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Transparent control of three-body selective destruction of tunneling via unusual states

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Abstract

We study transparent control of quantum tunneling via unusual analytical solutions for three bosons held in a driven double-well. Under high-frequency approximation, we analytically obtain the fine band structure and general non-Floquet state. At some collapse points of the quasi-energy spectra, the latter becomes the unusual special states. Based on the analytical results and their numerical correspondences, we clearly reveal the mechanism of coherent tunneling and suggest a scheme to transparently control the tunneling of three bosons from one well to another. The results can be observed with the current experimental capability (Chen et al 2011 Phys. Rev. Lett. 107 210405)

Keywords: transparent control, unusual state, selective destruction of tunneling, three-body system

(Some figures may appear in colour only in the online journal)

1. Introduction

Coherent control of quantum tunneling in a driven few-well or lattice potential has attracted a lot of interest in both the theoretical and experimental sides of the discipline [1–8]. By using a time-periodic driving field, many interesting tunneling phenomena have been demonstrated, such as the enhancement or suppression of tunneling [9–12]. One of the recent topics is the effect known as coherent destruction of tunneling (CDT) [10], which may be associated with collapse points of the Floquet quasi-energy spectrum [13]. The CDT effect has been verified experimentally in different systems [6, 7]. Then a selective CDT phenomenon was found numerically in the quantum-dot array, in which the tunneling of a single particle between dots can be suppressed selectively [14, 15]. Recently, different routes to the selective CDT were displayed by considering, respectively, the distinguishable intersite separations of a bipartite lattice [16, 17], selective in-phase modulation in a lattice array [18], \textit{a priori} prescribed number of particles for a many-body system [19, 20]. The two-site Bose–Hubbard model of a few-body system held in a double well supplies a fairly useful model for investigating tunneling dynamics in low-dimensional Hilbert space [21–24]. A direct research motivation comes from the theoretical applicability in quantum information processing and the experimental feasibility of the quantum manipulation in the considered system. In fact, the double-well confined few-body system with controllable numbers of atoms within a given well has been realized experimentally [25]. The dynamical control of tunneling processes was even extended to the regime of selective correlated tunneling of strongly interacting particles by adding a periodic shaking to the existing experimental setup [25, 26]. Many works focus on the single- and two-particle cases with Hamiltonian being equivalent to that of the well-known two-state and three-state systems, respectively [27–31]. Investigations on the quantum control of three particles in a periodically driven double well are extremely rare, even though they deserve study for further understanding tunneling dynamics in a four-dimensional Hilbert space. In general, a periodically driven few-body system is not exactly solvable, but the high-frequency
approximation makes the system the effective equivalent of the undriven system, which is analytically solvable [19, 20, 30, 31]. Particularly, at some collapse points of the Floquet quasi-energy spectrum [13], the analytical solutions can be reduced to the celebrated NOON-like (Schrödinger’s-cat-like) state [29, 32] and the dark NOON-like state with zero quasi-energy [33]. It is well known that analytical solutions can provide a deeper understanding of the underlying physics than can straight numerical calculations [28, 34, 35]. Therefore, we are interested in the high-frequency situations where the unusual analytical solutions of simple forms exist, enabling a better understanding of the CDT and selective CDT effects and rendering the control strategies of quantum tunneling more transparent [36].

In this paper, we investigate the tunneling dynamics of three bosons held in a driven double-well potential. In high-frequency approximation, a fine band structure and a set of analytical Floquet states are obtained, and a general non-Floquet state is constructed by coherent superposition of the Floquet solutions in four-dimensional Hilbert space. At some collapse points of the quasi-energy spectra, the general non-Floquet state is reduced to some special states of simple forms, which contain the interesting CDT single states, three-body selective CDT states, and dark NOON-like states. Hereafter, these special states are called the unusual states, due to the transparent connections between them and the control mechanism of the coherent tunneling effects. Based on the analytical results and their numerical correspondences, we suggest a scheme to transport three bosons from one well to another by adjusting the driving parameters to make the transitions between the unusual states. These results presented here can be observed in the current experimental setup [25, 26], and may be useful for the design of quantum devices [37].

2. Analytical solutions in the high-frequency approximation

We consider the tunneling dynamics of three ultracold bosons held in a driven double-well governed by the single-band two-site Bose-Hubbard Hamiltonian [19–21]

\[ \hat{H}(t) = -\Omega \left( \hat{c}^\dagger_1 \hat{c}^\dagger_2 + \hat{c}^\dagger_2 \hat{c}^\dagger_1 \right) + \frac{U}{2} \left( \hat{c}^\dagger_1 \hat{c}_1 \hat{c}^\dagger_2 \hat{c}_2 + \hat{c}^\dagger_2 \hat{c}_2 \hat{c}^\dagger_1 \hat{c}_1 \right) + \epsilon(t) \left( \hat{c}^\dagger_1 \hat{c}_1 - \hat{c}^\dagger_2 \hat{c}_2 \right), \]

(1)

where \( \hat{c}^\dagger_1 \) and \( \hat{c}_1 \) are, respectively, the atom creation and annihilation operators in well 1 and well 2. Constants \( \Omega \) and \( U \) denote the tunneling coupling between the two wells and the on-site interatom interaction, respectively. The time-dependent bias reads \( \epsilon(t) = \epsilon_0 \cos(\omega_0 t) \), with \( \epsilon_0 \) and \( \omega_0 \) being the strength and frequency of the ac field, respectively. The difference between equation (1) and the model of [21] is only a static bias that was used to investigate the fractional-photon-assisted tunneling for few interacting ultracold atoms in the double wells [21]. Throughout this paper, \( \hbar = 1 \) and the reference frequency \( \omega_0 \) on the order of \( 10^2 \) Hz are adopted [38] such that \( \epsilon_0, \alpha, U, \) and \( \Omega \) are in units of \( \omega_0^{-1} \). Using the Fock basis \(|n\rangle = |n\rangle \) with \( i \) atoms in the left well and \( 3 - i \) atoms in the right well, quantum state \( |\psi(t)\rangle \) of the three-body system can be expanded as

\[ |\psi(t)\rangle \equiv \frac{1}{\sqrt{2}} \sum_{i=0}^{2} a_i(t)|i, 3 - i\rangle, \]

(2)

where \( a_i(t) \) denote the time-dependent probability amplitudes with \( i \) atoms in the left well, which obey the normalization condition \( \sum_{i=0}^{2} |a_i(t)|^2 = 1 \).

Inserting equations (1) and (2) into the Schrödinger equation \( \frac{\partial \psi(t)}{\partial t} = \hat{H}(t) \psi(t) \) results in the coupling equations

\[ i\dot{a}_0(t) = -\sqrt{3} \Omega a_1(t) + [3U + 3\epsilon(t)]a_0(t), \]

(3)

\[ i\dot{a}_1(t) = -\sqrt{3} \Omega a_0(t) - 2\Omega a_2(t) + [U + \epsilon(t)]a_1(t), \]

\[ i\dot{a}_2(t) = -2\Omega a_1(t) - \sqrt{3} \Omega a_2(t) + [U - \epsilon(t)]a_2(t), \]

\[ i\dot{a}_3(t) = -\sqrt{3} \Omega a_2(t) + [3U - 3\epsilon(t)]a_3(t). \]

It is difficult to obtain the exact solutions of equation (3), because of the periodically varying coefficients, which is analytically solvable. To do so, we set \( U = \omega_0 \alpha + \alpha \) with integer \( n = 0, 1, 2 \ldots \) and reduced interaction strength \( \omega_1 \approx \omega_2 \), where case \( n = 0 \) is called the photon resonance for historical reasons [39]. In the conditions of high-frequency approximation [40, 41], we introduce the slowly varying functions \( b_j(t) \) through the transformation

\[ a_0(t) = b_0(t) e^{i \int [3(n\omega_0 + 3\omega(t))]/\omega_0 dt}, \]

\[ a_1(t) = b_1(t) e^{i \int [n\omega_0 - \epsilon(t)]/\omega_0 dt}, \]

\[ a_2(t) = b_2(t) e^{i \int [n\omega_0 - \epsilon(t)]/\omega_0 dt}, \]

\[ a_3(t) = b_3(t) e^{i \int [3(n\omega_0 - 3\omega(t))]/\omega_0 dt} \]

and use the Fourier expansion \( \int f(\pm \epsilon + \omega_0)dt = \sum_{n=\pm \infty} \mathcal{J}_n(e^{\omega_0}) \exp \left[i(\pm n - n\omega_0)\right] \). After neglecting the rapidly oscillating terms with \( n' \approx n \neq 0 \) [30, 42], equation (3) is transformed to the form

\[ i\dot{b}_0(t) = -\sqrt{3} J_{2n} b_1(t) + 3b_0(t), \]

\[ i\dot{b}_1(t) = -\sqrt{3} J_{2n} b_0(t) - 2J_n b_2(t) + ub_1(t), \]

\[ i\dot{b}_2(t) = -2J_{2n} b_1(t) - \sqrt{3} J_{3n} b_3(t) + ub_2(t), \]

\[ i\dot{b}_3(t) = -\sqrt{3} J_{2n} b_2(t) + 3b_3(t), \]

(4)

where the coupling constant \( \Omega \) has been renormalized by the effective constants \( J_{2n} = \Omega J_0 (2n) / \omega_0, J_0 = \Omega J_0 (2\alpha) / \omega_0 \), with \( J_0(x) \) being the n-order Bessel function of \( x \) and \( n \) being determined by the interaction strength. Solving such a set of linear equations with constant coefficients, one can obtain many interesting analytical results, as follows.

2.1. Quasienergy bands and Floquet states

Based on the Floquet theorem [43], the quasi-energies and Floquet states of equation (3) can be expected, by rewriting equation (2) as the Floquet form \( |\psi(t)\rangle = |l\rangle \phi(t) e^{i\lambda t} \), where \( \lambda \) is the quasi-energy and \( |l\rangle \phi(t) \) is called...
the Floquet state. Noticing the relations between \( a_i(t) \) and \( b_j(t) \), the Floquet state can be represented as

\[
|\varphi(t)\rangle = A e^{-i \int (3 nu + 3 \varepsilon(t)) dt} |0, 3\rangle \\
+ B e^{-i \int (nu + \varepsilon(t)) dt} |1, 2\rangle \\
+ C e^{-i \int (nu - \varepsilon(t)) dt} |2, 1\rangle \\
+ D e^{-i \int (3 nu - 3 \varepsilon(t)) dt} |3, 0\rangle,
\]

through the stationary solutions of equation (4), \( b_0(t) = A e^{-i E \varepsilon}, \quad b_1(t) = B e^{-i E \varepsilon}, \quad b_2(t) = C e^{-i E \varepsilon}, \quad b_3(t) = D e^{-i E \varepsilon} \), where \( A, B, C, \) and \( D \) are constants satisfying the normalization condition \( |A|^2 + |B|^2 + |C|^2 + |D|^2 = 1 \). Inserting the stationary solutions into equation (4), we obtain the Floquet quasi-energies \( E_l \) and the corresponding constants \( A_l, B_l, C_l, D_l \) for \( l = 1, 2, 3, 4 \) as

\[
E_1 = 2u + J_0 - \sqrt{(-u + J_0)^2 + 3J_{2n}^2}, \\
A_1 = D_1 = \frac{K_1}{2(K_1^2 + 1)}, \\
B_1 = C_1 = \frac{1}{\sqrt{2(K_1^2 + 1)}}, \\
E_2 = 2u + J_0 + \sqrt{(-u + J_0)^2 + 3J_{2n}^2}, \\
A_2 = D_2 = \frac{K_2}{2(K_2^2 + 1)}, \\
B_2 = C_2 = \frac{1}{\sqrt{2(K_2^2 + 1)}}, \\
E_3 = 2u - J_0 + \sqrt{(u + J_0)^2 + 3J_{2n}^2}, \\
A_3 = -D_3 = \frac{K_3}{\sqrt{2(K_3^2 + 1)}}, \\
B_3 = -C_3 = -\frac{1}{\sqrt{2(K_3^2 + 1)}}, \\
E_4 = 2u - J_0 - \sqrt{(u + J_0)^2 + 3J_{2n}^2}, \\
A_4 = -D_4 = \frac{K_4}{\sqrt{2(K_4^2 + 1)}}, \\
B_4 = -C_4 = -\frac{1}{\sqrt{2(K_4^2 + 1)}},
\]

with constants

\[
K_{1,2} = (u - J_0) \pm \sqrt{(-u + J_0)^2 + 3J_{2n}^2}, \\
K_{3,4} = -(u + J_0) \pm \sqrt{(u + J_0)^2 + 3J_{2n}^2}.
\]

Analytical Floquet states. Given equation (5), we immediately arrive at the four Floquet states

\[
|\varphi_l(t)\rangle = A_l \left[ \left( \frac{u + J_0}{\sqrt{3J_{2n}}} \right)^{\mathrm{sgn}((2.5)\varepsilon)} \right]|0, 3\rangle \\
+ \mathrm{sgn}((2.5 - l)\varepsilon)|1, 2\rangle \\
+ B_l \left[ \left( \frac{u - J_0}{\sqrt{3J_{2n}}} \right)^{\mathrm{sgn}((2.5 - l)\varepsilon)} \right]|1, 2\rangle. \tag{6}
\]

for \( l = 1, 2, 3, 4 \), where the sign function obeys \( \mathrm{sgn}(2.5 - l) = 1 \) (or -1) for \( 2.5 - l > 0 \) (or <0). Clearly, these Floquet states are the stationary-like states (SLSs) with variable probability amplitudes and invariant populations [29]. They may be in a NOON-like state for \( B_l = 0 \) or in a non-NOON state for \( A_l = 0 \). Quantum tunneling cannot happen in a single Floquet state, but can occur in coherent superposition states of the Floquet states. The considered three-body system may occupy any Floquet state only for some fixed initial conditions and restricted system parameters, since the constants \( A_l \) and \( B_l \) determined by the parameters have to satisfy the additional normalization restriction.

Fine band structures. On the other hand, from equation (5) we can illustrate the different quasi-energy spectra for selected system parameters. As an example, we take \( \omega = 50, \Omega = 0.5 \), and (a) \( U = 50 \) (\( u = 0 \)), (b) \( U = 50.066 \) (\( u = 0.066 \)), (c) \( U = 50.54 \) (\( u = 0.54 \)), (d) \( U = 51.4 \) (\( u = 1.4 \)) to plot the four quasi-energy spectra with quasi-energies as functions of the driving parameters \( 2\varepsilon/\omega \), as in figure 1(a)–(d), respectively. Here the analytical results are described by the circles and the solid curves denote the numerical results based on the exact model (3), and good agreement is displayed. In figure 1(a) with \( u = 0 \) and in figure 1(b) with a small reduced interaction strength \( u = 0.066 \), it can be seen that the quasi-energy spectra show a single band with the approximately same width. As the reduced interaction strength increases to large enough, figures 1(c) and (d) exhibit that the quasi-energy curves are divided into two energy bands of different widths. With the increase of the \( u \) value, each band width is approximately maintained and the gap between two bands is raised, which is similar to the previous results for a driven triple-well system [44]. The fine band structures near some collapse points are amplified in the corresponding insets of figure 1. From these amplified curves we find some small deviations between the analytical collapse points labeled by the circles and the numerical ones described by the curves. The similar deviations have been revealed by the second-order effects of tunneling [44, 45].

Collapse points of the quasi-energy spectra. In figure 1, we observe that there exist some collapse points of the quasi-energy spectra at which the quasi-energies crossing or avoided-crossing appear. We divide the collapse points into two series, which correspond to the roots of equations \( J^2_l(2\varepsilon_0/\omega) = 0 \) and \( J^4_l(2\varepsilon_0/\omega) = 0 \), respectively. In the parameter region of figure 1, only the first root...
$2\epsilon_0/\omega = 5.1356$ of the former equation and the first two roots $2\epsilon_0/\omega = 2.405$, 5.5201 of the latter equation are included.

**Series 1.** At the collapse point $2\epsilon_0/\omega = 5.1356$ of the first series, noticing $J_0(5.1356) = \Omega J_2(5.1356) = 0$ and $J_0(5.1356) = \Omega J_0(5.1356) \approx -0.066 < 0$, from equation (5) with $n = 1$, any quasi-energy spectrum of a fixed $u$ shows a single two- or three-level-crossing, as follows:

For $u < J_0 < 0$, there exists the two-level-crossing $E_1 = E_4 = 3u$ with $E_{2,3} = u \pm 2J_0$, which is not shown in figure 1.

For $J_0 < u < -J_0$, the two-level-crossing $E_2 = E_4 = 3u$ with $E_{1,3} = u \pm 2J_0$ is exhibited; e.g., see figure 1(a) with $u = 0$.

For $u > -J_0 \approx 0.066$, the two-level-crossings $E_2 = E_3 = 3u$ with $E_{1,4} = u \pm 2J_0$ and $u = 0.54$, 1.4 are shown in figures 1(c) and (d), respectively.

For $u = -J_0 = -0.066$, the three-level-crossing $E_2 = E_3 = E_4 = -3J_0$ with $E_1 = J_0$ is displayed by figure 1(b).

For $u = J_0 = -0.066$, the three-level-crossing $E_1 = E_2 = E_3 = 3J_0$ with $E_4 = -J_0$ exists, which is not shown in figure 1.

**Series 2.** At the collapse points $2\epsilon_0/\omega = 2.405$ and 5.5201 of the second series, from equation (5) we obtain the analytical quasi-energy $E_1 = E_4 = 2u - \sqrt{u^2 + 2 J_0^2}$ and $E_2 = E_3 = 2u + \sqrt{u^2 + 2 J_0^2}$. In figure 1 we show numerically that at any one of the collapse points there is a pair of two-level-crossing $E_1 \approx E_4$ and $E_2 \approx E_3$, and for a fixed $u$ value, quasi-energy at different collapse points are distinguished by the $J_2(2\epsilon_0/\omega)$ values. The small differences between the analytical results (circles) and the numerical curves can be eliminated by taking into account the second-order effects of tunneling [44, 45].

In the next section, we will reveal that at such collapse points some unusual states can be derived from the following coherent superposition of the Floquet states.

### 2.2. General coherent non-Floquet state

In order to study tunneling dynamics, we have to consider the coherent superposition of the Floquet states. When the superposition states do not satisfy the well-known definition of the Floquet state, we call them the non-Floquet states [29, 46]. The superposition principle of quantum mechanics indicates that the Floquet states (6) constitutes a set of complete bases in a four-dimensional Hilbert space [47, 48], and the non-Floquet states can be constructed by the linear superposition of the Floquet states. Directly employing equations (5) and (6) to the linear superposition yields

$$
\psi(t) = \sum_{l=1}^{4} \left| \phi_l(t) \right> e^{-i\Omega t} = b_0(t)e^{-\left[\frac{\sqrt{50} \sin(\omega t) + 3\omega t}{5\omega t}ight]}|0, 3\rangle
+ b_1(t)e^{-\left[\frac{\sqrt{50} \sin(\omega t) + 3\omega t}{5\omega t}ight]}|1, 2\rangle
+ b_2(t)e^{\left[\frac{\sqrt{50} \sin(\omega t) - 3\omega t}{5\omega t}ight]}|2, 1\rangle
+ b_3(t)e^{\left[\frac{\sqrt{50} \sin(\omega t) - 3\omega t}{5\omega t}ight]}|3, 0\rangle
$$

(7)

which is the general non-Floquet state for some different quasi-energy. The first part of equation (7) means that the general non-Floquet state can be reduced to a new Floquet
state if and only if the four quasi-energy take the forms $E_l = n_l \omega + \delta$ with integers $n_l$ and $\delta < \omega$. In equation (7), $s_l$ for $l = 1, 2, 3, 4$ are arbitrary superposition coefficients adjusted by the initial conditions and normalization, and the probability amplitudes are renormalized as

$$
b_0'(t) = \sum_{l=1}^{4} s_l A_l e^{-iE_l t},
$$

$$
b_1'(t) = \sum_{l=1}^{4} s_l B_l e^{-iE_l t},
$$

$$
b_2'(t) = \sum_{l=1}^{4} s_l B_l \text{sgn}(2.5 - l) e^{-iE_l t},
$$

$$
b_3'(t) = \sum_{l=1}^{4} s_l A_l \text{sgn}(2.5 - l) e^{-iE_l t},
$$

with constants $A_l, B_l$ given in equation (5). The occupy probabilities of $i$ bosons in the left well read as $P_l = |b_i'(t)|^2$ for $i = 0, 1, 2, 3$. The linear superposition state of equation (7) implies a quantum interference effect among the four Floquet states with different quasi-energy. It may cause the coherent enhancement or suppression of quantum tunneling, whose degree is adjusted by the values of the effective coupling parameters [49]. Such an interference effect will be applied to manipulations of the quantum tunneling.

### 3. Transparent control of three-body tunneling via unusual states

It is well known that many interesting tunneling phenomena, such as CDT and selective CDT, emerge at the collapse points of the quasi-energy spectra. Why? Here we will analytically demonstrate that the reason is from the existence of the unusual states of equation (7) at these collapse points.

#### 3.1. Unusual states for the first series of collapse points

At collapse point $2\epsilon_0/\omega = 5.1356$ of the first series with $J_0(5.1356) = 0, J_0(5.1536) \approx -0.066$ and in the range $J_0(5.1356) < \omega < -J_0(5.1536)$, equation (5) gives $E_2 = E_4 = 3\omega$, $E_{1,3} = \pm 2J_0(5.1356)$ and $A_{2,4} = D_{2,4} = -D_4 = \frac{1}{\sqrt{2}}$, $A_{1,3} = D_{1,3} = 0$. $B_{2,4} = C_{2,4} = 0$, $B_1 = -B_3 = C_{1,3} = \frac{1}{\sqrt{2}}$. Inserting these results into equation (6), we get the four Floquet states

$$\Psi_{1,3}(t) = \frac{1}{\sqrt{2}} e^{-\frac{i\omega}{2} \sin(\omega t) + \text{at}^2} \left[ e^{\frac{2i\omega}{3} \sin(\omega t)} |2, 1\rangle \pm |1, 2\rangle \right],$$

$$\Psi_{2,4}(t) = \frac{1}{\sqrt{2}} e^{-\frac{i\omega}{2} \sin(\omega t) + 3\text{at}^2} \left[ |0, 3\rangle \pm e^{\frac{4i\omega}{3} \sin(\omega t)} |3, 0\rangle \right].$$

Clearly, in the above case, due to $A_1 = 0, B_1 \neq 0$ for $l = 1, 3$ and $A_1 \neq 0, B_1 = 0$ for $l = 2, 4$, the four Floquet states of equation (6) are reduced to the two groups of equation (8).

Here $|\psi_{2,4}(t)\rangle$ are the NOON-like states associated with quasi-energy $E_2 = E_4$, and $|\psi_{1,3}(t)\rangle$ are the non-NOON states corresponded to $E_1 \neq E_3$. Thus, the non-Floquet state of equation (7) is reduced to the unusual state

$$\Psi(t) = \frac{1}{\sqrt{2}} (s_2 + s_4) e^{-\frac{i\omega}{2} \sin(\omega t) + 3\text{at}^2} |0, 3\rangle,$$

$$+ \frac{1}{\sqrt{2}} e^{\frac{2i\omega}{3} \sin(\omega t) + \text{at}^2} [s_1 e^{-iE_1 t} - s_3 e^{-iE_3 t}] |1, 2\rangle,$$

$$+ \frac{1}{\sqrt{2}} e^{\frac{4i\omega}{3} \sin(\omega t) + 3\text{at}^2} [s_1 e^{-iE_1 t} + s_3 e^{-iE_3 t}] |2, 1\rangle,$$

$$+ \frac{1}{\sqrt{2}} (s_2 - s_4) e^{\frac{4i\omega}{3} \sin(\omega t) - 3\text{at}^2} |3, 0\rangle. \quad (9)$$

We will select different sets of the arbitrary constants $s_l$ to produce the special unusual states, which contain three types of SLSs (stationary-like states) and one non-stationary state (selective CDT states of the first kind).

**SLSs of type 1: CDT single states.** When $s_3 = s_3 = 0$ and $s_2 = \pm s_4$ are selected, equation (9) gives two different states, respectively, consisting of a single Fock basis and possessing invariant populations, which transparently describe the CDT and are called the CDT single states:

$$\Psi_{10}(t) = e^{-\frac{i\omega}{2} \sin(\omega t) + 3\text{at}^2} |0, 3\rangle,$$

$$\Psi_{30}(t) = -e^{\frac{4i\omega}{3} \sin(\omega t) - 3\text{at}^2} |3, 0\rangle,$$

with $P_1(t) = 1$, $P_{\neq 1}(t) = 0$ being shown in figure 2(a).

**SLSs of type 2: NOON-like states.** When $s_2 = s_3 = 0$ and $s_2 \neq \pm s_4$ are taken, equation (9) becomes the general NOON-like state

$$\Psi_{0003}(t) = \frac{1}{\sqrt{2}} (s_2 + s_4) e^{\frac{4i\omega}{3} \sin(\omega t) + 3\text{at}^2} |0, 3\rangle,$$

$$+ \frac{1}{\sqrt{2}} (s_2 - s_4) e^{\frac{4i\omega}{3} \sin(\omega t) - 3\text{at}^2} |3, 0\rangle,$$

with the invariant populations $P_1(t) = \frac{1}{2} |s_2 + s_4|^2$, $P_2(t) = \frac{1}{2} |s_2 - s_4|^2$, and $P_1(t) = P_2(t) = 0$, which are exhibited in figures 2(c) and (d), respectively, for the initial conditions (c) $P_{10}(0) = P_{20}(0) = \frac{1}{2}$ with $s_2 = 1$, $s_3 = s_4 = 0$, and (d) $P_{10}(0) = P_{20}(0) = \frac{3}{4}$ with $s_{2,4} = \frac{\sqrt{3}(1 + \sqrt{3})}{4}$, $s_1 = s_3 = 0$. Specifically, in the photon resonance case with $U = \omega (u = 0)$, the two-level-crossing $E_2 = E_4 = 3\omega = 0$ occurs such that the general NOON-like state becomes the interesting dark NOON-like state of zero quasi-energy [33].

**SLSs of type 3: non-NOON states.** When $s_2 = s_4 = 0$ and $s_3 = 0, s_4 \neq 0$, or $s_2 \neq 0, s_3 = 0$ are given, equation (9) becomes the following non-NOON states with the same...
Obviously, these SLSs are new Floquet states that can be called the unusual Floquet states.

Selective CDT state of kind 1. Now we consider the special non-stationary state

\[
\left| \Psi_{2112}(E_1, E_3, t) \right> = \frac{1}{\sqrt{2}} e^{-i \left( \frac{\omega}{2} \sin(\alpha t) + \alpha t - E_1 t \right)} \left| 2, 1 \right> + \frac{1}{\sqrt{2}} e^{-i \left( \frac{\omega}{2} \sin(\alpha t) + \alpha t - E_3 t \right)} \left| 1, 2 \right>
\]

of equation (9) with \( s_2 = s_4 = 0, s_1, s_3 \neq 0 \), and \( E_{1,3} = \alpha \pm 2J_0(5.1356) \). Such a state can be called the selective CDT state [18], due to the transparent descriptions on the selective tunneling between states \( |11, 2 \rangle \) and \( |12, 1 \rangle \), and
on the three-body selective CDT from (1, 2), (12, 1) to the states (10, 3) and (13, 0). As an example, we consider the initial state \( \psi(0) = |11, 2\rangle \) for the initial constants \( s_1 = s_3 = \frac{1}{\sqrt{2}}, \ s_2 = s_4 = 0 \) and the system parameters \( U = \omega = 50 \ (u = 0), \ \Omega = 0.5, \ 2\epsilon_0/\omega = 5.1356 \) to plot the time evolutions of probabilities \( P_z(t) \), as in figure 3(a). The alternate changes of \( P_1(t) \) and \( P_2(t) \) between 0 and 1 imply that the selective tunneling of a single boson occurs between the states (11, 2) and (12, 1). Noticing \( P_1(t) = \left| a_1(t) \right|^2 \), we know the oscillating frequency \( \alpha_1 \) of \( P_1(t) \) to be two times that of \( a_1(t) \), and we analytically obtain \( \alpha_1 = 2 \left| E_{1, 1} \right| = 4 \left| \epsilon_0 \right| (5.1356) \approx 0.264 \). The corresponding period reads \( T_1 = 2\pi/\alpha_1 \approx 23.8 \). In figure 3(a) it can be seen that the analytical results (circles) are in good agreement with the numerical curves.

3.2. Unusual states for the second series of collapse points

Now we consider the collapse points \( 2\epsilon_0/\omega \approx 2.405 \) of the second series and the parameters \( U = \omega + u \) with \( u = 1 \). The corresponding quasi-energy and constants of equation (5) read

\[
E_1 = E_4 = 2u - \sqrt{u^2 + 3J_z^2}, \ E_2 = E_3 = 2u + \sqrt{u^2 + 3J_z^2}
\]

and

\[
A_1 = A_4 = D_1 = D_4 = -\frac{u - \sqrt{u^2 + 3J_z^2}}{\sqrt{6J_z^2 + 2(u - \sqrt{u^2 + 3J_z^2})^2}},
\]

\[
A_2 = A_3 = D_2 = D_3 = -\frac{u + \sqrt{u^2 + 3J_z^2}}{\sqrt{6J_z^2 + 2(u + \sqrt{u^2 + 3J_z^2})^2}},
\]

\[
B_1 = B_4 = C_1 = C_4 = \frac{\sqrt{3}J_z}{\sqrt{6J_z^2 + 2(u - \sqrt{u^2 + 3J_z^2})^2}},
\]

\[
B_2 = B_3 = C_2 = C_3 = \frac{\sqrt{3}J_z}{\sqrt{6J_z^2 + 2(u + \sqrt{u^2 + 3J_z^2})^2}},
\]

for \( \Omega = 0.5 \) and \( J_z = J_z(2.405) = \Omega J_z(2.405) \approx 0.216 \). Inserting these into equation (7), the non-Floquet state becomes the new unusual state

\[
\psi(t) = \left[ A_1(s_1 - s_4)e^{-i\epsilon_1t} + A_2(s_2 - s_3)e^{-i\epsilon_2t} + B_1(s_1 - s_4)e^{-i\epsilon_1t} + B_2(s_2 - s_3)e^{-i\epsilon_2t} + A_1(s_1 + s_4)e^{i\epsilon_1t} + A_2(s_2 + s_3)e^{i\epsilon_2t} + B_1(s_1 + s_4)e^{i\epsilon_1t} + B_2(s_2 + s_3)e^{i\epsilon_2t} \right] \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 + \alpha_2)} \right][0, 3]}
\]

\[
+ \left[ B_1(s_1 - s_4)e^{-i\epsilon_1t} + B_2(s_2 - s_3)e^{-i\epsilon_2t} + B_1(s_1 + s_4)e^{-i\epsilon_1t} + B_2(s_2 + s_3)e^{-i\epsilon_2t} \right] \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 - \alpha_2)} \right][1, 2]}
\]

\[
+ A_1(s_1 + s_4)e^{i\epsilon_1t} + A_2(s_2 + s_3)e^{i\epsilon_2t} + A_1(s_1 + s_4)e^{i\epsilon_1t} + A_2(s_2 + s_3)e^{i\epsilon_2t} \right] \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 - \alpha_2)} \right][2, 1]}
\]

\[
+ \left[ A_1(s_1 + s_4)e^{-i\epsilon_1t} + A_2(s_2 + s_3)e^{-i\epsilon_2t} + A_1(s_1 + s_4)e^{-i\epsilon_1t} + A_2(s_2 + s_3)e^{-i\epsilon_2t} \right] \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 + \alpha_2)} \right][3, 0]}
\].

From equation (10) we easily find the different kinds of the selective CDT states as follows.

Selective CDT state of kind 2. For the initial constants \( s_1 = -s_4 \) and \( s_2 = -s_3 \), equation (10) is reduced as

\[
\psi_{0312}(t = E_1, E_2, t) = 2(A_1s_1e^{-i\epsilon_1t} + A_2s_2e^{i\epsilon_2t})
\]

\[
\times \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 + \alpha_2)} \right][0, 3]}
\]

\[
+ 2(B_1s_1e^{-i\epsilon_1t} + B_2s_2e^{i\epsilon_2t})
\]

\[
\times \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 - \alpha_2)} \right][1, 2]},
\]

which contains different states for different \( s_1 \) and \( s_2 \) values. In such a state, the population transfer can periodically occur only between the states \( |0, 3\rangle \) and \( |11, 2\rangle \), which means the selective CDT to the states \( |0, 3\rangle \) and \( |12, 1\rangle \).

Selective CDT state of kind 3. When the superposition coefficients are set as \( s_1 = s_4 \) and \( s_2 = s_3 \), equation (10) gives another kind of selective CDT state:

\[
\psi_{0310}(t = E_1, E_2, t) = 2(B_1s_1e^{-i\epsilon_1t} + B_2s_2e^{i\epsilon_2t})
\]

\[
\times \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 - \alpha_2)} \right][2, 1]},
\]

\[
\times \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 + \alpha_2)} \right][3, 0]},
\]

In such a kind of state, the selective tunneling of a boson between states \( |12, 1\rangle \) and \( |13, 0\rangle \) can periodically occur that means the selective CDT to the states \( |0, 3\rangle \) and \( |11, 2\rangle \).

Particularly, applying the photon-resonance condition \( U = \omega \ (u = 0) \) to the above case, we obtain the quasi-energy \( E_1 = -E_2 = -E_3 = E_4 = -\sqrt{3}J_z(2.405) \), and the corresponding constants \( A_1 = A_3 = D_1 = -D_3 = \frac{1}{2}, \ A_2 = A_4 = D_2 = -D_4 = \frac{1}{2}, \ B_1 = -B_3 = C_1 = C_3 = \frac{1}{2}, \ B_2 = -B_4 = C_2 = C_4 = \frac{1}{2} \). Inserting these results into equation (10), the unusual state is reduced as the evident form

\[
\psi(t) = \frac{1}{2} \left[ (s_1 - s_4)e^{-i\epsilon_1t} - (s_2 - s_3)e^{i\epsilon_2t} \right]
\]

\[
\times \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 + \alpha_2)} \right][0, 3]}
\]

\[
+ \frac{1}{2} \left[ (s_1 - s_4)e^{-i\epsilon_1t} + (s_2 - s_3)e^{i\epsilon_2t} \right]
\]

\[
\times \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 - \alpha_2)} \right][1, 2]}
\]

\[
+ \frac{1}{2} \left[ (s_1 + s_4)e^{-i\epsilon_1t} + (s_2 + s_3)e^{i\epsilon_2t} \right]
\]

\[
\times \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 - \alpha_2)} \right][2, 1]}
\]

\[
- \frac{1}{2} \left[ (s_1 + s_4)e^{-i\epsilon_1t} - (s_2 + s_3)e^{i\epsilon_2t} \right]
\]

\[
\times \epsilon^{-\left[ \frac{\Omega}{\sin(\alpha_1 + \alpha_2)} \right][3, 0]}. \quad (11)
\]

Simultaneously, the selective CDT states of kinds 2 and 3 are simplified as the corresponding simple forms. For example, taking the initial constants \( s_1 = -s_4 = \frac{1}{2} \) and \( s_2 = -s_3 = -\frac{1}{2} \)
leads the selective CDT state of kind 2 to the form
\[
|\psi_{0312}(E_1, t)\rangle = -\cos \left( E_1 t \right) e^{-i \frac{3 \omega}{8} \sin (\omega t) + 3 \omega t} |0, 3\rangle + \sin \left( E_1 t \right) e^{-i \frac{3 \omega}{8} \sin (\omega t) + \omega t} |1, 2\rangle.
\]

Such a state directly describes the selective tunneling between states \( |0\rangle, |3\rangle \) and \( |1\rangle, |2\rangle \) and the three-body selective CDT to the states \( |3\rangle, |0\rangle \) and \( |2\rangle, |1\rangle \). The time evolutions of the related probabilities \( P_2(t) \) are plotted in figure 3(b) for the parameters \( \Omega = 0.5 \) and \( \omega = U = 50 \). The periodic evolutions of \( P_2(t) \) and \( R(t) \) mean the Rabi oscillation is between states \( |0\rangle, |3\rangle \) and \( |1\rangle, |2\rangle \), where the oscillating frequency and period read
\[
\omega_2 = 2 |E_1| = 2 \sqrt{3} J_2(2.405) \approx 0.748.
\]

Similarly, selecting the initial constants
\[
s_1 = s_2 = s_3 = s_4 = \frac{1}{2},
\]
the selective CDT state of kind 3 is reduced to
\[
|\psi_{2130}(E_1, t)\rangle = \cos \left( E_1 t \right) e^{i \frac{\pi}{8} \sin (\omega t) - \omega t} |2, 1\rangle + \sin \left( E_1 t \right) e^{i \frac{\pi}{8} \sin (\omega t) - 3 \omega t} |3, 0\rangle.
\]

The time evolutions of the corresponding probabilities are shown in figure 3(c). The alternate changes of \( P_2(t) \) and \( P_3(t) \) mean that the single-particle tunneling occurs periodically between states \( |2\rangle, |1\rangle \) and \( |3\rangle, |0\rangle \) with the same period \( T_2 \) as that of the selective tunneling of kind 2.

3.3. Coherent manipulation of quantum tunneling

Coherent control of quantum tunneling is very important and interesting in the area of quantum technologies [50, 51]. In the above two subsections, we have demonstrated both analytically and numerically the existence of the interesting unusual states. Now we are interested in the transparent control of three-body tunneling via these unusual states. As an example, by using some of the analytical states we directly give a scheme for manipulating the tunneling of three bosons from the right well to the left well, which is equivalent to the quantum transition from the initial state \( |\psi(0)\rangle \) to the final state \( |\psi_f(0)\rangle \), as shown in table 1 and figure 4. In the manipulation process, the parameters \( \Omega = 0.5 \), \( \omega = U = 50 \) (\( \omega = 0 \)) have been fixed so that only the driving strength is adjusted between values \( \epsilon_{0.1} = 2.5678 \) of collapse point 1 and \( \epsilon_{0.2} = 1.0205 \) of collapse point 2.

In table 1 and figure 4(a), we show that the three bosons are prepared initially in the CDT single state \( |\psi_{0312}(E_1, t)\rangle \) with the probability of occupying the right well \( P_0(0) = 1 \) by setting the initial driving strength as \( \epsilon_{0.1} = 2.5678 \). At an arbitrarily given time \( t_1 = 2 \), we change the driving strength to \( \epsilon_{0.2} = 1.0205 \) which results in the transition from \( |\psi(0)\rangle \) to the selective CDT state \( |\psi_{0312}(E_1, t)\rangle \) of kind 2. Then we return to the driving strength \( \epsilon_{0.1} \) at time \( t_2 = t_1 + T_2/2 = 2 + 4.2 = 6.2 \) such that transition from \( |\psi_{0312}(E_1, t)\rangle \) to the selective CDT state \( |\psi_{2130}(E_1, t)\rangle \) of kind 1 occurs. Further, when we tune \( \epsilon_0 \) to \( \epsilon_{0.1} \) at time \( t_3 = t_2 + T_2/2 = 6.2 + 11.9 = 18.1 \), the system is transited to the selective CDT state \( |\psi_{2130}(E_1, t)\rangle \) of kind 3. Finally, at \( t_4 = t_3 + T_2/2 = 18.1 + 4.2 = 22.3 \) the adjustment from \( \epsilon_{0.2} \) to \( \epsilon_{0.1} \) leads the system to the final state \( |\psi_f(t)\rangle \). Time evolutions of the probabilities corresponding to the unusual states in table 1 are described by figure 4(a). The spatial distributions of the three particles at the tuning moments \( t_i \), \( i = 1, 2, 3, 4 \), are exhibited in figure 4(b), where \( \Delta t_i \) denotes transferring times between the different populations—namely, the time differences between the moments of the four spatial distributions, \( \Delta t_1 = t_2 - t_1 = t_3 - t_1 = T_2/2 = 4.2 \) and \( \Delta t_2 = t_3 - t_2 = T_2/2 = 11.9 \). Thus, we successfully transport the three bosons from the right well to the left one by adjusting the driving strength of the ac field to induce the transitions among some of the unusual states.

<table>
<thead>
<tr>
<th>Starting and ending times</th>
<th>( t_1 \sim t_2 )</th>
<th>( 0 \sim t_1 )</th>
<th>( t_1 \sim t_2 = t_1 + T_2/2 )</th>
<th>( t_2 \sim t_3 = t_2 + T_2/2 )</th>
<th>( t_3 \sim t_4 = t_3 + T_2/2 )</th>
<th>( t_4 \sim t_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Driving parameters</td>
<td>( \epsilon_{0.1} )</td>
<td>2.5678</td>
<td>1.0205</td>
<td>2.5678</td>
<td>1.0205</td>
<td>2.5678</td>
</tr>
<tr>
<td>Unusual states</td>
<td>(</td>
<td>\psi(0)\rangle )</td>
<td>(</td>
<td>\psi_{0312}(E_1, t)\rangle )</td>
<td>(</td>
<td>\psi_{2130}(E_1, t)\rangle )</td>
</tr>
</tbody>
</table>
4. Conclusion

In summary, we have investigated transparent control of quantum tunneling via unusual analytical solutions for three bosons held in a driven double-well. Under the high-frequency approximation, we analytically obtain the fine Floquet quasi-energy bands and general non-Floquet state. For two different series of quasi-energy collapse points, the latter becomes the unusual special states that include the three types of stationary-like states such as the CDT single states, NOON-like states and non-NOON states, and the three kinds of nonstationary states such as the different selective CDT states. Based on the analytical results and their numerical correspondence, we clearly reveal the mechanism of coherent tunneling and suggest a scheme to transparently transport the three bosons from one well to the other by applying the selective CDT states to realize the transitions between two CDT single states. Such a scheme may be useful in the designs of quantum devices. We expect that the future experiments along the lines sketched here can be considered, which may help us further to obtain new insight into the quantum tunneling dynamics of a many-body multi-well system.

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