in different field of science and engineering. Wavelets permit the accurate representation of variety of functions and operators, and establish a connection in dealing with various problems. The main characteristic of this method is that it reduces these problems to those of solving systems of algebraic equations, thus it greatly simplifies the problems.

The interested reader is referred to the recent published paper by Abd-Elhameed, et al. [12], in which a numerical method based on the second kind Chebyshev wavelets operational matrix of differentiation is derived and used to transform differential equation with their initial and boundary conditions to systems of algebraic equations. Another numerical method based on the third and fourth kinds Chebyshev wavelets is fully discussed in Abd-Elhameed, et al. [13].

In this paper, we consider the following convection diffusion equation with variable coefficients:

\[
\frac{\partial \mu}{\partial t} + a(x) \frac{\partial \mu}{\partial x} = b(x) \frac{\partial^2 \mu}{\partial x^2} + g(x, t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1,
\]

with initial condition

\[\text{Initial condition:} \quad \mu(x, 0) = \mu_0(x).\]
\( \mu(x, 0) = f(x), \; 0 \leq x \leq 1, \)

and boundary conditions
\[ \mu(0, t) = g_0(t), \; \mu(1, t) = g_1(t), \; 0 \leq t \leq 1, \]

where \( a(x) \) and \( b(x) (\neq 0) \) are continuous functions. The aim of the present work is to develop Chebyshev wavelets method with the operational matrix of integration to solve the above convection diffusion equation. The rest part of this paper is organized as follows. Section 2 introduces the second kind Chebyshev wavelets and their properties. In Section 3, Chebyshev wavelets operational matrix of integration is derived. In Section 4, the proposed method is applied to approximate the solution of the problem. Section 5 gives some examples to test the proposed method. A conclusion is drawn in Section 6.

2. Properties of the second kind Chebyshev wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter \( a \) and the translation parameter \( b \) vary continuously, we have the following family of continuous wavelets:
\[ \psi_{a,b}(t) = |a|^{-1/2} \psi \left( \frac{t-b}{a} \right), \quad a, b \in \mathbb{R}, a \neq 0. \]

If we restrict the parameters \( a \) and \( b \) to discrete values as \( a = a_0^j, b = mb_0a_0^j \), where \( a_0 > 1, b_0 > 0, \) and \( j, m \) are positive integers, we have the following family of discrete wavelets:
\[ \psi_{j,m}(t) = |a_0|^{j/2} \psi(a_0^j t - mb_0), \]

which form a wavelet basis for \( L^2(\mathbb{R}) \). In particular, when \( a_0 = 2 \) and \( b_0 = 1 \), then \( \psi_{j,m}(t) \) form an orthonormal basis.

The second kind Chebyshev wavelets \( \psi_{n,m}(t) = \psi(k, n, m, t) \) have four arguments \([14]\): \( k \) can assume any positive integer, \( n = 1, 2, 3, \ldots, 2^k - 1 \), \( m \) is the degree of the second kind Chebyshev polynomials and \( t \) is the normalized time. They are defined on the interval \([0, 1]\) as
\[ \psi_{n,m}(t) = \begin{cases} 2^k \bar{U}_m(2^k t - 2n + 1), & \frac{n-1}{2^k} \leq t < \frac{n}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \]

where
\[ \bar{U}_m(t) = \sqrt{\frac{2}{n}} U_m(t), \quad (1) \]

\( m = 0, 1, 2, \ldots, M - 1 \) and \( M \) is a fixed positive integer. The coefficient in relation (1) is for orthonormality. Here \( U_m(t) \) are the second kind Chebyshev polynomials of degree \( m \) which are orthogonal with respect to the weight function \( \omega(t) = \sqrt{1 - t^2} \) on the interval \([-1, 1]\) and satisfy the following recursive formula:
\[ U_0(t) = 1, \quad U_1(t) = 2t, \]
\[ U_{m+1}(t) = 2t U_m(t) - U_{m-1}(t), \quad m = 1, 2, 3, \ldots. \]

Note that when dealing with the second kind Chebyshev wavelets the weight function has to be dilated and translated as
\[ \omega_k(t) = \omega(2^k t - 2n + 1). \]

A function \( f(x) \in L^2(\mathbb{R}) \) defined on the interval \([0, 1]\) may be expanded by the second kind Chebyshev wavelets as
\[ f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x), \quad (2) \]

where
\[ c_{n,m} = \langle f(x), \psi_{n,m}(x) \rangle_{L^2(\mathbb{R})} = \int_0^1 f(x) \psi_{n,m}(x) \omega_k(x) dx, \]

in which \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{R})} \) denotes the inner product in \( L^2(\mathbb{R}) \). If the infinite series in Eq. (2) is truncated, then it can be written as
\[ f(x) \approx \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) = C^T \Psi(x), \]

where \( C \) and \( \Psi(x) \) are \( 2^{k-1}M \times 1 \) matrices given by
\[ C = \begin{pmatrix} C_{1,0} & C_{1,1} & \ldots & C_{1,M-1} & C_{2,0} & C_{2,1} & \ldots & C_{2,M-1} & \ldots & C_{2^{k-1},0} & C_{2^{k-1},1} & \ldots & C_{2^{k-1},M-1} \end{pmatrix}^T, \]
Let $f(x)$ be a second-order derivative square-integrable function defined on $[0, 1]$ with bounded second-order derivative, say $|f''(x)| \leq B$ for some constant $B$, then

(i) $f(x)$ can be expanded as an infinite sum of the second kind Chebyshev wavelets and the series converges to $f(x)$ uniformly, that is

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x),$$

where $c_{n,m} = \langle f(x), \psi_{n,m}(x) \rangle_{L^2([0,1])}$.

(ii)

$$\sigma_{f,k,M} \leq \frac{\sqrt{\pi B}}{2} \left( \sum_{n=2^{k-1}}^{\infty} \frac{1}{n^2} \sum_{m=0}^{\infty} \frac{1}{(m-1)^4} \right)^{\frac{1}{2}},$$

where $\sigma_{f,k,M} = \left( \int_0^1 |f(x) - \sum_{n=0}^{2^{k-1}-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x)|^2 \omega_n(x)dx \right)^{\frac{1}{2}}$. 

Theorem 1.
Proof 1

(i) $c_{n,m} = \langle f(x), \psi_{n,m}(x) \rangle_{L^2([0,1])} = \int_0^1 f(x) \psi_{n,m}(x) \omega_n(x) dx = \int_{-\frac{\pi}{2k}}^{\frac{\pi}{2k}} f(x) \sqrt{\frac{2}{\pi}} U_m(2^k x - 2n + 1) \omega(2^k x - 2n + 1) dx.$

Let $2^k x - 2n + 1 = t$, then

$$c_{n,m} = \sqrt{\frac{2}{\pi}} \frac{1}{2^k} \int_{-1}^{1} f\left(\frac{t + 2n - 1}{2^k}\right) U_m(t) \sqrt{1 - t^2} dt.$$  

By letting $t = \cos \theta$ and the definition of the second kind Chebyshev wavelets, it follows

$$c_{n,m} = \sqrt{\frac{2}{\pi}} \frac{1}{2^k} \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \sin (m + 1) \theta \sin \theta d\theta$$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{1}{2^k} \left( \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \cos m \theta \sin \theta d\theta - \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \cos (m + 2) \theta d\theta \right).$$

Using the integration by parts, we have

$$c_{n,m} = \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{1}{2^k} \frac{1}{m} \left( \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \sin m \theta \sin \theta d\theta - \frac{1}{m + 2} \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \sin (m + 2) \theta \sin \theta d\theta \right)$$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{1}{2^k} \left( I_1 - I_2 \right),$$  \hspace{1cm} (6)

where $I_1 = \frac{1}{m} \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \sin m \theta \sin \theta d\theta$ and $I_2 = \frac{1}{m + 2} \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \sin (m + 2) \theta \sin \theta d\theta$. Next we estimate $I_1$ and $I_2$, respectively. A simple computation shows that

$$I_1 = \frac{1}{m} \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \sin m \theta \sin \theta d\theta = \frac{1}{2m} \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) (\cos (m - 1) \theta - \cos (m + 1) \theta) d\theta$$

$$= \frac{1}{2m} \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \cos (m - 1) \theta d\theta - \frac{1}{2m} \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \cos (m + 1) \theta d\theta = I_{11} - I_{12},$$

where $I_{11} = \frac{1}{2m} \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \cos (m - 1) \theta d\theta$ and $I_{12} = \frac{1}{2m} \int_{0}^{\pi} f\left(\cos \theta + \frac{2n - 1}{2^k}\right) \cos (m + 1) \theta d\theta$. By using the integration by parts, we get

$$I_{11} = \frac{2^k}{2m(m - 1)} \int_{0}^{\pi} f'\left(\cos \theta + \frac{2n - 1}{2^k}\right) \sin (m - 1) \theta \sin \theta d\theta$$

$$= \frac{2^k}{4m(m - 1)} \int_{0}^{\pi} f'\left(\cos \theta + \frac{2n - 1}{2^k}\right) (\cos (m - 2) \theta - \cos m \theta) d\theta,$$

for $m > 1$, and

$$I_{12} = \frac{2^k}{2m(m + 1)} \int_{0}^{\pi} f'\left(\cos \theta + \frac{2n - 1}{2^k}\right) \sin (m + 1) \theta \sin \theta d\theta$$

$$= \frac{2^k}{4m(m + 1)} \int_{0}^{\pi} f'\left(\cos \theta + \frac{2n - 1}{2^k}\right) (\cos m \theta - \cos (m + 2) \theta) d\theta.$$

Thus, for $m > 1$, we have

$$I_1 = \frac{2^k}{4m} \int_{0}^{\pi} f'\left(\cos \theta + \frac{2n - 1}{2^k}\right) \left( \frac{\cos (m - 2) \theta - \cos m \theta}{m - 1} - \frac{\cos m \theta - \cos (m + 2) \theta}{m + 1} \right) d\theta.$$  

So

$$|I_1|^2 = \frac{2^k}{4m} \int_{0}^{\pi} f'\left(\cos \theta + \frac{2n - 1}{2^k}\right) \left( \frac{\cos (m - 2) \theta - \cos m \theta}{m - 1} - \frac{\cos m \theta - \cos (m + 2) \theta}{m + 1} \right) d\theta^2$$

$$= \frac{2^{2k}}{16m^2} \int_{0}^{\pi} f'\left(\cos \theta + \frac{2n - 1}{2^k}\right) \left( \frac{m + 1}{(m - 1)(m + 1)} \cos (m - 2) \theta - 2m \cos m \theta + (m - 1) \cos (m + 2) \theta \right) d\theta^2.$$
By the fact that $|f''(x)| \leq B$ and Schwarz inequality, it follows

$$
|I_1|^2 \leq \frac{2^{-2k}}{16m^2} \int_0^\pi f'(\frac{\cos \theta + 2n - 1}{2^k})^2d\theta \cdot \int_0^\pi \left| (m + 1) \cos(m - 2)\theta - 2m \cos m\theta + (m - 1) \cos(m + 2)\theta \right|^2d\theta
$$

$$\leq \frac{2^{-2k} \pi B^2}{16m^2(m - 1)^2} \int_0^\pi \left| (m + 1) \cos(m - 2)\theta - 2m \cos m\theta + (m - 1) \cos(m + 2)\theta \right|^2d\theta
$$

$$= \frac{2^{-2k} \pi B^2}{16m^2(m - 1)^2} \int_0^\pi \left[ \int_0^\pi \left( (m + 1)^2 \cos^2(m - 2)\theta d\theta + \int_0^\pi 4m^2 \cos^2 m\theta d\theta + \int_0^\pi (m - 1)^2 \cos^2(m + 2)\theta d\theta \right) \right]
$$

$$= \frac{2^{-2k} \pi B^2}{16m^2(m - 1)^2} \left( \frac{\pi}{2} (m + 1)^2 + \frac{\pi}{2} 4m^2 + \frac{\pi}{2} (m - 1)^2 \right) = \frac{2^{-2k} \pi B^2 (3m^2 + 1)}{16m^2(m - 1)^2} \leq \frac{2^{-2k} \pi B^2}{4(m - 1)^2},
$$

where the fourth equality follows for $m > 2$. Hence, for $m > 2$, we obtain

$$|I_1| \leq \frac{2^{-k} \pi B}{2(m - 1)^{\frac{3}{2}}}.
$$

Similar analysis, enables one to get

$$|I_2| \leq \frac{2^{-k} \pi B}{2(m - 1)^{\frac{3}{2}}}.
$$

Therefore, for $m > 2$, we have

$$|c_{n,m}| = \left| \frac{1}{\sqrt{\pi}} \int_0^\pi \frac{\sqrt{2}}{\pi} \frac{nB}{2} \left( (I_1 - I_2) \right) \right| \leq \frac{1}{\sqrt{\pi}} \frac{\sqrt{2}}{\pi} \frac{nB}{2} \left( \left| I_1 \right| + \left| I_2 \right| \right) \leq \frac{1}{\sqrt{\pi}} \frac{\sqrt{2}}{\pi} \frac{nB}{2} \left( \left| I_1 \right| + \left| I_2 \right| \right) \leq \frac{1}{\sqrt{\pi}} \frac{\sqrt{2}}{\pi} \frac{nB}{2} \left( \left| I_1 \right| + \left| I_2 \right| \right) = \frac{1}{\sqrt{\pi}} \frac{\sqrt{2}}{\pi} \frac{nB}{2} \left( \left| I_1 \right| + \left| I_2 \right| \right) \leq \frac{\sqrt{nB}}{2} \frac{1}{\sqrt{\pi}} \frac{1}{m(m - 1)^{\frac{3}{2}}}. \tag{7}
$$

Note that $f'(x)$ is bounded on $[0, 1]$ due to the fact that $|f''(x)| \leq B$. Indeed, by the Differential Mean Value Theorem and for any $x \in (0, 1)$, there exists some $\xi_x \in (0, x)$ such that

$$f'(x) - f'(0) = f''(\xi_x)x.
$$

So

$$|f'(x)| \leq |f'(0)| + B,
$$

for $x \in (0, 1)$. Thus $f'(x)$ is bounded on $[0, 1)$, say $|f'(x)| \leq \tilde{B}$ for some constant $\tilde{B}$. Hence, by (6), we have

$$|c_{n,1}| \leq \frac{1}{\sqrt{\pi}} \frac{\sqrt{2}}{\pi} \frac{nB}{2} \left( \int_0^\pi \left| f'(\frac{\cos \theta + 2n - 1}{2^k}) \right|^2d\theta + \frac{1}{3} \int_0^\pi \left| f'(\frac{\cos \theta + 2n - 1}{2^k}) \right|^2d\theta \right)
$$

$$= \frac{1}{\sqrt{\pi}} \frac{\sqrt{2}}{\pi} \frac{nB}{2} \left( \int_0^\pi \left| f'(\frac{\cos \theta + 2n - 1}{2^k}) \right|^2d\theta + \frac{1}{3} \int_0^\pi \left| f'(\frac{\cos \theta + 2n - 1}{2^k}) \right|^2d\theta \right) \leq \frac{1}{\sqrt{\pi}} \frac{\sqrt{2}}{\pi} \frac{nB}{2} \left( \int_0^\pi \left| f'(\frac{\cos \theta + 2n - 1}{2^k}) \right|^2d\theta + \frac{1}{3} \int_0^\pi \left| f'(\frac{\cos \theta + 2n - 1}{2^k}) \right|^2d\theta \right) \leq \frac{\sqrt{\pi}}{2} \frac{\sqrt{2}}{\pi} \frac{nB}{3} \leq \frac{3\sqrt{\pi}B}{16} \frac{1}{n^{\frac{3}{2}}}, \tag{8}
$$

and

$$|c_{n,2}| \leq \frac{1}{\sqrt{\pi}} \frac{\sqrt{2}}{\pi} \frac{nB}{2} \left( \int_0^\pi \left| f'(\frac{\cos \theta + 2n - 1}{2^k}) \right|^2d\theta + \frac{1}{4} \int_0^\pi \left| f'(\frac{\cos \theta + 2n - 1}{2^k}) \right|^2d\theta \right) \leq \frac{1}{\sqrt{\pi}} \frac{\sqrt{2}}{\pi} \frac{nB}{2} \left( \int_0^\pi \left| f'(\frac{\cos \theta + 2n - 1}{2^k}) \right|^2d\theta + \frac{1}{4} \int_0^\pi \left| f'(\frac{\cos \theta + 2n - 1}{2^k}) \right|^2d\theta \right) \leq \frac{3\sqrt{\pi}B}{16} \frac{1}{n^{\frac{3}{2}}}. \tag{9}
$$

Relations (7)–(9) show that the series $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}$ is absolutely convergent. For $m = 0$ and according to the definition of $\psi_{n,0}(x)$, the series $\sum_{n=1}^{\infty} c_{n,0} \psi_{n,0}(x)$ is convergent. Therefore, the series $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x)$ converges to $f(x)$ uniformly.

(ii) $\sigma_{f,k}^2 = \int_0^1 f(x) - \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x)^2 \omega_n(x)dx = \int_0^1 \left| \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} c_{n,m} \psi_{n,m}(x)^2 \omega_n(x)dx = \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} |c_{n,m}|^2, \right.$

where the last equality follows due to the orthonormality of $\psi_{n,m}(x)$. Together with (7), we get

$$\sigma_{f,k}^2 \leq \frac{\pi B^2}{2} \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m(m - 1)^{\frac{3}{2}}},
$$

which completes the proof of the theorem.
3. The second kind Chebyshev wavelets operational matrix of integration

In this section, we will derive precise integral of the second kind Chebyshev wavelets functions which play a great role in dealing with the problem of convection diffusion equations. First, we figure out the precise integral of the second kind Chebyshev wavelets functions with \( k = 2 \) and \( M = 3 \). In this case, the six basis functions are given by

\[
\begin{align*}
\psi_{1.0}(t) & = 2 \sqrt{\frac{2}{\pi}} t, \\
\psi_{1.1}(t) & = 2 \sqrt{\frac{2}{\pi}} (8t - 2), \\
\psi_{1.2}(t) & = 2 \sqrt{\frac{2}{\pi}} (64t^2 - 32t + 3)
\end{align*}
\]

on \( 0 \leq t < \frac{1}{2} \) and

\[
\begin{align*}
\psi_{2.0}(t) & = 2 \sqrt{\frac{2}{\pi}} (8t - 6), \\
\psi_{2.1}(t) & = 2 \sqrt{\frac{2}{\pi}} (64t^2 - 96t + 35),
\end{align*}
\]

on \( \frac{1}{2} \leq t < 1 \). Let \( \Psi_6(t) = (\psi_{1.0}(t) \ \psi_{1.1}(t) \ \psi_{1.2}(t) \ \psi_{2.0}(t) \ \psi_{2.1}(t) \ \psi_{2.2}(t))^T \). By integrating (10) and (11) from 0 to \( t \) and representing them in the matrix form, we obtain

\[
\int_0^t \psi_{1.0}(s) ds = \begin{cases} 
2 \sqrt{\frac{2}{\pi}} t, & 0 \leq t < \frac{1}{2}, \\
2 \sqrt{\frac{2}{\pi}} \frac{1}{2} \leq t < 1, 
\end{cases} \quad \int_0^t \psi_{1.1}(s) ds = \begin{cases} 
4 \sqrt{\frac{2}{\pi}} (2t^2 - t), & 0 \leq t < \frac{1}{2}, \\
0, & \frac{1}{2} \leq t < 1, 
\end{cases} \quad \int_0^t \psi_{1.2}(s) ds = \begin{cases} 
2 \sqrt{\frac{2}{\pi}} \left(64t^2 - 16t^2 + 3t\right), & 0 \leq t < \frac{1}{2}, \\
2 \sqrt{\frac{2}{\pi}} \left(16t^2 - 16t^2 + 3t\right), & \frac{1}{2} \leq t < 1, 
\end{cases}
\]

\[
\int_0^t \psi_{2.0}(s) ds = \begin{cases} 
0, & 0 \leq t < \frac{1}{2}, \\
2 \sqrt{\frac{2}{\pi}} \left(t^2 - \frac{1}{2}\right), & \frac{1}{2} \leq t < 1, 
\end{cases} \quad \int_0^t \psi_{2.1}(s) ds = \begin{cases} 
0, & 0 \leq t < \frac{1}{2}, \\
2 \sqrt{\frac{2}{\pi}} \left(2t^2 - 3t + 1\right), & \frac{1}{2} \leq t < 1, 
\end{cases} \quad \int_0^t \psi_{2.2}(s) ds = \begin{cases} 
0, & 0 \leq t < \frac{1}{2}, \\
2 \sqrt{\frac{2}{\pi}} \left(48t^2 - 48t^2 + 35t - 48\right), & \frac{1}{2} \leq t < 1, 
\end{cases}
\]

Thus

\[
\int_0^t \Psi_6(s) ds = P_{6 \times 6} \Psi_6(t) + \Psi_6(t),
\]

where

\[
P_{6 \times 6} = \frac{1}{4} \begin{pmatrix}
1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & -\frac{1}{6} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 1 & \frac{1}{2} & 0 \\
0 & 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} \\
0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & 0 
\end{pmatrix}
\]
and \( \tilde{\Psi}_6(t) = \frac{1}{2^3} (0 \ 0 \ \psi_{1,3}(t) \ 0 \ 0 \ \psi_{2,3}(t))^T \). In fact, the matrix \( P_{6 \times 6} \) can be written as
\[
P_{6 \times 6} = \frac{1}{4} \begin{pmatrix} L_{3 \times 3} & F_{3 \times 3} \\ 0_{3 \times 3} & L_{3 \times 3} \end{pmatrix},
\]
where
\[
L_{3 \times 3} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{8} & 0 \end{pmatrix} \quad \text{and} \quad F_{3 \times 3} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.
\]
In general, when \( M \geq 4 \), we have
\[
\int_0^t \Psi(s) \, ds = P\Psi(t) + \tilde{\Psi}(t),
\]
where \( \Psi(t) \) is given in (3) and \( P \) is a \( 2^{k-1}M \times 2^{k-1}M \) matrix given by
\[
P = \frac{1}{2^k} \begin{pmatrix} L & F & F & \cdots & F \\ 0 & L & F & \cdots & F \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & \cdots & L \end{pmatrix},
\]
here \( L \) and \( F \) are \( M \times M \) matrices given by
\[
L = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & -\frac{1}{8} & 0 & \frac{1}{8} & \cdots & 0 & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{8} & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ (-1)^{M-2} & \frac{1}{2^{M-1}} & 0 & 0 & \cdots & 0 & \frac{1}{2^{M-1}} \\ (-1)^{M-1} & \frac{1}{2^M} & 0 & 0 & \cdots & -\frac{1}{2^M} & 0 \end{pmatrix},
\]
\[
F = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 0 & 0 & \cdots & 0 \\ \alpha_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_M & 0 & 0 & \cdots & 0 \end{pmatrix},
\]
where \( \alpha_i = \frac{1}{2^i}, \ i \ odd, \quad \alpha_i = 0, \ i \ even, \quad i = 1, 2, 3, \ldots, M \), and \( \tilde{\Psi}(t) \) in Eq. (12) is called as omitted item given by
\[
\tilde{\Psi}(t) = \frac{1}{2^k} (L_1 \ L_2 \ L_3 \ \cdots \ L_{2^{k-1}})^T,
\]
where \( L_i \) are \( 1 \times M \) matrices given by
\[
L_i = \frac{1}{2^M} (0 \ 0 \ 0 \ \cdots \ 0 \ \psi_{i,M}),
\]
i = 1, 2, 3, \ldots, \ 2^{k-1}. It is worthy to say that \( \tilde{\Psi}(t) \) in Eq. (12) is often omitted in many literatures for simplicity when performing numerical calculations [15–17].

4. Description of the proposed method

In this section, we will use the second kind Chebyshev wavelets operational matrix of integration for solving the initial boundary value problem of convection diffusion equation with variable coefficients. Consider the convection diffusion equation of the following form:
\[
\frac{\partial \mu}{\partial t} + a(x) \frac{\partial \mu}{\partial x} = b(x) \frac{\partial^2 \mu}{\partial x^2} + g(x, t), \quad 0 \leq x \leq 1, \ 0 \leq t \leq 1,
\]
with initial condition
\[ \mu(x, 0) = f(x), \quad 0 \leq x \leq 1, \quad (14) \]
and boundary conditions
\[ \mu(0, t) = g_0(t), \quad \mu(1, t) = g_1(t), \quad 0 \leq t \leq 1, \quad (15) \]
where \( a(x) \) denotes diffusion coefficient and \( b(x) \) denotes convection coefficient, and \( g(x, t) \) denotes source item, \( g_0(t) \) and \( g_1(t) \) are continuous functions with first-order derivative. In order to solve this problem, we assume
\[ \frac{\partial^3 \mu}{\partial \xi^3} = \Psi^T(\xi) D \Psi(t), \quad (16) \]
where \( D = (d_{ij})_{2\times 2} \) is an unknown matrix which should be determined, and \( \Psi(\cdot) \) is the vector defined in (3). By integrating (16) from 0 to \( t \), we obtain
\[ \frac{\partial^2 \mu}{\partial \xi^2} = \Psi^T(\xi) D \int_0^t \Psi(s) ds. \quad (17) \]
Making use of the initial condition (14), enables one to put Eq. (17) in the form
\[ \frac{\partial^2 \mu}{\partial \xi^2} = f''(x) + \Psi^T(\xi) D \int_0^t \Psi(s) ds. \quad (18) \]
By integrating (18) two times from 0 to \( x \), we obtain
\[ \mu(x, t) = \mu(0, t) + \frac{x}{2} \frac{\partial \mu}{\partial \xi} \bigg|_{\xi=0} + f(x) - f(0) - xf'(0) + \int_0^x \int_0^u \Psi^T(v) dv du \cdot D \cdot \int_0^t \Psi(s) ds, \quad (19) \]
and by putting \( x = 1 \) in (19), we get
\[ \mu(x, t) = \mu(0, t) + x H(t) + f(x) - f(0) - xf'(0) + \int_0^x \int_0^u \Psi^T(v) dv du \cdot D \cdot \int_0^t \Psi(s) ds, \quad (20) \]
where
\[ H(t) = \mu(1, t) - \mu(0, t) + f(0) + f'(0) - f(1) - \int_0^1 \int_0^u \Psi^T(v) dv du \cdot D \cdot \int_0^t \Psi(s) ds. \]

Fig. 1. Approximate solution of Example 1.
By one time differentiation of (20) with respect to \( x \) and \( t \), respectively, we obtain

\[
\frac{\partial \mu}{\partial x} = H(t) + f'(x) - f'(0) + \int_0^x \Psi'(v)dv \cdot D \cdot \int_0^t \Psi(s)ds,
\]

\[
\frac{\partial \mu}{\partial t} = \mu'(0, t) + xH'(t) + \int_0^x \int_0^t \Psi'(v)dv \cdot D \cdot \Psi(t),
\]

where

Fig. 2. Absolute errors of Example 1.

Fig. 3. Numerical and exact solutions in different values of \( t \) for Example 1.
\[ \dot{H}(t) = \mu'(1,t) - \mu'(0,t) - \int_0^1 \int_0^t \Psi'(v)d\nu \cdot D \cdot \Psi(t). \]

Now by substituting (18), (21), (22) into (13) and combining (12), (15) and taking collocation points \( x_i = \frac{2i-1}{2^{k+1}}, t_j = \frac{2j-1}{2^{k+1}}, i,j = 1,2,3,\ldots,2^{k-1}M, \) we obtain the following linear system of algebraic equations.
\[ g_0'(t_j) + \lambda \left( g_0'(t_j) - g_0'(t_j) \right) - \int_0^1 \int_0^u \Psi^T(v)dvdu \cdot D \cdot \Psi(t_j) \right) + \left( (\Psi^T(\lambda)P^T + \Psi^T(\lambda))P + \int_0^1 \Psi^T(u)du \cdot D \cdot \Psi(t_j) \right) \\
+ a(\lambda) \left( H(t_j) + f'(\lambda) - f'(0) + (\Psi^T(\lambda)P^T + \Psi^T(\lambda)) \cdot D \cdot (P\Psi(t_j) + \Psi(t_j)) \right) \\
= b(\lambda) \left( f''(\lambda) + \Psi^T(\lambda) \cdot D \cdot (P\Psi(t_j) + \Psi(t_j)) \right) + g(\lambda, t_j), \]

Fig. 5. Approximate solution of Example 2.

Fig. 6. Absolute errors of Example 2.
\[i, j = 1, 2, 3, \ldots, 2^{k-1}M.\] By solving this system to determine \(D\), we can get the numerical solution of this problem by substituting \(D\) into (20).

5. Numerical examples

In this section, we will use the proposed method to solve the initial boundary value problem of convection diffusion equation with variable or constant coefficients. The following numerical examples are given to show the effectiveness and practicality of the method and the results have been compared with the exact solution.

![Numerical solutions](image1)

Fig. 7. Numerical and exact solutions in different values of \(t\) for Example 2.

![Absolute errors](image2)

Fig. 8. Absolute errors in different values of \(t\) for Example 2.
**Example 1.** Consider convection diffusion Eq. (13) with \( a(x) = -0.1 \), \( b(x) = 0.01 \) and \( g(x, t) = 0 \). The boundary conditions are given by \( \mu(x, 0) = e^{-x} \), \( \mu(0, t) = e^{-0.09t} \), \( \mu(1, t) = e^{-1-0.09t} \). The exact solution of this problem is \( \mu(x, t) = e^{-x-0.09t} \). The space–time diagram of the numerical solution for \( M = 6 \), \( k = 2 \) is shown in Fig. 1. Absolute errors between the numerical and analytical solutions are shown in Fig. 2. The graph of analytical and approximate solutions for some nodes on \( [0, 1] \times [0, 1] \) is presented in Fig. 3. Absolute errors between the numerical and analytical solutions at different times are shown in Fig. 4.

![Approximate solution](image1)

Fig. 9. Approximate solution of Example 3.

![Absolute errors](image2)

Fig. 10. Absolute errors of Example 3.
From Table 1 and Table 2, we can see that the approximate solutions obtained by adding omitted item $\Psi(t)$ in (12) are more accurate than the case that $\Psi(t)$ in (12) is omitted. So the following examples are all the case that $\Psi(t)$ in (12) is not omitted.

Example 2. Consider the convection diffusion Eq. (13) with $a(x) = -\frac{x}{2}$, $b(x) = \frac{x^2}{12}$ and $g(x, t) = 0$. The boundary conditions are given by $\mu(x, 0) = x^3$, $\mu(0, t) = 0$, $\mu(1, t) = e^t$. The exact solution of this problem is $\mu(x, t) = x^3e^t$. The space–time diagram of

Fig. 11. Numerical and exact solutions in different values of $t$ for Example 3.

Fig. 12. Absolute errors in different values of $t$ for Example 3.
the numerical solution for \( M = 6, k = 2 \) is shown in Fig.5. Absolute errors between the numerical and analytical solutions are shown in Fig.6. The graph of analytical and approximate solutions for some nodes on \([0, 1] \times [0, 1]\) is presented in Fig.7. Absolute errors between the numerical and analytical solutions at different times are shown in Fig.8.

**Example 3.** Consider the convection diffusion Eq. (13) with \( a(x) = 2, b(x) = 1 \) and \( g(x, t) = -2e^{-x} \). The boundary conditions are given by \( \mu(x, 0) = e^{-x}, \mu(0, t) = e^t, \mu(1, t) = e^{1-t} \). The exact solution of this problem is \( \mu(x, t) = e^{-x} \). The space–time diagram of the numerical solution for \( M = 6, k = 2 \) is shown in Fig.9. Absolute errors between the numerical and analytical solutions are shown in Fig.10. The graph of analytical and approximate solutions for some nodes on \([0, 1] \times [0, 1]\) is presented in Fig.11. Absolute errors between the numerical and analytical solutions at different times are shown in Fig.12.

### 6. Conclusion

In this paper, we have derived the second kind Chebyshev wavelets operational matrix of integration and proposed a numerical method to approximate the solution of the initial boundary value problem of convection diffusion equation with variable or constant coefficients. The method is computationally efficient and the algorithm can be implemented easily on a computer. The advantage of the method is that only small size operational matrix is required to provide the solution at high accuracy. Numerical examples are given to show that the proposed method is applicable, efficient and accurate.

### Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.amc.2014.08.091.

### References


