Quasi-Whittaker modules for the Schrödinger algebra

Yan-an Cai a, Yongsheng Cheng b, Ran Shen c,*

a School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China
b School of Mathematics and Information Sciences & Institute of Contemporary Mathematics, Henan University, Kaifeng 475004, China
c College of Science, Donghua University, Shanghai 201620, China

A R T I C L E   I N F O

Article history:
Received 10 December 2013
Accepted 1 September 2014
Available online xxxx
Submitted by M. Bresar

MSC:
17B10
17B65
17B68

Keywords:
Schrödinger algebra
Locally finite modules
Quasi-Whittaker modules
Simple modules

A B S T R A C T

In this paper, we construct a class of new modules for the Schrödinger algebra $\mathfrak{g}$, called quasi-Whittaker modules. Different from [30], the quasi-Whittaker module is not induced by the Borel subalgebra of the Schrödinger algebra related with the triangular decomposition, but by its Heisenberg subalgebra $\mathcal{H}$. We prove that, a simple $\mathfrak{g}$-module $V$ is a quasi-Whittaker module if and only if $V$ is a locally finite $\mathcal{H}$-module. Furthermore, we classify simple quasi-Whittaker modules by central character of $U(\mathfrak{g})$ and their quasi-Whittaker functions. Finally, we characterize arbitrary quasi-Whittaker modules.

© 2014 Elsevier Inc. All rights reserved.

* Supported by the National Science Foundation of China (Nos. 11047030, 11101388, 11171055, 11001046) and the Fundamental Research Funds for the Central Universities.
* Corresponding author.
E-mail addresses: yatsai@mail.ustc.edu.cn (Y.-a. Cai), yscheng.math@gmail.com (Y. Cheng), rshen@dhu.edu.cn (R. Shen).

http://dx.doi.org/10.1016/j.laa.2014.09.001
0024-3795/© 2014 Elsevier Inc. All rights reserved.
1. Introduction

In [15], B. Kostant introduced a class of modules for a finite-dimensional complex semisimple Lie algebra. He called these modules Whittaker modules because of their connection with the Whittaker equations that arise in the study of the corresponding representations of the associated Lie group. The traditional definition of Whittaker modules is closely tied to the triangular decomposition of a Lie algebra. Results for the complex semisimple Lie algebras have been extended to quantum groups for $\mathcal{U}_q(g)$ [25], and $\mathcal{U}_q(\mathfrak{sl}_2)$ [23], Virasoro algebra [12,16,24], Schrödinger–Witt algebra [31], Heisenberg algebras and affine Kac–Moody algebras [8,13], Heisenberg–Virasoro algebra [9,19], Weyl algebras [6] and some other infinite dimensional Lie algebras [26,27]. In particular, in [7], the author proved that all the simple $\mathfrak{sl}_2$-modules fall into three families: highest (lowest) weight modules, Whittaker modules and a third family obtained by localization; in [8], the author studied a class of new modules, which were called imaginary Whittaker modules over the non-twisted affine Lie algebra induced from its parabolic subalgebras. Inspired by recent activities on Whittaker modules over various algebras, the authors of [5] described a general framework for the study of Lie algebra modules which were locally finite over some subalgebra. In [21], the authors studied and classified the simple Virasoro modules which are locally finite over a positive part, up to simple modules over a family of finite dimensional solvable Lie algebras.

The Schrödinger algebra is the Lie algebra of the Schrödinger group, which is the symmetry group of the free particle Schrödinger equation. It is a Lie algebra that has attracted considerable interest in mathematical physics and its applications [1–4, 11] since its introduction in [14,22]. Let $\mathfrak{S}$ denote the Schrödinger algebra. Then $\mathfrak{S} = \text{span}_C \{ e, h, f, p, q, z \}$ with the following Lie brackets:

\[
\begin{align*}
[h, e] &= 2e, & [h, f] &= -2f, & [e, f] &= h, \\
[h, p] &= p, & [h, q] &= -q, & [p, q] &= z, \\
[e, q] &= p, & [p, f] &= -q, & [f, q] &= 0, \\
[e, p] &= 0, & [z, \mathfrak{S}] &= 0.
\end{align*}
\]  
(1.1)

We see that $\mathfrak{S}$ contains two subalgebras: the Heisenberg subalgebra $\mathcal{H} = \text{span}_C \{ p, q, z \}$ and $\mathfrak{sl}_2 = \text{span}_C \{ e, h, f \}$. The Schrödinger algebra $\mathfrak{S}$ can be viewed as a semidirect product $\mathfrak{S} = \mathcal{H} \rtimes \mathfrak{sl}_2$. $\mathfrak{S}$ has a triangular decomposition

\[\mathfrak{S} = \mathfrak{S}_+ \oplus \mathfrak{S}_0 \oplus \mathfrak{S}_-,\]  
(1.2)

where $\mathfrak{S}_+ = \text{span}_C \{ e, p \}$, $\mathfrak{S}_0 = \text{span}_C \{ h, z \}$ and $\mathfrak{S}_- = \text{span}_C \{ f, q \}$.

The representation theory of the Schrödinger algebra $\mathfrak{S}$ has attracted many authors’ attention. For example, using the technique of singular vectors, a classification of the irreducible lowest weight representations of the Schrödinger algebra were given in [11]. A classification of simple weight modules with finite dimensional weight spaces and
finite-dimensional indecomposable modules over the Schrödinger algebra were shown in [10,28,29], respectively. Recently, all simple weight modules over the Schrödinger algebra were classified in [18]. In particular, in [30], the authors studied the Whittaker modules induced by the triangular decomposition (1.2) of $\mathcal{S}$, simple Whittaker modules and related Whittaker vectors were determined. The irreducible representations of conformal Galilei algebras which are closely related with Schrödinger algebra in $l$-spatial dimension were studied in [17].

In this paper, we generalize the concept of Whittaker modules for the Schrödinger algebra $\mathcal{S}$. Different from the method of Ref. [30], this type of modules are neither induced by its Borel subalgebra, nor by its parabolic subalgebra, but by its Heisenberg subalgebra $\mathcal{H}$. We call this kind of new modules as quasi-Whittaker modules. We prove that, a simple $\mathcal{S}$-module $V$ is a quasi-Whittaker module if and only if $V$ is a locally finite $\mathcal{H}$-module. Using central character of $U(\mathcal{S})$ and their quasi-Whittaker functions, we classify the simple quasi-Whittaker modules for the Schrödinger algebra $\mathcal{S}$.

The paper is organized as follows. In Section 2, we define quasi-Whittaker modules induced by its Heisenberg subalgebra $\mathcal{H}$, give some notations and formulaes and show that a simple $\mathcal{S}$-module $V$ is a quasi-Whittaker module if and only if $V$ is a locally finite $\mathcal{H}$-module. We characterize the quasi-Whittaker vectors of quasi-Whittaker modules in Section 3. In Section 4, we classify simple quasi-Whittaker modules for the Schrödinger algebra; In the last section, we describe arbitrary quasi-Whittaker modules with generating quasi-Whittaker vectors.

Throughout this paper, we denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$ and $\mathbb{C}$ the sets of all integers, nonnegative integers, positive integers, and complex numbers, respectively. Also, we denote $\mathbb{C}[x]$ the polynomial ring in $x$ over $\mathbb{C}$.

2. Preliminaries

First of all, we give the basic definitions for this paper.

**Definition 2.1.** Let $\phi : \mathcal{H} \to \mathbb{C}$ be any Lie algebra homomorphism, which we call a quasi-Whittaker function. Let $V$ be an $\mathcal{S}$-module.

(i) A nonzero vector $v \in V$ is called a quasi-Whittaker vector of type $\phi$ if $xv = \phi(x)v$ for all $x \in \mathcal{H}$.

(ii) $V$ is called a quasi-Whittaker module for $\mathcal{S}$ of type $\phi$ if $V$ contains a cyclic quasi-Whittaker vector $v$ of type $\phi$.

**Remark 2.2.** From the definition, it is easy to see that $\phi(z) = 0$.

**Lemma 2.3.** If $\phi$ is a zero homomorphism, then $V$ is a simple quasi-Whittaker module of type $\phi$ if and only if it is a simple $\mathfrak{sl}_2$-module, i.e., $\mathcal{H}V = 0$.

**Proof.** Since $\mathcal{H}$ is an ideal of $\mathcal{S}$, so it is clear that $\{v \mid \mathcal{H}v = 0\}$ is an $\mathcal{S}$-submodule. $\square$
Definition 2.4. Let \( \phi : \mathcal{H} \rightarrow \mathbb{C} \) be any nonzero Lie algebra homomorphism. Define a one-dimensional \( \mathcal{H} \)-module \( \mathbb{C}_\phi = \mathbb{C}w \) by \( pw = \phi(p)w, qw = \phi(q)w, zw = 0 \). The induced module

\[
M_\phi = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathcal{H})} \mathbb{C}_\phi, \tag{2.1}
\]

is called the universal quasi-Whittaker module of type \( \phi \).

The following notations will be used to describe bases for \( \mathcal{U}(\mathfrak{g}) \) and for quasi-Whittaker modules. Fix \( \phi : \mathcal{H} \rightarrow \mathbb{C} \) be any nonzero Lie algebra homomorphism, define the following elements in \( \mathcal{U}(\mathfrak{g}) \):

\[
X = \delta_{\phi(q)}oe + \delta_{\phi(p)}of, \\
C = \phi(p)^2f - \phi(q)^2e - \phi(p)\phi(q)h, \\
P_+ = \delta_{\phi(p)}op + \delta_{\phi(q)}oq, \\
P_- = \delta_{\phi(q)}op + \delta_{\phi(p)}oq.
\]

Remark 2.5. It is easy to see that \( X, h, C \) forms a basis for \( \mathfrak{sl}_2 \) when \( \phi \) is nonzero with \( \phi(p)\phi(q) = 0 \).

Definition 2.6. Let \( M_\phi \) be the universal quasi-Whittaker module with cyclic quasi-Whittaker vector \( w \). For \( \xi \in \mathbb{C} \), define a submodule \( W_{\phi, \xi} \) as

\[
W_{\phi, \xi} = \mathcal{U}(\mathfrak{g})(C - \xi)w.
\]

And let

\[
L_{\phi, \xi} = M_\phi / W_{\phi, \xi}.
\]

Denote “\( - \)” the canonical projection from \( M_\phi \) to \( L_{\phi, \xi} \).

Recall that a locally finite module for a Lie algebra is defined as follows.

Definition 2.7. Let \( L \) be a Lie algebra. An \( L \)-module is called locally finite if any nonzero element \( v \) in \( V \) is contained in a finite dimensional submodule.

From the following lemma, we can identify any simple \( \mathfrak{g} \)-module on which \( \mathcal{H} \) acts locally finite with a quasi-Whittaker module.

Theorem 2.8. Let \( V \) be a simple \( \mathfrak{g} \)-module. Then the following two conditions are equivalent:

(i) \( V \) is a locally finite \( \mathcal{H} \)-module;
(ii) \( V \) is a quasi-Whittaker module for \( \mathfrak{g} \).
Proof. (ii) ⇒ (i). Let \( V \) be a quasi-Whittaker module of type \( \phi \) with cyclic Whittaker vector \( w \). For any \( x \in V \), we have \( x = w.y \) for some \( y \in \mathcal{U}(\mathfrak{S}) \). Lemma 3 in [10] shows that \( p \) and \( q \) act locally nilpotently on \( \mathcal{U}(\mathfrak{S}) \), which means that there exist \( n_1, n_2 \in \mathbb{N} \) such that

\[(\text{ad } p)^{n_1}y = (\text{ad } q)^{n_2}y = 0\]

Hence, we have

\[(p - \phi(p))^{n_1}x = (\text{ad } p)^{n_1}y.w = 0,\]
\[(q - \phi(q))^{n_2}x = (\text{ad } q)^{n_2}y.w = 0\]

So, \( \{p^iq^jx \mid 0 \leq i \leq n_1 - 1, 0 \leq j \leq n_2 - 1\} \) span a finite dimensional \( \mathcal{H} \)-submodule containing \( v \) by the definition of \( \mathcal{H} \) and Remark 2.2. Thus \( V \) is a locally finite \( \mathcal{H} \)-module.

(i) ⇒ (ii). Take any nonzero \( v_0 \in V \), then there exists a submodule \( V_1 \subseteq V \) such that \( V_1 \) is a finite dimensional \( \mathcal{H} \)-module containing \( v_0 \). Note that \( \mathcal{H} \) is solvable, following from Lie’s theorem, \( p \) and \( q \) have common eigenvector \( v \) in \( V_1 \), and \( zv = [p, q]v = 0 \). Since \( V \) is simple as \( \mathfrak{S} \)-module, we see that \( V = \mathcal{U}(\mathfrak{S})v \) and \( zV = 0 \). By definition, \( V \) is a quasi-Whittaker module for \( \mathfrak{S} \). \( \square \)

By (1.1), we can deduce by induction the following identities which will be used later.

**Lemma 2.9.** For any \( m, n, k \in \mathbb{N} \), we have

\[p^n h^m = \sum_{i=0}^{m} \binom{m}{i} (-1)^i n! h^{m-i} p^n, \quad (2.2)\]

\[q^n h^m = \sum_{i=0}^{m} \binom{m}{i} n^i h^{m-i} q^n, \quad (2.3)\]

\[pf^k = f^k p - k f^{k-1} q, \quad (2.4)\]

\[qe^k = e^k q - k e^{k-1} p. \quad (2.5)\]

**Remark 2.10.** In the following sections, we discuss the reducibility of the quasi-Whittaker modules for the Schrödinger algebra. There are four conditions according to the choice of \( \phi \) in **Definition 2.1**., that is, \( \phi(p) = \phi(q) = 0 \); \( \phi(p) = 0 \) while \( \phi(q) \neq 0 \); \( \phi(p) \neq 0 \) while \( \phi(q) = 0 \); \( \phi(p) \neq 0 \) and \( \phi(q) \neq 0 \). The simplest condition of the four cases is \( \phi(p) = \phi(q) = 0 \). In this case, from Lemma 2.3, any \( \mathfrak{s}_{\mathbb{Z}} \)-module is a quasi-Whittaker module. While the reducibility of \( \mathfrak{s}_{\mathbb{Z}} \)-module was completely determined by R. Block in [7,20]. Thus, in this paper, we do not deal with this case. We will discuss the quasi-Whittaker module for \( \mathfrak{S} \) according to the remaining three conditions. When \( \phi(p)\phi(q) = 0 \), we take a basis for \( M_{\phi} \) as \( \{X^i h^j C^k w \mid i, j, k \in \mathbb{Z}_+\} \); while, when \( \phi(p)\phi(q) \neq 0 \), we take it as \( \{h^i f^j C^k w \mid i, j, k \in \mathbb{Z}_+\} \), where \( w \) is a cyclic quasi-Whittaker vector of \( M_{\phi} \).
3. Quasi-Whittaker vectors in $M_\phi$ and $L_{\phi,\xi}$

In this section, we characterize the quasi-Whittaker vectors of the quasi-Whittaker modules $M_\phi$ and $L_{\phi,\xi}$.

**Lemma 3.1.** Let $w$ be a quasi-Whittaker vector of type $\phi$. For any $k \in \mathbb{Z}_+$, we have

\[
(p - \phi(p)) C^k w = 0,
\]

\[
(q - \phi(q)) C^k w = 0.
\]

Thus, any vector in $\mathbb{C}[C]w$ is a quasi-Whittaker vector of type $\phi$.

**Proof.** The lemma follows from a direct computation. □

From (2.2)–(2.5), we know that

**Lemma 3.2.** Suppose $\phi(p)\phi(q) = 0$, then

\[
(P_+ - \phi(P_+)) X^i = X^i P_+ - i X^{i-1} P_-,
\]

\[
(P_- - \phi(P_-)) h^m = \sum_{i=1}^{m} \binom{m}{i} (-1)^{i+1} h^{m-i} P_-,
\]

where $i, m \in \mathbb{N}$.

**Lemma 3.3.** Let $w$ be a quasi-Whittaker vector of type $\phi$ with $\phi(p)\phi(q) = 0$. Suppose

\[
x = \sum_{i=0}^{n} X^i \sum_{j=0}^{m} h^i a_{ij}(C) w \in M_\phi, \quad a_{ij}(C) \in \mathbb{C}[C], \quad m, n \in \mathbb{Z}_+.
\]

Then

\[
(P_+ - \phi(P_+))^n x = (-1)^{n} n! \sum_{j=0}^{m} P^n h^j a_{nj}(C) w
\]

\[
= (-1)^{n} n! (\phi(p) + \phi(q))^n \sum_{j=0}^{m} h^j b_j(C) w.
\]

**Proof.** By (2.2)–(2.5), we see that

\[
(P_+ - \phi(P_+)) h^k C^i w = 0,
\]

\[
(P_- - \phi(P_-)) C^k w = 0.
\]

Hence,
\[(P_+ - \phi(P_+))^n x = (P_+ - \phi(P_+))^{n-1} \sum_{i=0}^{n} P_+ X^i \sum_{j=0}^{m} h^j a_{ij}(C)w\]

\[= (P_+ - \phi(P_+))^{n-1} \sum_{i=0}^{n} (X^i P^+ - iX^{i-1} P_-) \sum_{j=0}^{m} h^j a_{ij}(C)w\]

\[= -(P_+ - \phi(P_+))^{n-1} \sum_{i=1}^{n} iX^{i-1} P_- \sum_{j=0}^{m} h^j a_{ij}(C)w\]

\[= (P_+ - \phi(P_+))^{n-2} \sum_{i=1}^{n} i(i - 1)X^{i-2} P^2 \sum_{j=0}^{m} h^j a_{ij}(C)w\]

\[= (-1)^n n! \sum_{j=0}^{m} P^n h^j a_{nj}(C)w\]

\[= (-1)^n n! \sum_{j=0}^{m} \sum_{s=0}^{j} \binom{j}{s} h^j-s P^n a_{nj}(C)w\]

\[= (-1)^n n! (\phi(p) + \phi(q))^{n} \sum_{j=0}^{m} h^j b_j(C)w. \quad \Box\]

For later use, we also need the following lemmas.

**Lemma 3.4.** Let \( w \) be a quasi-Whittaker vector of type \( \phi \) with \( \phi(p)\phi(q) = 0 \). Then

\[ (P_- - \phi(P_-))^s (h^t b(C)w) = \begin{cases} 0, & \text{if } s > t; \\ (\phi(p) + \phi(q))^t (-1)^{i\phi(p),a+1} t! b(C)w, & \text{if } s = t, \end{cases} \]

where \( s, t \in \mathbb{N}, b(C) \in \mathbb{C}[C] \).

**Proof.** We prove the second identity firstly. If \( t = 1 \), we have

\[ (P_- - \phi(P_-))^2 (h b(C)w) = (P_- - \phi(P_-))((-1)^{i\phi(p),a+1} P_- b(C)w)\]

\[= (-1)^{i\phi(p),a+1} \phi(P_-)(P_- - \phi(P_-)) (b(C)w)\]

\[= 0.\]

Hence, \( (P_- - \phi(P_-))^s (h b(C)w) = 0, \forall s > 1. \) Assume that the identity holds for \( t \leq k \), then for \( t = k+1 \), we have

\[ (P_- - \phi(P_-))^{k+2} (h^{k+1} b(C)w)\]

\[= (P_- - \phi(P_-))^{k+1} \left( \sum_{i=1}^{k+1} \binom{k+1}{i} (-1)^{i\phi(p),a+1} h^{k+1-i} P_- b(C)w \right)\]
Thus it is easy to check that

$$(P_- - \phi(P_-))(hb(C)w) = P_-b(C)w = \phi(P_-)b(C)w.$$  

Suppose the identity holds for $t \leq k$, then for $t = k + 1$, we have

$$(P_- - \phi(P_-))^{k+1}(h^{k+1}b(C)w)$$

$$= (P_- - \phi(P_-))^{k+1}(h^{k+1}b(C)w)$$

$$= (P_- - \phi(P_-))^{k+1}(h^{k+1}b(C)w)$$

$$= (P_- - \phi(P_-))^{k+1}(h^{k+1}b(C)w)$$

$$= (P_- - \phi(P_-))^{k+1}(h^{k+1}b(C)w)$$

$$= (P_- - \phi(P_-))^{k+1}(h^{k+1}b(C)w)$$

$$= (P_- - \phi(P_-))^{k+1}(h^{k+1}b(C)w)$$

$$= (P_- - \phi(P_-))^{k+1}(h^{k+1}b(C)w)$$

$$= (P_- - \phi(P_-))^{k+1}(h^{k+1}b(C)w)$$

$$= (P_- - \phi(P_-))^{k+1}(h^{k+1}b(C)w)$$

The lemma holds. □

**Lemma 3.5.** Let $w$ be a quasi-Whittaker vector of type $\phi$ with $\phi(p)\phi(q) \neq 0$. Then

$$(q - \phi(q))^s\left(h^t \sum_{i=0}^m f^i a_i(C)w\right) = \begin{cases} 0, & \text{if } s > t; \\
 t!(\phi(q))^t \sum_{i=0}^m f^i a_i(C)w, & \text{if } s = t, \end{cases}$$

where $s, t \in \mathbb{Z}_+, a_i(C) \in \mathbb{C}[C], i = 1, \ldots, m.$

**Proof.** It is easy to check that $(q - \phi(q))f^j a(C)w = 0$, $\forall j \in \mathbb{Z}_+, a(C) \in \mathbb{C}[C]$. We use the induction on $t$. If $t = 1$, then

$$(q - \phi(q)) \left( h \sum_{i=0}^m f^i a_i(C)w \right) = q \sum_{i=0}^m f^i a_i(C)w = \phi(q) \sum_{i=0}^m f^i a_i(C)w,$$

$$(q - \phi(q))^2 \left( h \sum_{i=0}^m f^i a_i(C)w \right) = (q - \phi(q)) \left( \phi(q) \sum_{i=0}^m f^i a_i(C)w \right) = 0.$$
\[
(q - \phi(q))^{k+1} \left( h^{k+1} \sum_{i=0}^{m} f^i a_i(C)w \right) \\
= (q - \phi(q))^k \left( \sum_{j=1}^{k+1} \left( \begin{array}{c} k+1 \\ j \end{array} \right) h^{k+1-j} q \sum_{i=0}^{m} f^i a_i(C)w \right) \\
= (q - \phi(q))^k \left( \phi(q) \sum_{j=1}^{k+1} \left( \begin{array}{c} k+1 \\ j \end{array} \right) h^{k+1-j} \sum_{i=0}^{m} f^i a_i(C)w \right) \\
= (q - \phi(q))^k \left( \phi(q)(k+1) h^k \sum_{i=0}^{m} f^i a_i(C)w \right) \\
= (k+1)! (\phi(q))^{k+1} \sum_{i=0}^{m} f^i a_i(C)w.
\]

Therefore the lemma holds. \(\square\)

**Lemma 3.6.** Let \(w\) be a quasi-Whittaker vector of type \(\phi\) with \(\phi(p)\phi(q) \neq 0\). Then

\[
(p - \phi(p))^m \left( \sum_{i=0}^{m} f^i a_i(C)w \right) = (-1)^m m! (\phi(q))^m a_m(C)w, \quad \forall m \in \mathbb{Z}_+.
\]

**Proof.**

\[
(p - \phi(p))^m \left( \sum_{i=0}^{m} f^i a_i(C)w \right) = (p - \phi(p))^{m-1} \left( -\sum_{i=1}^{m} if^{i-1} a_i(C)w \right) \\
= (p - \phi(p))^{m-1} \left( -\phi(q) \sum_{i=1}^{m} if^{i-1} a_i(C)w \right) \\
= (-1)^m m! (\phi(q))^m a_m(C)w. \quad \square
\]

Now, we can determine the quasi-Whittaker vectors for \(M_\phi\) and \(L_{\phi, \xi}\).

**Proposition 3.7.** Let \(M_\phi\) be the universal quasi-Whittaker module generated by quasi-Whittaker vector \(w\). Suppose \(w' \in M_\phi\) is a quasi-Whittaker vector, then \(w' \in \mathbb{C}[C]w\).

**Proof.** (1) Assume that \(\phi(p)\phi(q) = 0\). Suppose

\[
w' = \sum_{i=0}^{n} X^i \sum_{j=0}^{m} h^j a_{ij}(C)w
\]

with \(a_{ij}(C) \in \mathbb{C}[C]\). Then by Lemma 3.3, we see that if \((P_+ - \phi(P_+))w' = 0\), then \(n = 0\), that is \(w' = \sum_{j=0}^{m} h^j a_j(C)w\). Applying Lemma 3.4, we get \(m = 0\). Thus, we have \(w' = b(C)w \in \mathbb{C}[C]w\). The proposition holds by Lemma 3.1.
(2) If \( \phi(p)\phi(q) \neq 0 \), we may assume that

\[
w' = \sum_{i=0}^{m} h^i \sum_{j=0}^{n} f^j a_{ij}(C)w
\]

with \( a_{ij}(C) \in \mathbb{C}[C] \). Then by Lemma 3.5, we see that if \( (q - \phi(q))w' = 0 \), then \( m = 0 \), that is \( w' = \sum_{j=0}^{n} f^j a_{ij}(C)w \). Applying Lemma 3.6, we get \( n = 0 \). Thus, we have \( w' = b(C)w \in \mathbb{C}[C]w \). The proposition holds by Lemma 3.1. \( \square \)

**Proposition 3.8.** Let \( w \) be the cyclic quasi-Whittaker vector of \( M_\phi \), \( \bar{w} \in L_\phi,\xi \). If \( w' \in L_\phi,\xi \) is a quasi-Whittaker vector, then \( w' = cw \) for some \( c \in \mathbb{C} \).

**Proof.** We only show the statement for \( \phi(p)\phi(q) = 0 \) since the proof is similar when \( \phi(p)\phi(q) \neq 0 \). Note that \( \{X^ih^j\bar{w} \mid i, j \in \mathbb{Z}_+ \} \) span \( L_\phi,\xi \). We claim that this set is linearly independent and thus a basis for \( L_\phi,\xi \). To check this, suppose

\[
0 = \sum_{i,j} a_{ij}X^ih^j\bar{w} = \sum_{i,j} a_{ij}X^ih^jw.
\]

Then \( \sum_{i,j} a_{ij}X^ih^jw \in \mathcal{U}(\mathcal{S})(C - \xi)w \), and so

\[
\sum_{i,j} a_{ij}X^ih^jw = \sum_{i,j} \sum_{k=0}^{m} b_{ij}^k X^ih^jC^k(C - \xi)w
\]

for some \( m \in \mathbb{Z}_{>0} \) and \( b_{ij}^k \in \mathbb{C} \). This expression can be rewritten as

\[
\sum_{i,j} \left(a_{ij} + \xi b_{ij}^0\right)X^ih^jw + \sum_{i,j} \sum_{k=1}^{m} \left(\xi b_{ij}^k - b_{ij}^{k-1}\right)X^ih^jC^kC^{m+1}w = 0,
\]

From this we conclude that \( b_{ij}^m = 0 \), \( \xi b_{ij}^k - b_{ij}^{k-1} = 0 \), \( a_{ij} + \xi b_{ij}^0 = 0 \), and thus \( a_{ij} = 0 \) for all \( i, j \). With this fact now established, it is possible to use the same argument as in Proposition 3.7 to complete the proof. \( \square \)

**4. Simple quasi-Whittaker modules for \( \mathcal{G} \)**

In this section we will determine all simple quasi-Whittaker modules of type \( \phi \), up to isomorphism. After this, following from Theorem 2.8, we can classify all simple \( \mathcal{G} \)-modules on which \( \mathcal{H} \) acts locally finitely.

**Theorem 4.1.** Let \( V \) be a quasi-Whittaker module of type \( \phi \) for \( \mathcal{G} \), and let \( W \subseteq V \) be a nonzero submodule. Then there is a nonzero quasi-Whittaker vector \( w' \in W \).
Proof. Let \( w \) be the cyclic quasi-Whittaker vector of \( V \).

Case 1. If \( \phi(p)\phi(q) = 0 \), let

\[
x = \sum_{i=0}^{n} X^i \sum_{j=0}^{m} h^j a_{ij}(C) w \in W.
\]

Then by Lemma 3.3, we see that \( \sum_{j=0}^{m} h^j b_j(C) w \in W \). Applying Lemma 3.4, we deduce that \( W \) contains an element \( w' = b(C) w \neq 0 \). From Proposition 3.7, we know that \( w' \) is a quasi-Whittaker vector.

Case 2. If \( \phi(p)\phi(q) \neq 0 \), let

\[
x = \sum_{i=0}^{n} h^i \sum_{j=0}^{m} f^j a_{ij}(C) w \in W.
\]

Then by Lemma 3.5, we see that \( \sum_{j=0}^{m} f^j b_j(C) w \in W \). Applying Lemma 3.6, we deduce that \( W \) contains an element \( w' = b(C) w \neq 0 \). From Proposition 3.7, we know that \( w' \) is a quasi-Whittaker vector. \( \square \)

Theorem 4.2. Let \( \phi : \mathcal{H} \to \mathbb{C} \) be a nonzero Lie algebra homomorphism. Then the simple quasi-Whittaker modules of type \( \phi \) are exactly the modules \( \{ L_{\phi, \xi} \mid \xi \in \mathbb{C} \} \).

Proof. By Theorem 4.1, any nonzero submodule of \( L_{\phi, \xi} \) has a nonzero quasi-Whittaker vector, which is a nonzero multiple of \( \bar{w} \) by Proposition 3.8 and hence this submodule is \( L_{\phi, \xi} \) itself which must be simple.

Now let \( V \) be a simple quasi-Whittaker module of type \( \phi \) and let \( w_1 \) be a cyclic quasi-Whittaker vector corresponding to \( \phi \). From [18], we know that

\[
[\mathcal{U}(\mathfrak{G}), p^2 f - q^2 e - hpq] V = 0.
\]

By Schur’s lemma, we have \( C_0 = p^2 f - q^2 e - hpq \) acts on \( V \) as a scalar \( \xi \). Then \( \xi w_1 = C_0 w_1 = C w_1 \). Thus \( C \) acts on \( w \) as the scalar \( \xi \). Now by the universal property of \( M_\phi \), there exists a homomorphism \( \varphi : M_\phi \to V \) with \( uw \mapsto uw_1 \). This is a surjective map since \( V \) is generated by \( w_1 \). But \( \varphi(W_{\phi, \xi}) = \mathcal{U}(\mathfrak{G})(C - \xi)w_1 = 0 \), so we have

\[
W_{\phi, \xi} \subseteq \ker \varphi \subset M_\phi.
\]

Since \( L_{\phi, \xi} \) is simple, \( W_{\phi, \xi} \) is maximal, hence \( \ker \varphi = W_{\phi, \xi} \). That is \( V \cong L_{\phi, \xi} \). \( \square \)

Note that if \( w \) is a quasi-Whittaker vector of type \( \phi \), then we have \( Cw = C_0 w \). Hence, in the theorems and properties above, \( C \) can be replaced by \( C_0 \). For the remaining part of this section, we will discuss the annihilator of quasi-Whittaker vectors for simple quasi-Whittaker modules. The following proposition gives a characterization of simple quasi-Whittaker modules using annihilators.
Proposition 4.3. Fix a nonzero $\phi: \mathcal{H} \to \mathbb{C}$.
(1) Define the left ideal $L$ of $U(\mathcal{S})$ by $L = U(\mathcal{S})(C_0 - \xi 1) + U(\mathcal{S})(p - \phi(p)1) + U(\mathcal{S})(q - \phi(q)1)$, and regard $V = U(\mathcal{S})/L$ as a left $U(\mathcal{S})$-module. Then $V \cong L_{\phi, \xi}$, consequently $V$ is simple.
(2) Let $V$ be a quasi-Whittaker module of type $\phi$ such that $C_0$ acts on the cyclic quasi-Whittaker vector by the scalar $\xi \in \mathbb{C}$. Then $V$ is simple. Moreover, if $w$ is a cyclic quasi-Whittaker vector for $V$, then
\[ \text{Ann}_{U(\mathcal{S})}(w) = U(\mathcal{S})(C_0 - \xi 1) + U(\mathcal{S})(p - \phi(p)1) + U(\mathcal{S})(q - \phi(q)1). \]

Proof. (1) For $u \in U(\mathcal{S})$, let $\bar{u} = u + L \in U(\mathcal{S})/L$. Then we may regard $V$ as a quasi-Whittaker module of type $\phi$ with cyclic quasi-Whittaker vector $\bar{1}$. By the universal property of $M_\phi$, there exists a homomorphism $\varphi: M_\phi \to V$ with $uw \mapsto u\bar{1}$. This is a surjective map since $\bar{1}$ is the generator of $V$. However, for any $u(C_0 - \xi)w \in W_{\phi, \xi}$, we have $\varphi(u(C_0 - \xi)w) = u(C_0 - \xi)\bar{1} = 0$. Hence,
\[ W_{\phi, \xi} \subseteq \text{Ker} \varphi \subseteq M_\phi. \]
Since $W_{\phi, \xi}$ is maximal, it follows that $V \cong M_\phi/\text{Ker} \varphi \cong L_{\phi, \xi}$.

(2) Let $K$ denote the kernel of the natural surjective map $U(\mathcal{S}) \to V$ given by $u \mapsto uw$. Then $K$ is a proper left ideal containing
\[ L = U(\mathcal{S})(C_0 - \xi 1) + U(\mathcal{S})(p - \phi(p)1) + U(\mathcal{S})(q - \phi(q)1). \]
By (1), $L$ is maximal, thus $K = L$ and $V \cong U(\mathcal{S})/L$ is simple. □

5. Arbitrary quasi-Whittaker modules

In this section, we always assume that $V$ is an arbitrary quasi-Whittaker module of type $\phi$ with cyclic quasi-Whittaker vector $w$, similar to [24], we will describe its reducibility and its quasi-Whittaker vectors in terms of $\text{Ann}_{\mathbb{C}[C_0]}(w)$.

Lemma 5.1. Let $V$ be a quasi-Whittaker module of type $\phi$ with cyclic quasi-Whittaker vector $w$. Assume that $\text{Ann}_{\mathbb{C}[C_0]}(w) = (C_0 - \xi 1)^k$ for some $k > 0$, and the sequence of submodules
\[ V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_k = 0 \]
are defined by $V_i = U(\mathcal{S})(C_0 - \xi 1)^i w$, $i = 0, 1, 2, \cdots, k$. Then
(i) $V_i$ is a quasi-Whittaker module of type $\phi$, with cyclic quasi-Whittaker vector $v_i = (C_0 - \xi 1)^i w$ for $0 \leq i < k$.
(ii) $V_i/V_{i+1}$ is simple for $0 \leq i < k$.
Particularly,
(iii) the submodules $V_0, \cdots, V_k$ are the only submodules of $V$. 
Proof. It is easy to see that $V_i$ is a quasi-Whittaker module with cyclic quasi-Whittaker vector $w_i$ by (3.2) and (3.4). Obviously, $V_i/V_{i+1}$ is a quasi-Whittaker module of type $\varphi$ with quasi-Whittaker vector $\bar{w}_i$. Since $C_0$ acts by the scalar on $\bar{w}_i$, following from Proposition 4.3, we deduce that $V_i/V_{i+1}$ is simple, and thus isomorphic to $L_{\varphi, \xi}$. Thus $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_k$ form a composition series for $V_i$, and any simple subquotient of $V$ is isomorphic to $V/V_1 \cong L_{\varphi, \xi}$.

Let $M$ be any maximal submodule of $V$, then $V/M$ is a simple quasi-Whittaker module of type $\phi$ with quasi-Whittaker vector $\bar{w}$. By Theorem 4.2, we see that $C_0$ acts on $w$ by some scalar $\kappa \in \mathbb{C}$. While, $(C_0 - \xi_1)^k$ acts as 0 on $w$, and therefore on $\bar{w}$. So we have $(\kappa - \xi)^k w \in M$. Since $w \notin M$, we deduce that $\kappa = \xi$. It follows that $(C_0 - \xi_1)w \in M$, thus $V_1 = U(\mathcal{S})(C_0 - \xi_1)w \subseteq M$. Since $V_i$ is a maximal submodule of $M$, we get $V_1 = M$. Similarly, we can show that $V_{i+1}$ is the unique maximal submodule of $V_i$ for every $i < k$. Thus $\{V_i, 1 \leq i \leq k\}$ are the only submodules of $V$. \( \square \)

**Theorem 5.2.** Let $V$ be a quasi-Whittaker module of type $\phi$ generated by the cyclic quasi-Whittaker vector $w$. Assume that $\text{Ann}_{\mathbb{C}[C_0]}(w) \neq 0$ and $d(C_0) = \prod_{i=1}^k (C_0 - \xi_i)^{a_i}$ for distinct $\xi_1, \cdots, \xi_k \in \mathbb{C}$ is the unique monic generator of $\text{Ann}_{\mathbb{C}[C_0]}(w)$ in $\mathbb{C}[C_0]$.

(i) Define $V_{\phi, \xi_i} = U(\mathcal{S})(C_0 - \xi_i)w$, $i = 1, \cdots, k$, then $V_{\phi, \xi_1}, \cdots, V_{\phi, \xi_k}$ are the only maximal submodules of $V$.

(ii) Define

$$w_j = d_j(C_0)w, \quad \text{where } d_j(C_0) = \prod_{i \neq j} (C_0 - \xi_i)^{a_i}, \quad \text{and } V_j = U(\mathcal{S})w_j.$$ 

Then $V_i$ is a quasi-Whittaker module of type $\phi$ with cyclic quasi-Whittaker vector $w_i$ for $0 \leq i < k$, and $V = V_1 \oplus \cdots \oplus V_k$. Furthermore, the submodules $V_1, \cdots, V_k$ are indecomposable; $V_j$ is simple if and only if $a_j = 1$; and $a_j$ is the composition length of $V_j$. In particular, $V$ has a composition series of length $\sum_{i=1}^k a_i$.

**Proof.** (i) Let $M$ be a maximal submodule of $V$, then $V/M$ is simple and thus $C_0$ acts on $w$ by some scalar $\kappa \in \mathbb{C}$. On the other hand, $d(C_0)$ acts as 0 on $w$, and therefore on $\bar{w}$. Thus $d(\kappa) = 0$, which implies that $\kappa = \xi_i$ for some $1 \leq i \leq k$. Hence,

$$V_{\phi, \xi_i} = U(\mathcal{S})(C_0 - \xi_i)w \subseteq M.$$ 

Since $V_{\phi, \xi_i}$ is the image of the maximal submodule $W_{\phi, \xi_i}$ under the epimorphism $M_{\phi} \to V$, $V_{\phi, \xi_i}$ is maximal. Thus, $M = V_{\phi, \xi_i}$.

(ii) Firstly, we show that: $V = V_1 + \cdots + V_k$. Since $\gcd(d_1, \cdots, d_k) = 1$, there exist polynomials $r_1(C_0), \cdots, r_k(C_0) \in \mathbb{C}[C_0]$ such that $\sum_{i=1}^k r_i(C_0)d_i(C_0) = 1$. Therefore,

$$w = 1w = \left( \sum_{i=1}^k r_i(C_0)d_i(C_0) \right)w \in V_1 + \cdots + V_k.$$
To show that the sum $V = V_1 + \cdots + V_k$ is direct, note that for $i \neq j$, $d(C_0)$ is a factor of $d_i(C_0)d_j(C_0)$, which implies that $d_j(C_0)w_i = 0$. Following from this, we have

$$w_i = 1w_i$$

$$= (r_1(C_0)d_1(C_0) + \cdots + r_k(C_0)d_k(C_0))w_i$$

$$= r_i(C_0)d_i(C_0)w_i.$$  

Suppose that $u_1w_1 + \cdots + u_kw_k = 0$ for $u_1, \ldots, u_k \in \mathcal{U}(\mathcal{S})$, then

$$0 = r_i(C_0)d_i(C_0)\left(\sum_{j=1}^k u_jw_j\right) = u_ir_i(C_0)d_i(C_0)w_i = u_iw_i.$$  

Thus, the sum is direct.

To finish the proof, by Lemma 5.1, we know that the submodules $V_1, \ldots, V_k$ are indecomposable with the stated composition length. □

If we consider the annihilator of $w$ in $\mathbb{C}[C]$, then we can get similar result as following:

**Theorem 5.2’.** Let $V$ be a quasi-Whittaker module of type $\phi$ with cyclic quasi-Whittaker vector $w$. Assume that $\text{Ann}_{\mathbb{C}[C]}(w) \neq 0$ and $d(C) = \prod_{i=1}^k (C - \xi_i)^{a_i}$ for distinct $\xi_1, \ldots, \xi_k \in \mathbb{C}$, be the unique monic generator of the ideal $\text{Ann}_{\mathbb{C}[C]}(w)$ in $\mathbb{C}[C]$.

(i) Define $V_{\phi,\xi_i} = \mathcal{U}(\mathcal{S})(C - \xi_i)w$, $i = 1, \ldots, k$, then $V_{\phi,\xi_1}, \ldots, V_{\phi,\xi_k}$ are the only maximal submodules of $V$.

(ii) Let $n = \deg d(C)$, then $V$ has a unique composition series (up to permutation): $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$ with $V_i/V_{i+1} \cong L_{\phi,\xi_j}$ for some $j = 1, \ldots, k$. And the composition factors are $a_i$ copies of $L_{\phi,\xi_i}$, $i = 1, \ldots, k$. □

Indeed, suppose $V$ and $d(C)$ are as in Theorem 5.2’, then $d(C)$ being the unique monic generator of the ideal $\text{Ann}_{\mathbb{C}[C]}(w)$ indicates $d(C_0)$ being the unique monic generator of the ideal $\text{Ann}_{\mathbb{C}[C_0]}(w)$, since $Cw = C_0w$ in $V$. And Theorem 5.2 applies.

**Corollary 5.3.** Let $V$ be as in Theorem 5.2. Assume that $\text{Ann}_{\mathbb{C}[C_0]}(w) \neq 0$ and $d(C_0)$ be the unique monic generator of $\text{Ann}_{\mathbb{C}[C_0]}(w)$. Then

$$\text{Ann}_{\mathcal{U}(\mathcal{S})}(w) = \mathcal{U}(\mathcal{S})d(C_0) + \mathcal{U}(\mathcal{S})(p - \phi(p)1) + \mathcal{U}(\mathcal{S})(q - \phi(q)1).$$

**Proof.** We use induction on the composition length $n$ of $V$ (or equivalently, the degree of $d(C_0)$). If $n = 1$, then $d(C_0) = C_0 - \xi$, thus $V$ is simple, therefore the result is true by Proposition 4.3. Assume that $n > 1$, write $d(C_0) = (C_0 - \xi)d'(C_0)$ for some $\xi \in \mathbb{C}$ and some monic polynomial $d'(C_0) \in \mathbb{C}[C_0]$ with positive degree. Then $w' = (C_0 - \xi)w' \neq 0$.

Let $V' = \mathcal{U}(\mathcal{S})w' \subseteq V$. Then $V'$ is a quasi-Whittaker module with cyclic quasi-Whittaker vector $w'$, and $\text{Ann}_{\mathbb{C}[C_0]}(w') = \mathbb{C}[C_0]d'(C_0)$. Theorem 5.2 therefore implies that the composition length of $V'$ is $n - 1$, and by induction hypothesis, we deduce that
Ann_{\mathbb{C}[C_0]}(w') = \mathcal{U}(\mathfrak{S})d'(C_0) + \mathcal{U}(\mathfrak{S})(p - \phi(p)1) + \mathcal{U}(\mathfrak{S})(q - \phi(q)1).

Let \bar{w} = w + V' \in V/V', note that Ann_{\mathbb{C}[C_0]}(\bar{w}) = \mathbb{C}[C_0](C_0 - \xi).

Let \(u \in \text{Ann}_{\mathcal{U}(\mathfrak{S})}(w)\). Since Ann_{\mathcal{U}(\mathfrak{S})}(w) \subseteq \text{Ann}_{\mathcal{U}(\mathfrak{S})}(\bar{w})\), following from Proposition 4.3, we have

\[ u = u_0(C_0 - \xi 1) + u_1(p - \phi(p)) + u_2(q - \phi(q)) \in \mathcal{U}(\mathfrak{S})(C - \xi 1) + \mathcal{U}(\mathfrak{S})(p - \phi(p)) + \mathcal{U}(\mathfrak{S})(q - \phi(q)). \] (5.1)

While \(u_1(p - \phi(p)) + u_2(q - \phi(q)) \in \text{Ann}_{\mathcal{U}(\mathfrak{S})}(w)\), thus \(u_0(C_0 - \xi 1) \in \text{Ann}_{\mathcal{U}(\mathfrak{S})}(w)\). Observe that \(0 = u_0(C_0 - \xi 1)w = u_0w'\), we have

\[ u_0 \in \text{Ann}_{\mathcal{U}(\mathfrak{S})}(w') = \mathcal{U}(\mathfrak{S})d'(C_0) + \mathcal{U}(\mathfrak{S})(p - \phi(p)1) + \mathcal{U}(\mathfrak{S})(q - \phi(q)1). \]

(5.1) implies that \(u\) has the required form. \(\square\)

**Theorem 5.4.** Let \(M_\phi\) be the universal quasi-Whittaker module of type \(\phi\) with cyclic quasi-Whittaker vector \(w\). If \(V \subseteq M_\phi\) is a submodule, then \(V \cong M_\phi\). Furthermore, \(V\) is generated by a quasi-Whittaker vector of the form \(d(C)w\) for some \(d(x) \in \mathbb{C}[x]\).

To prove this theorem, we need the following lemma.

**Lemma 5.5.** Let \(V\) be a quasi-Whittaker module of type \(\phi\) with cyclic quasi-Whittaker vector \(w\). If \(\text{Ann}_{\mathbb{C}[C_0]}(w) = 0\), then \(V \cong M_\phi\).

**Proof.** By the universal property of \(M_\phi\), there exists an epimorphism \(\varphi : M_\phi \to V\). It is clear that \(\text{Ker} \varphi\) is a submodule of \(M_\phi\). If \(\text{Ker} \varphi \neq 0\), then Theorem 4.1 implies that there is a nonzero quasi-Whittaker vector \(w' \in \text{Ker} \varphi\). It follows from Proposition 3.7, \(0 \neq w' = d(C_0)1 \otimes 1\) and thus \(0 \neq d(C_0) \in \text{Ann}_{\mathbb{C}[C_0]}(w)\). This is impossible. Therefore, \(\text{Ker} \varphi\) must be \(0\) and \(\varphi\) is an isomorphism. \(\square\)

**Proof of Theorem 5.4.** Since \(W_{\phi, \xi}\) is a maximal submodule and a quasi-Whittaker module of type \(\phi\) with cyclic quasi-Whittaker vector \((C_0 - \xi)w\), using Lemma 5.5, we obtain that \(W_{\phi, \xi} \cong M_\phi\).

Let \(\text{Wh}(V)\) be the set of quasi-Whittaker vectors in \(V\), which is nonempty because of Theorem 4.1. Since \(V\) is a submodule, we see that the set \(I = \{d \in \mathbb{C}[C_0] \mid dw \in \text{Wh}(V)\}\) is an ideal of \(\mathbb{C}[C_0]\), and hence can be generated by a polynomial. Let \(d_0\) be the monic generator of \(I\), then \(V = \mathcal{U}(\mathfrak{S})d_0w\). Since \(d_0\) can be written as a product of linear factors from above, we know that there exists a chain of universal quasi-Whittaker modules between \(V\) and \(M_\phi\), satisfying that each quotient is simple. Thus, \(V\) is a universal quasi-Whittaker module of type \(\phi\). \(\square\)
**Remark 5.6.** From Proposition 3.7, we know that the set of quasi-Whittaker vectors in $M_\phi$ is $\mathbb{C}[C]w$, hence $\mathbb{C}[C_0]w$. For distinct $p, q \in \mathbb{C}[C_0]$, we should have $pw \neq qw$. Otherwise, $(p-q)w = 0$, which contradicts with the fact that $\{C^k w \mid k \in \mathbb{Z}_+\}$ is a subset of a basis for $M_\phi$. So, there is a one-to-one correspondence between the quasi-Whittaker vectors and the polynomials in $\mathbb{C}[C_0]$, and hence, from the proof of Theorem 5.4, a one-to-one correspondence between the submodules of $M_\phi$ and the ideals of $\mathbb{C}[C_0]$ since $\mathbb{C}[C_0]$ is a principal ideal domain.

**Theorem 5.7.** Let $V$ be an arbitrary quasi-Whittaker module of type $\phi$ with cyclic quasi-Whittaker vector $w$. Then the set of quasi-Whittaker vectors in $V$ is $\mathbb{C}[C_0]w$.

**Proof.** If $\text{Ann}_{\mathbb{C}[C_0]}(w) = 0$, using Proposition 3.7 and Lemma 5.5 we know that the conclusion holds. If $\text{Ann}_{\mathbb{C}[C_0]}(w) \neq 0$, by Theorem 5.2, we obtain that $V$ has finite composition length $n$.

Next we proceed by induction on $n$. If $n = 1$, then $V$ is simple, using Proposition 3.8 we obtain the conclusion. Suppose that $V$ is an arbitrary module with the composition length $n$. Let

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$$

be a composition series of $V$, and assume $V_1$ has cyclic quasi-Whittaker vector $w_1 = (C_0 - \xi_1)w$. Let $w' \in V$ be a quasi-Whittaker vector. Since $V/V_1$ is simple, by Proposition 3.8 we obtain that the image of $w'$ in $V/V_1$ is a scalar multiple of $\bar{w}$. Therefore, in $V$, $w' = cw + w''$ for some $c \in \mathbb{C}$ and $w'' \in V_1$. Note that $w'' = w' - cw$ is also a quasi-Whittaker vector. Since $V_1$ has composition length $n - 1$, by induction, we have

$$w'' = d(C_0)w_1 = d(C_0)(C_0 - \xi_1)w$$

for some $d(C_0) \in \mathbb{C}[C_0]$.

Therefore, $w' = cw + d(C_0)(C_0 - \xi_1)w$, the statement holds. $\square$

**Acknowledgements**

We would like to thank the referee for a number of helpful comments that greatly improved the presentation of this paper. This work was done during authors’ visit at Wilfrid Laurier University. The authors thank Prof. Kaiming Zhao for formulating the problem, stimulating discussions and help in preparation of this paper. The authors also thank the China Scholar Council for the financial support.

**References**


