Greedy Orthogonal Matching Pursuit Algorithm for Sparse Signal Recovery in Compressive Sensing

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Abstract—The sparse signal recovery problem has been the subject of extensive research in several different communities. Tractable recovery algorithm is a crucial and fundamental theme of compressive sensing (CS), which has drawn significant interests in the last few years. In this paper, we firstly analyze the iterative residual in Orthogonal Matching Pursuit (OMP) algorithm. Secondly, a greedier algorithm is introduced, which is called Greedy OMP (GOMP) algorithm. This algorithm iteratively identifies more than one atoms using greedy atom identification, and then discards some atoms, which are of high similarity with the optimal atom. Compared with OMP algorithm, the experiments conducted on Gaussian and Zero-one sparse signal demonstrate that the proposed GOMP algorithm can provide better recovery performance. Finally, we experimentally investigate the effect of greedy constant in GOMP upon the recovery performance.

Keywords—Compressive sensing; orthogonal matching pursuit; sparse signal reconstruction; measurement matrix

I. INTRODUCTION

Compressive sensing [1], has received considerable attention recently and has shown significant advantage in signal sampling and compression. Let \( x \in \mathbb{R}^n \) be \( k \) sparse signal, which means the number of non-zero components in the vector \( x \) and is not greater than \( k \), i.e., \( \|x\|_0 = k \ll n \). CS theory indicates that \( x \) can be sampled and compressed simultaneously via compressive measurement \( y = \Phi x \in \mathbb{R}^M \) with \( M = o(k \log(n/k)) \). \( \Phi \) and \( y \) respectively denote the \( m \times n \) measurement matrix and measurement signal, where \( \Phi \) is generated by arranging a collection of sampling vectors \( \{\phi_m\}_{m=1}^n \) as its column, and the sampling vectors are also known as atoms of the matrix. Note that \( \Phi \) is always an overcomplete matrix \( (m \ll n) \), which makes the recovery of original sparse signal \( x \) underdetermined. In CS, the original sparse signal can be recovered or estimated by solving the following \( l_0 \) norm problem:

\[
\min_x \|x\|_0 \quad \text{s.t.} \quad y = \Phi x
\]  
(1)

However, the minimum \( l_0 \) solution is NP-hard, so instead we use the minimum \( l_1 \) solution

\[
\min_x \|x\|_1 \quad \text{s.t.} \quad y = \Phi x
\]  
(2)

In order to keep the energy or information unchanged during the compressive measurement, the measurement matrix \( \Phi \) should satisfy some properties. One important and widely mentioned property is Restricted Isometry Property (RIP) [2], which is defined with respect to an isometry constant \( 0 < \delta \leq 1 \). For all \( k \) sparse signal \( x \), the \( k \) order Restricted Isometry Constant (RIC) of \( \Phi \) is the smallest \( \delta_k \) that satisfies

\[
\sqrt{1 - \delta_k} \|x\|_2 \leq \|\Phi x\|_2 \leq \sqrt{1 + \delta_k} \|x\|_2
\]  
(3)

where \( \|x\|_2 \) represents norm two operator. Meanwhile the coherence of measurement matrix has been used to indicate the property of it, which is defined as the maximum absolute inner product of any two distinct atoms, i.e.,

\[
\mu = \min_{i \neq j} \left| \phi_i^T \phi_j \right|
\]  
(4)

The signal recovery problem can be solved via convex optimization, including linear programming (LP) and Basis Pursuit (BP) [3], which has higher recover accuracy but more complexity burden. Another set of algorithms, which can efficiently reconstruct sparse signal from linear measurement, are based on the idea of iterative greedy pursuit. A traditional method is Orthogonal Matching Pursuit (OMP) [4], which adds one atom to the approximation of support, recovers the signal and renews the residual in each iteration. Even recently, the Stagewise OMP (StOMP) [5] and Subspace Pursuit (SP) [6] were introduced and analyzed for Compressive Sensing signal reconstruction. Currently, convex optimization obtains the best recovery performance in theory, while its greedy counterparts offer desirable computational complexity around \( o(kn\log n) \), which is significantly lower than that of BP about \( o(n^3) \). In this way, they overcome the limit in practical application.

The main contribution of this paper is that we propose a greedier algorithm, Greedy Orthogonal Matching Pursuit (GOMP), to recover the original sparse signal. This algorithm inherits the atom identification from OMP algorithm. Moreover, GOMP algorithm adopts Greedy atom identification process to identify more atoms in each iteration, and then optimizes the selected atoms using similarity analysis. The simulation results indicate the excellent performances in recovering Gaussian and zero-one sparse signal.

The remainder of this paper is organized as follows. Section II introduces the original Orthogonal Matching Pursuit (OMP) algorithm and some theoretical results. Section III provides the detailed description of the proposed Greedy OMP (GOMP) algorithm. Simulation results are given in Section IV.
II. OMP Algorithms

OMP algorithm is the most widely used signal recovery algorithm in compressive sensing and is proved to be practical and easy to be implemented. OMP algorithm is a classical greedy algorithm and its performance is dependent heavily on the properties of the measurement matrix. The procedure of OMP algorithm can be divided into three steps: identification, recovery and renewal, which are presented as followed.

**OMP Algorithm**

**Input:** measurement signal \( y \), measurement matrix \( \Phi \)

**Initialize:** residual \( y_r = y \), support \( \Lambda = \emptyset \), recovered signal \( \hat{x} = 0 \)

**Iteration:** at the \( l \)th iteration, until stopping criterion is met.

1. \( \alpha = \arg \max_{j \in \Lambda} \| \Phi_{j}^T y \| \); 
2. \( \Lambda = \Lambda \cup \{i\} \), \( \hat{x} = \arg \min \| y - \Phi_{\Lambda} x_{\Lambda} \| \); 
3. \( y_r = y - \Phi \hat{x} \);

**Output:** signal approximation \( \hat{x} \)

According to OMP algorithm, in each iteration the atom that can maximize its inner product with the residual signal is selected as one of those that span the measurement signal. In each iteration, a sparse signal \( \hat{x} \) is used to estimate the original sparse signal \( x \). However, without any constraints, there is no concrete guarantee that atoms selected in this way are always the right ones. The following gives detailed analysis of OMP algorithm in each iteration.

In OMP algorithm, after \( l \)th (0 \( \leq \) l \( \leq \) k-1) iteration, there are totally \( l \) atoms selected. The set of the selected atom index is written as \( \Lambda \), and the dictionary \( \Phi_{\Lambda} \) is the sub dictionary containing atoms with the corresponding index of its columns in \( \Lambda \). For a \( k \) sparse signal, we use \( \Gamma \) to denote the set of index of the right atoms. In accordance with OMP algorithm, the residual can be represented as

\[
y_r = y - \Phi \hat{x} = y - proj(\Phi_{\Lambda}, y)
\]

\[
= (\Phi_{\Gamma \setminus \Lambda} x_{\Gamma \setminus \Lambda} + \Phi_{\Gamma \setminus \Lambda} x_{\Gamma \setminus \Lambda})
\]

\[
- \Phi_{\Lambda} \Phi_{\Lambda}^T y
\]

\[
= \Phi_{\Gamma \setminus \Lambda} x_{\Gamma \setminus \Lambda} - \Phi_{\Lambda} \Phi_{\Lambda}^T y
\]

\[
= [\Phi_{\Gamma \setminus \Lambda} \Phi_{\Lambda}]^{-1} \Phi_{\Gamma \setminus \Lambda} x_{\Gamma \setminus \Lambda}
\]

where \( proj(\Phi_{\Lambda}, y) \) denotes the projection of \( y \) onto the space span(\( \Phi_{\Lambda} \)). According to the projection property, \( x_p \) can be written as

\[
x_p = (\Phi_{\Lambda}^T \Phi_{\Lambda})^{-1} \Phi_{\Lambda}^T \Phi_{\Gamma \setminus \Lambda} x_{\Gamma \setminus \Lambda}
\]

Let \( x_r = \begin{bmatrix} x_{\Gamma \setminus \Lambda} \\ -x_p \end{bmatrix} \), thus the residual \( y_r \) can be represented as

\[
y_r = \Phi_{\Gamma \setminus \Lambda} \Phi_{\Lambda} x_r.
\]

It can be seen that \( y_r \) is composed of \( \Phi_{\Gamma \setminus \Lambda} x_{\Gamma \setminus \Lambda} \) and \( -\Phi_{\Lambda} x_p \). This indicates that the signal \( x_r \) also contains the index of atoms that are already selected by OMP algorithm in the former iterations.

III. Greedy OMP (GOMP) Algorithm

The original OMP algorithm selects only one atom to the approximation of support in each iteration, which makes number of iteration high when coping with large scale data and the application impractical. Here, we attempt to identify more atoms in each iteration based on the following approach. The Greedy OMP (GOMP) algorithm applies the greedy atom identification and optimization of selected atoms to OMP algorithm, which is described as followed.

**A. Greedy atom identification**

In the \( l \)th iteration, we firstly generate identification vector \( h_t = \Phi y_t \), with the \( l \)th component denoted as \( h_t(i) \). Then a small set of \( I \) atoms are selected through

\[
I = \{i \mid h_t(i) \geq \alpha \cdot \max \{ |h_t(j)| \} \}
\]

where \( \alpha (0 < \alpha \leq 1) \) is a greedy constant to restrain the size of set \( I \). Obviously, only one atom is identified in each iteration when \( \alpha = 1 \), which makes the identification the same as that in original OMP algorithm.

The smaller \( \alpha \) is, the more atoms will be selected, which makes the identification of atom “greedier”. The recovery process will perform more efficiently because of the reduced number of iteration. However, the percentage of identifying right atom, the recover accuracy, will decrease due to the high coherence between distinct columns from set \( I \). On the contrary, when \( \alpha \) becomes bigger, less atoms will be identified in each iteration, which accordingly makes the number of iteration increase. Moreover, orthogonal projection will iteratively proceeds in signal recovery, which consumes most of the computational resources in signal recovery.

**B. Optimization of selected atoms**

According to the above analysis, we should adopt another way to balance the recovery accuracy and computational burden. In this paper, we optimize the set of selected atoms based on the similarity analysis. Define the similarity \( \rho(i,j) \) between \( \phi_i \) and \( \phi_j \) as followed [7]:

\[
\rho(i,j) = \frac{\langle \phi_i, \phi_j \rangle}{\| \phi_i \| \| \phi_j \|}
\]

A different way to understand similarity is by considering the Gram matrix \( G \) which is defined as \( G = \Phi^T \Phi \). The off-diagonal entries in \( G \) are the similarity values defined in (7). The coherence in (4) is the off-diagonal entry with the largest magnitude.
In this paper, we propose a parameter $\xi$ to judge the selected atom. Considering the optimal atom $\arg \max \{ |h_i(j)| \}$, other atoms in set $I$, whose absolute inner product with the optimal atom is bigger than $\xi$, will be abandoned, i.e.,

$$I = I - \left\{ i \mid \rho(i, \arg \max_j |h_i(j)|) \geq \xi, i \in I \right\}$$  \hspace{1cm} (8)

C. The description of GOMP algorithm

The detailed GOMP algorithm is presented here.

<table>
<thead>
<tr>
<th>GOMP Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> measurement signal $y$, measurement matrix $\Phi$</td>
</tr>
<tr>
<td><strong>Initialize:</strong> residual $y_r = y$, support $\Lambda = \emptyset$, recovered signal $\hat{x} = 0$</td>
</tr>
<tr>
<td><strong>Iteration:</strong> at the $h$th iteration, until stopping criterion is met.</td>
</tr>
<tr>
<td>1. $h_r = \Phi y_r$ ;</td>
</tr>
<tr>
<td>2. $I = \left{ i \mid</td>
</tr>
<tr>
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</tr>
<tr>
<td>4. $\Lambda = \Lambda \cup I$ , $\hat{x} = \arg \min_{\hat{x}} | y - \Phi \hat{x} |_2$ ;</td>
</tr>
<tr>
<td>5. $y_r = y - \Phi \hat{x}$ ;</td>
</tr>
<tr>
<td><strong>Output:</strong> signal approximation $\hat{x}$</td>
</tr>
</tbody>
</table>

IV. SIMULATION RESULT

This section illustrates experimentally that the proposed GOMP algorithm is a powerful method for sparse signal recovery, when it was compared with OMP algorithms.

In the compressive sensing framework, all sparse signals are expected to be exactly recovered as long as the level of the sparsity is below a certain threshold. Meanwhile we adopted the following strategy for the simulation.

1) Randomly generate an $m \times n$ measurement matrix $\Phi$ with norm column from the standard i.i.d. Gaussian ensemble.

2) Select a support set $\Gamma$ of size $|\Gamma| = k$ uniformly at random, and generate the sparse signal vector $x$ with length $n$ by either one of the following two methods:

   a) draw the elements of the vector $x$ restricted to $\Gamma$ from the standard Gaussian distribution; we refer to this type of signal as a Gaussian signal; or

   b) set all entries of $x$ supported on $\Gamma$ to ones; we refer to this type of signal as a Zero-one (or 0-1) signal.

3) Obtain the measurement signal $y = \Phi x$, apply recovery algorithm to obtain $\hat{x}$, the estimate of $x$, and compare $\hat{x}$ with $x$.

4) Repeat the process 200 times for each $k$.

The parameters in GOMP algorithm is listed: $\alpha = 0.7$ and $\xi = 0.8$. To investigate the percentage of exact recovery as a function of the signal sparsity $k$ for given measurement $m = 128$, several signal sparsity $k$ were chosen from 5 to 7 in steps 3. The evaluation criteria used in the simulation is the percentage of exact recovery. For each case, the support set was selected uniformly at random. The recovery was considered to be exact when the original signal $x$ and the recovered one $\hat{x}$ satisfies

$$\frac{\| x - \hat{x} \|_2}{\| x \|_2} < 10^{-4}$$  \hspace{1cm} (9)

The stopping criterion of the recovery is the following constraint

$$\frac{\| y - \Phi \hat{x} \|_2}{\| y \|_2} < 10^{-5}$$  \hspace{1cm} (10)

The original signals were Gaussian signal with length $n = 256$, whose recovery results can be seen from Fig.1.

![Fig.1 The percentage of exact recovery of Gaussian sparse signal versus the signal sparsity $k$ by OMP and GOMP algorithm.](image)

As can be seen from Fig. 1, for Gaussian sparse signal, the performance of the proposed GOMP algorithm is obviously much better than that of OMP algorithm. Specially, while OMP algorithms start to fail when $k > 17$, GOMP algorithm still gives nearly 100% exact reconstruction until $k > 26$.

![Fig.2 The percentage of exact recovery of 0-1 sparse signal versus the signal sparsity $k$ by OMP and GOMP algorithm.](image)

In the second experiment, the only difference is that the original signal is zero-one signal. The simulation results can be seen from Fig. 2. Compared with OMP algorithm, the proposed GOMP algorithm started to fail when $k > 23$, which is a noticeable improvement.
In the third experiment, we try to investigate the effect of $\alpha$ upon the recovery performance of GOMP algorithm. The setup and parameters of this experiment are same as those of the second one. Keep $\xi=0.8$ unchanged, we vary the value of $\alpha$ from {0.5, 0.6, 0.7, 0.8, 1}. The recovery performances of GOMP for different $\alpha$ can be seen from Fig.3.

From Fig.3, we can see that GOMP can provides the best recovery when $\alpha=0.7$. Moreover, when alpha is big enough, for instance, $\alpha=0.8$ and 1, or small enough, for instance, $\alpha=0.5$ and 0.6, the recovery performance of GOMP algorithm will greatly drop. Thus, the selection of alpha is of significance in sparse signal recovery when adopting GOMP algorithm.

V. CONCLUSION

Recovery algorithm, as a core problem in CS theory, draws a widespread attention. In this paper, OMP algorithm and some related theoretical result about iterative residual is firstly introduced. And then, GOMP algorithm is proposed to improve the recovery performance. The procedure of greedy atom identification is iteratively performed to identify more than one atom, which decreases the computational burden through the reduced number of iteration. The procedure of optimization of selected atoms is iteratively operated to discard some atom of high similarity with the optimal atom, which can greatly increase the recovery accuracy. Compared with OMP algorithm, several simulation results verify its efficiency in recovering Gaussian and zero-one sparse signal. In addition, the effect of greedy constant upon the recovery performance is experimentally investigated.

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REFERENCES