The optimal group continuous logarithm compatibility measure for interval multiplicative preference relations based on the COWGA operator

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\textbf{Abstract}

The calculation of compatibility measures is an important technique employed in group decision-making with interval multiplicative preference relations. In this paper, a new compatibility measure called the continuous logarithm compatibility, which considers risk attitudes in decision-making based on the continuous ordered weighted geometric averaging (COWGA) operator, is introduced. We also develop a group continuous compatibility model (GCC Model) by minimizing the group continuous logarithm compatibility measure between the synthetic interval multiplicative preference relation and the continuous characteristic preference relation. Furthermore, theoretical foundations are established for the proposed model, such as the sufficient and necessary conditions for the existence of an optimal solution, the conditions for the existence of a superior optimal solution and the conditions for the existence of redundant preference relations. In addition, we investigate certain conditions for which the optimal objective function of the GCC Model guarantees its efficiency as the number of decision-makers increases. Finally, practical illustrative examples are examined to demonstrate the model and compare it with previous methods.

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\section{Introduction}

Group decision-making (GDM) is a common task in human activities that consists of determining the most preferred alternative(s) from a given set of possibilities, as performed by a group of decision-makers (DMs). In recent decades, GDM with preference relations has received considerable attention [3,5,7,9,28]. However, because of time pressure, lack of knowledge and limited expertise, the preference relations provided by DMs are typically given in the form of numerical intervals rather than exact values. Saaty and Vargas [17] formulated the concept of interval multiplicative preference relations, which are a very important type of preference relation in the GDM process. In the literature, we find a wide range of techniques developed to address interval multiplicative preference relations [6,8,24].

The key to GDM with interval multiplicative preference relations is to know how to effectively aggregate all of the individual preference relations, which involves two procedures. The first is to determine whether all of the individual preference relations...
The objective of this paper is to develop a new compatibility measure for interval multiplicative preference relations based on the COWGA operator and to determine the optimal weights of DMs by constructing a group continuous compatibility model (GCC Model). We present a new compatibility measure called the continuous logarithm compatibility measure for interval multiplicative preference relations, in which the risk attitudes of the DMs are considered using the COWGA operator. We study several desirable properties of the compatibility measure and investigate the relationship between each DM’s synthetic interval multiplicative preference relation and the continuous characteristic preference relation.

To determine the weights of the DMs in group decision-making with interval multiplicative preference relations, we construct our GCC Model by minimizing the group continuous logarithm compatibility measure between the synthetic interval multiplicative preference relation and the continuous characteristic preference relation. Next, the sufficient and necessary for the existence of an optimal solution are studied. Further attention is also given to the conditions for the existence of a superior optimal solution and of redundant preference relations in the GCC Model. Moreover, we investigate several properties of the GCC Model by adding additional DMs into the GDM process.

The remainder of the paper is organized as follows. In Section 2, we briefly describe several preliminaries. Section 3 presents the new compatibility measure for interval multiplicative preference relations, and we develop the GCC Model to determine the DMs’ weights and investigate certain properties of the GCC Model. Section 4 provides an illustrative example, and in Section 5, we conclude the paper by summarizing the main conclusions.
2. Preliminaries

This section is devoted to describing multiplicative preference relations, interval multiplicative preference relations, the OWA operator, the COWGA operator and the expected multiplicative preference relation.

2.1. Multiplicative preference relations

Multiplicative preference relations, first introduced by Saaty [14], are the most widely used type of preference relation. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set of alternatives. Then, a multiplicative preference relation [14] can be defined as follows.

**Definition 1.** Let \( A = (a_{ij})_{n \times n} \) be a matrix. If

\[
 a_{ij}a_{ji} = 1, \quad a_{ii} = 1, \quad a_{ij} > 0, \quad \forall i, j = 1, 2, \ldots, n, \tag{1}
\]

then \( A \) is called a multiplicative preference relation on set \( X \), where \( a_{ij} \) denotes the preference degree of alternative \( x_i \) over \( x_j \).

Specifically, \( a_{ij} = 1 \) indicates indifference between \( x_i \) and \( x_j \), \( a_{ij} > 1 \) indicates that \( x_i \) is preferred over \( x_j \), and \( a_{ij} < 1 \) indicates that \( x_j \) is preferred over \( x_i \).

For a multiplicative preference relation \( A = (a_{ij})_{n \times n} \), consistency is a vital basis for GDM. If \( a_{ij} = a_{ik}a_{kj}, \forall i, j, k = 1, 2, \ldots, n \), then \( A \) is called a consistent multiplicative preference relation, and such a multiplicative preference relation is given by

\[
 a_{ij} = w_i/w_j, \quad \forall i, j = 1, 2, \ldots, n, \tag{2}
\]

where \( w = (w_1, w_2, \ldots, w_n)^T \) is the vector of priority weights of \( A \); \( w_i \) reflects the importance degree of alternative \( x_i \) and satisfies \( w_i > 0 \) for \( i = 1, 2, \ldots, n \) and \( \sum_{i=1}^{n} w_i = 1 \).

2.2. Interval multiplicative preference relations

**Definition 2.** Let \( \ddot{a} = [\ddot{a}^L, \ddot{a}^U] = \{x|0 \leq \ddot{a}^L \leq x \leq \ddot{a}^U\} \). Then, \( \ddot{a} \) is called a nonnegative interval number [18], where \( \ddot{a}^L \) and \( \ddot{a}^U \) are the lower and upper limits of \( \ddot{a} \). If \( \ddot{a}^L = \ddot{a}^U \), \( \ddot{a} \) is a nonnegative real number.

Let \( \ddot{a} = [\ddot{a}^L, \ddot{a}^U] \) and \( \ddot{b} = [\ddot{b}^L, \ddot{b}^U] \) be two nonnegative interval numbers. If \( \mu \geq 0 \), then we have the following operational laws [18]: (1) \( \ddot{a} = \ddot{b} \) if \( \ddot{a}^L = \ddot{b}^L \) and \( \ddot{a}^U = \ddot{b}^U \); (2) \( \ddot{a} + \ddot{b} = [\ddot{a}^L + \ddot{b}^L, \ddot{a}^U + \ddot{b}^U] \); (3) \( \mu\ddot{a} = [\mu\ddot{a}^L, \mu\ddot{a}^U] \); in particular, \( \mu \ddot{a} = 0 \) if \( \mu = 0 \); (4) \( \ddot{a}\ddot{b} = [\ddot{a}^L\ddot{b}^L, \ddot{a}^U\ddot{b}^U] \).

Interval multiplicative preference relations [17] can be used to describe DMs’ preferences regarding objects in GDM in the presence of uncertainty and are defined as follows:

**Definition 3.** An interval multiplicative preference relation is defined as \( \dddot{A} = (\dddot{a}_{ij})_{n \times n} \), which satisfies

\[
 \dddot{a}_{ij} = [\dddot{a}_{ij}^L, \dddot{a}_{ij}^U], \quad \dddot{a}_{ij}^U \geq \dddot{a}_{ij}^L > 0, \quad \dddot{a}_{ij}^L \times \dddot{a}_{ji}^U = \dddot{a}_{ij}^L \times \dddot{a}_{ji}^U = 1, \quad \dddot{a}_{ij}^U = \dddot{a}_{ij}^U = 1 \tag{3}
\]

for all \( i, j = 1, 2, \ldots, n \), where \( \dddot{a}_{ij} \) indicates the interval-valued preference degree of the \( i \)th alternative over the \( j \)th alternative and \( \dddot{a}_{ij}^L \) and \( \dddot{a}_{ij}^U \) are the lower and upper limits of \( a_{ij} \), respectively.

Let \( \dddot{A} = (\dddot{a}_{ij})_{n \times n} \) be an interval multiplicative preference relation. Similar to the case of multiplicative preference relations, if \( \dddot{a}_{ij} = \dddot{a}_{ik}\dddot{a}_{kj}, \forall i, j, k = 1, 2, \ldots, n \), then \( \dddot{A} \) is called a consistent interval multiplicative preference relation [33].

Note that for convenience, throughout this paper, we let \( M_n \) be the set of all \( n \times n \) interval multiplicative preference relations.

2.3. The COWGA operator

The OWA operator [30] is an aggregation function that provides a parameterized family of aggregation operators between the minimum and maximum. The properties, applications and extensions of the OWA operator can be found in [10–12,30].

The COWGA operator was developed by Yager and Xu [32] as a combination of the COWA operator with the geometric mean. It is defined as follows:

**Definition 4.** A COWGA operator is a mapping \( g : \Sigma^+ \to R^+ \) associated with a basic unit interval monotonic (BUM) function \( Q \) such that

\[
 g_Q(\dddot{a}) = g_Q([\dddot{a}^L, \dddot{a}^U]) = \dddot{a}^U \left( \frac{\dddot{a}^U}{\dddot{a}^L} \right) \int_{\dddot{a}^L}^{\dddot{a}^U} \frac{dy}{Q(y)} \quad \text{.} \tag{4}
\]

where \( \Sigma^+ \) is the set of all closed intervals whose lower limits are positive and \( R^+ \) is the set of positive real numbers. The BUM function [31] \( Q \) is a function \([0, 1] \to [0, 1]\) such that \( Q(0) = 0, Q(1) = 1 \), and if \( x > y \), then \( Q(x) \geq Q(y) \).

If \( \lambda = \int_{\dddot{a}^L}^{\dddot{a}^U} Q(y) dy \) is the attitudinal character of \( Q \), then a general formulation of \( g_Q(\dddot{a}) \) can be obtained as follows:

\[
 g_\lambda(\dddot{a}) = g_Q([\dddot{a}^L, \dddot{a}^U]) = (\dddot{a}^U)^\lambda(\dddot{a}^L)^{1-\lambda} \quad \text{.} \tag{5}
\]

As seen, the COWGA operator \( g_\lambda(\dddot{a}) \) is a linear convex exponential combination of the end points \( \dddot{a}^U \) and \( \dddot{a}^L \) based on the attitudinal character. In particular, if \( \dddot{a} \) is a real number, then \( g_\lambda(\dddot{a}) = \dddot{a} \).
2.4. The expected multiplicative preference relation

Using Eq. (4), Yager and Xu defined an expected preference relation based on the attitudinal character as follows [32].

**Definition 5.** Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ be an interval multiplicative preference relation, where $\tilde{a}_{ij} = [\hat{a}_{ij}^L, \hat{a}_{ij}^U]$, $\hat{a}_{ij}^L \geq \hat{a}_{ij}^U > 0$, $\hat{a}_{ij}^L \times \hat{a}_{ij}^U = \hat{a}_{ij}^L \times \hat{a}_{ij}^U = 1$, and $\hat{a}_{ij}^L = \hat{a}_{ij}^U = 1$, with $i, j = 1, 2, \ldots, n$.

Then, we call $g_\lambda(\tilde{A}) = (g_\lambda(\tilde{a}_{ij}))_{n \times n}$ the expected multiplicative preference relation corresponding to $\tilde{A}$, where $g_\lambda(\tilde{a}_{ij})$ is the expected value of the preference degree $\tilde{a}_{ij}$ determined by the COWGA operator:

$$g_\lambda(\tilde{a}_{ij}) = g_\lambda\left([\hat{a}_{ij}^L, \hat{a}_{ij}^U]\right) = \hat{a}_{ij}^U \cdot \left(\frac{\hat{a}_{ij}^L}{\hat{a}_{ij}^U}\right) \int_0^1 (\partial Q(y)/\partial y)dy, \quad g_\lambda(\tilde{a}_{ij}) = 1/g_\lambda(\tilde{a}_{ij}), \quad \text{for } i \leq j.$$  \hspace{1cm} (6)

where $\lambda = \int_0^1 Q(y)dy$ is the attitudinal character of $Q$ and $Q$ is a BUM function. Obviously, from Eq. (6), we have

$$g_\lambda(\tilde{a}_{ij}) > 0, \quad g_\lambda(\tilde{a}_{ij}) \times g_\lambda(\tilde{a}_{ij}) = 1, \quad g_\lambda(\tilde{a}_{ij}) = 1, \quad i, j = 1, 2, \ldots, n.$$  \hspace{1cm} (7)

which means that $g_\lambda(\tilde{A})$ is a multiplicative preference relation.

3. The continuous logarithm compatibility measure for interval multiplicative preference relations based on the COWGA operator

In this section, we introduce the continuous logarithm compatibility measure based on the COWGA operator. Then, we develop the GCC Model to determine the weights of the DMs. In addition, the sufficient and necessary conditions for the existence of an optimal solution as well as the conditions for the existence of a superior optimal solution and of redundant preference relations in the GCC Model are discussed based on reference [20]. Moreover, several properties of the GCC Model are investigated by adding additional DMs into the GDM process.

3.1. The continuous logarithm compatibility measure for interval multiplicative preference relations

**Definition 6.** Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n} \in M_n$ and $\tilde{B} = (\tilde{b}_{ij})_{n \times n} \in M_n$ be two interval multiplicative preference relations. Then,

$$C - LC(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \log^2(\tilde{a}_{ij}/\tilde{b}_{ij})$$  \hspace{1cm} (8)

is called the continuous logarithm compatibility degree of $\tilde{A}$ and $\tilde{B}$, where $\tilde{A} = (\tilde{a}_{ij})_{n \times n} = (g_\lambda(\tilde{a}_{ij}))_{n \times n}$ and $\tilde{B} = (\tilde{b}_{ij})_{n \times n} = (g_\lambda(\tilde{b}_{ij}))_{n \times n}$ are the expected multiplicative preference relations corresponding to $\tilde{A}$ and $\tilde{B}$, respectively.

From **Definition 5**, it follows that

$$C - LC(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \log^2(\tilde{a}_{ij}/\tilde{b}_{ij}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\log \tilde{a}_{ij} - \log \tilde{b}_{ij}\right)^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\log g_\lambda(\tilde{a}_{ij}) - \log g_\lambda(\tilde{b}_{ij})\right)^2.$$  \hspace{1cm} (9)

As seen, the continuous logarithm compatibility degree of $\tilde{A}$ and $\tilde{B}$ reflects the total logarithmic difference between the interval multiplicative preference relations $\tilde{A}$ and $\tilde{B}$, which considers the risk attitude of a DM based on the COWGA operator. A DM can choose a different value of the parameter $\lambda$, depending on his/her risk attitude, which makes the continuous logarithm compatibility degree more flexible than the conventional measure.

From Eq. (9), the following theorem is evident.

**Theorem 1.** Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n} \in M_n$, $\tilde{B} = (\tilde{b}_{ij})_{n \times n} \in M_n$ and $\tilde{C} = (\tilde{c}_{ij})_{n \times n} \in M_n$ be three interval multiplicative preference relations. Then,

1. $C - LC(\tilde{A}, \tilde{B}) \geq 0$;
2. $C - LC(\tilde{A}, \tilde{A}) = 0$;
3. $C - LC(\tilde{A}, \tilde{B}) = C - LC(\tilde{B}, \tilde{A})$; and
4. if $C - LC(\tilde{A}, \tilde{B}) = 0$ and $C - LC(\tilde{B}, \tilde{C}) = 0$, then $C - LC(\tilde{A}, \tilde{C}) = 0$. 
Theorem 3. Let \( \tilde{A} = (\tilde{a}_{ij})_{n \times n} \in M_n \) and \( \tilde{B} = (\tilde{b}_{ij})_{n \times n} \in M_n \) be two interval multiplicative preference relations. If \( C - LC(\tilde{A}, \tilde{B}) = 0 \) for all \( \lambda \), then \( \tilde{A} \) and \( \tilde{B} \) are perfectly continuously compatible.

Theorem 2. Let \( \tilde{A} = (\tilde{a}_{ij})_{n \times n} \in M_n \) and \( \tilde{B} = (\tilde{b}_{ij})_{n \times n} \in M_n \) be two interval multiplicative preference relations. Then, \( \tilde{A} \) and \( \tilde{B} \) are perfectly continuously compatible if and only if \( \tilde{a}_{ij} = \tilde{b}_{ij} \) for all \( i = 1, 2, \ldots, n \).

Definition 8. Let \( \tilde{A} = (\tilde{a}_{ij})_{n \times n} \in M_n \) and \( \tilde{B} = (\tilde{b}_{ij})_{n \times n} \in M_n \) be two interval multiplicative preference relations. Then,

\[
C - LCI(\tilde{A}, \tilde{B}) = \frac{1}{n^2} C - LC(\tilde{A}, \tilde{B})
\]

is called the continuous logarithm compatibility measure of \( \tilde{A} \) and \( \tilde{B} \).

Based on Theorems 1 and 2, the following results are apparent.

Theorem 3. Let \( \tilde{A} = (\tilde{a}_{ij})_{n \times n} \in M_n \) and \( \tilde{B} = (\tilde{b}_{ij})_{n \times n} \in M_n \) be two interval multiplicative preference relations. Then,

1. \( C - LCI(\tilde{A}, \tilde{B}) \geq 0 \) and
2. \( C - LCI(\tilde{A}, \tilde{B}) = 0 \) if and only if \( \tilde{A} \) and \( \tilde{B} \) are perfectly continuously compatible.

Definition 9. Let \( \tilde{A} = (\tilde{a}_{ij})_{n \times n} \in M_n \) and \( \tilde{B} = (\tilde{b}_{ij})_{n \times n} \in M_n \) be two interval multiplicative preference relations. If

\[
C - LCI(\tilde{A}, \tilde{B}) \leq \alpha
\]

for all \( \lambda \), then \( \tilde{A} \) and \( \tilde{B} \) exhibit acceptable continuous compatibility, where \( \alpha \) is the threshold of acceptable continuous compatibility.

In general, if we take 0.2 as the threshold of acceptable compatibility, as in [2], then we obtain \( \alpha = e^{\sqrt{0.2}} = 1.564 \).

Definition 10. Let \( \tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n \times n} \in M_n \) for \( k = 1, 2, \ldots, m \). Then,

\[
\tilde{A}^{(k)} = \prod_{k=1}^{m} (\tilde{a}_{ij}^{(k)})^{v_k}
\]

then \( \tilde{A} = (\tilde{a}_{ij})_{n \times n} \) is called the synthetic preference relation of \( \tilde{A}^{(k)} \), and \( v = (v_1, v_2, \ldots, v_m)^T \) is the weighting vector of \( \tilde{A}^{(k)} \), which satisfies \( v_k \geq 0 \) for \( k = 1, 2, \ldots, m \) and \( \sum_{k=1}^{m} v_k = 1 \).

Theorem 4. Let \( \tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n \times n} \in M_n \) and \( \tilde{B}^{(k)} = (\tilde{b}_{ij}^{(k)})_{n \times n} \in M_n \) for \( k = 1, 2, \ldots, m \), and let \( \tilde{A} = (\tilde{a}_{ij})_{n \times n} \) and \( \tilde{B} = (\tilde{b}_{ij})_{n \times n} \) be the synthetic preference relations of \( \tilde{A}^{(k)} \) and \( \tilde{B}^{(k)} \), respectively. Assume that \( v = (v_1, v_2, \ldots, v_m)^T \) is the weighting vector of \( \tilde{A}^{(k)} \) and \( \tilde{B}^{(k)} \), which satisfies \( v_k \geq 0 \) for \( k = 1, 2, \ldots, m \) and \( \sum_{k=1}^{m} v_k = 1 \). If \( C - LCI(\tilde{A}^{(k)}, \tilde{B}^{(k)}) \leq \alpha \) for \( k = 1, 2, \ldots, m \), then

\[
C - LCI(\tilde{A}, \tilde{B}) \leq \alpha.
\]

Proof. Assume that \( \tilde{a}_{ij} = \prod_{k=1}^{m} (\tilde{a}_{ij}^{(k)})^{v_k} \) and \( \tilde{b}_{ij} = \prod_{k=1}^{m} (\tilde{b}_{ij}^{(k)})^{v_k} \). Then,

\[
g_\lambda(\tilde{a}_{ij}) = (\tilde{a}_{ij})^\lambda (\tilde{a}_{ij})^{1-\lambda} = \left( \prod_{k=1}^{m} (\tilde{a}_{ij}^{(k)})^{v_k} \right)^\lambda \left( \prod_{k=1}^{m} (\tilde{a}_{ij}^{(k)})^{v_k} \right)^{1-\lambda} = \prod_{k=1}^{m} (\tilde{a}_{ij}^{(k)})^{\lambda v_k} (\tilde{a}_{ij}^{(k)})^{(1-\lambda v_k)} = \prod_{k=1}^{m} (g_\lambda(\tilde{a}_{ij}^{(k)}))^{v_k}
\]

and

\[
g_\lambda(\tilde{b}_{ij}) = \prod_{k=1}^{m} (g_\lambda(\tilde{b}_{ij}^{(k)}))^{v_k}.
\]

It follows that

\[
C - LCI(\tilde{A}, \tilde{B}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\log g_\lambda(\tilde{a}_{ij}) - \log g_\lambda(\tilde{b}_{ij}))^2
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \log \left( \prod_{k=1}^{m} (g_\lambda(\tilde{a}_{ij}^{(k)}))^{v_k} \right) - \log \left( \prod_{k=1}^{m} (g_\lambda(\tilde{b}_{ij}^{(k)}))^{v_k} \right) \right)^2
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sum_{k=1}^{m} v_k \log g_\lambda(\tilde{a}_{ij}^{(k)}) - \sum_{k=1}^{m} v_k \log g_\lambda(\tilde{b}_{ij}^{(k)}) \right)^2
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sum_{k=1}^{m} v_k (\log g_\lambda(\tilde{a}_{ij}^{(k)}) - \log g_\lambda(\tilde{b}_{ij}^{(k)})) \right)^2
\]
Theorem 4 indicates that if \( \hat{A}(k) \) and \( \hat{B}(k) \) are acceptably compatible for all \( k \), then the synthetic preference relations \( \hat{A} \) and \( \hat{B} \) are also acceptably compatible. Moreover, based on the proof of Theorem 4, we obtain

\[
g_{\hat{A}}(\hat{a}_{ij}) = \prod_{k=1}^{m} (g_{\hat{A}}(\hat{a}_{ik}))^{v_k}, \quad \forall i, j.
\]  

(14)

which means that the expected multiplicative preference relation \( g_{\hat{A}} = (g_{\hat{A}}(\hat{a}_{ij}))_{n \times n} \) corresponding to the synthetic preference relation \( \hat{A} \) can be regarded as the weighted geometric average of the expected multiplicative preference relation \( g_{\hat{A}}(\hat{A}(k)) = (g_{\hat{A}}(\hat{a}_{ik}))_{n \times n} \) corresponding to \( \hat{A}(k) \).

In particular, if \( \alpha = 0 \), then we obtain Corollary 1 as follows.

**Corollary 1.** Let \( \hat{A}(k) = (\hat{a}_{ik})_{n \times n} \in M_n \), \( \hat{B}(k) = (\hat{b}_{ik})_{n \times n} \in M_n \), \( \hat{A} = (\hat{a}_{ij})_{n \times n} \) and \( \hat{B} = (\hat{b}_{ij})_{n \times n} \) be as defined above. If \( C - \text{LCI}(\hat{A}, \hat{B}) = 0 \) for \( k = 1, 2, \ldots, m \), then

\[
C - \text{LCI}(\hat{A}, \hat{B}) = 0.
\]

**Definition 11.** Let \( A = (a_{ij})_{n \times n} \) be a multiplicative preference relation, and let \( w = (w_1, w_2, \ldots, w_n)^T \) be the priority vector of \( A \), which satisfies \( w_i > 0 \) for \( i = 1, 2, \ldots, n \) and \( \sum_{i=1}^{n} w_i = 1 \).

\[
\hat{w}_{ij} = w_i / w_j, \quad \forall i, j = 1, 2, \ldots, n,
\]

then \( \hat{W} = (\hat{w}_{ij})_{n \times n} \) is called the characteristic preference relation of \( A \).

**Definition 12.** Let \( \hat{A} = (\hat{a}_{ij})_{n \times n} \in M_n \) be an interval multiplicative preference relation, and let \( \hat{A} = (\hat{a}_{ij})_{n \times n} = (g_{\hat{A}}(\hat{a}_{ij}))_{n \times n} \) be the expected multiplicative preference relation corresponding to \( \hat{A} \). Assume that \( \hat{w} = (\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_n)^T \) is the priority vector of \( \hat{A} \), which satisfies \( \hat{w}_i > 0 \) for \( i = 1, 2, \ldots, n \) and \( \sum_{i=1}^{n} \hat{w}_i = 1 \).

\[
\hat{\hat{w}}_{ij} = \hat{w}_i / \hat{w}_j, \quad \forall i, j = 1, 2, \ldots, n,
\]

then \( \hat{\hat{W}} = (\hat{\hat{w}}_{ij})_{n \times n} \) is called the continuous characteristic preference relation of \( \hat{A} \).

**Definition 13.** Let \( \hat{A} = (\hat{a}_{ij})_{n \times n} \in M_n \) and \( \hat{A} = (\hat{a}_{ij})_{n \times n} = (g_{\hat{A}}(\hat{a}_{ij}))_{n \times n} \) be as defined above. If \( \hat{A} \) is consistent for all \( \lambda \), then \( \hat{A} \) is called a continuous consistent interval multiplicative preference relation.

**Theorem 5.** Let \( \hat{A} = (\hat{a}_{ij})_{n \times n} \in M_n \) and \( \hat{W} = (\hat{w}_{ij})_{n \times n} = (\hat{w}_{ij} / \hat{w}_j)_{n \times n} \) be as defined above. Then, \( \hat{A} \) and \( \hat{W} \) are perfectly continuously compatible if and only if \( \hat{A} \) is a consistent interval multiplicative preference relation.

**Proof.** If \( \hat{A} \) and \( \hat{W} \) are perfectly continuously compatible, then for any \( \lambda \), \( g_{\hat{A}}(\hat{a}_{ij}) = \hat{w}_{ij} / \hat{w}_j \) \( \forall i, j \). In particular, if \( \lambda = 1 \), then

\[
\hat{a}_{ij}^{(\lambda)} = \hat{w}_{ij} / \hat{w}_j \quad \text{for all } i, j.
\]

Similarly, we can obtain \( \hat{a}_{ij}^{(\lambda)} = \hat{a}_{ik}^{(\lambda)} / \hat{a}_{kj}^{(\lambda)} \). Therefore, \( \hat{A} \) is consistent.

By contrast, if \( \hat{A} \) is a consistent interval multiplicative preference relation, then for all \( i, j, k \), we find that \( \hat{a}_{ij} = \hat{a}_{ik} / \hat{a}_{kj} \), which means that \( \hat{a}_{ij} = \hat{a}_{ik}^{(\lambda)} / \hat{a}_{kj}^{(\lambda)} \) and \( \hat{a}_{ij} = \hat{a}_{ik}^{(\lambda)} / \hat{a}_{kj}^{(\lambda)} \). Thus, \( g_{\hat{A}}(\hat{a}_{ij}) = \hat{a}_{ij}^{(\lambda)} / \hat{a}_{ij}^{(\lambda)} / \hat{a}_{ij}^{(\lambda)} = \hat{w}_{ij} / \hat{w}_j \).

**Theorem 6.** Let \( \hat{A}(k) = (\hat{a}_{ik})_{n \times n} \in M_n \), and let \( \hat{\hat{A}} = (\hat{a}_{ij})_{n \times n} \) be as defined above. If \( \hat{\hat{A}}(k) \) is consistent for all \( k \), then \( \hat{\hat{A}} \) is continuously consistent.

**Definition 14.** Let \( \hat{A}(k) = (\hat{a}_{ik})_{n \times n} \in M_n \), and let \( \hat{\hat{W}}(k) = (\hat{w}_{ij}(k))_{n \times n} = (\hat{w}_{ij} / \hat{w}_j(k))_{n \times n} \) be the continuous characteristic preference relation of \( \hat{A}(k) \) for \( k = 1, 2, \ldots, m \), where \( \hat{\hat{w}}(k) = (\hat{w}_1(k), \hat{w}_2(k), \ldots, \hat{w}_n(k))^T \) is the priority vector of \( \hat{A}(k) = (\hat{a}_{ij}(k))_{n \times n} = (g_{\hat{A}}(\hat{a}_{ij}(k)))_{n \times n} \), which satisfies \( \hat{w}_{ij}(k) > 0 \) for \( i = 1, 2, \ldots, n \) and \( \sum_{i=1}^{n} \hat{w}_{ij}(k) = 1 \). Then, \( \hat{W} = (\hat{w}_{ij})_{n \times n} \) such that \( \hat{w}_{ij} = \prod_{k=1}^{m} (\hat{w}_{ij}(k))^{v_k} \) for \( i, j = 1, 2, \ldots, n \), then \( \hat{W} \) is called the continuous synthetic characteristic preference relation of \( \hat{A}(k) \).

Based on Theorems 5 and 6, we can obtain Theorem 7 as follows.

**Theorem 7.** Let \( \hat{A}(k) = (\hat{a}_{ik}(k))_{n \times n}, \hat{A} = (\hat{a}_{ij})_{n \times n}, \) and \( \hat{W} = (\hat{w}_{ij})_{n \times n} \) be as defined above. If \( \hat{A}(k) \) is consistent for all \( k \), then \( \hat{A} \) and \( \hat{W} \) are perfectly continuously compatible.
**Theorem 7** indicates that if \( \tilde{A}^{(k)} \) is consistent, then the consistency of the synthetic preference relation \( \hat{A} \) and the compatibility of \( \hat{A} \) with the continuous characteristic preference relation \( \hat{W} \) are of the same nature.

### 3.2. The optimal group continuous logarithm compatibility measure for interval multiplicative preference relations

Let \( D = \{d_1, d_2, \ldots, d_m\} \) be a finite set of DMs, and let \( \tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n \times n} \in M_n \) be the interval multiplicative preference relation provided by expert \( d_k \), where \( v = (v_1, v_2, \ldots, v_m)^T \) is the weighting vector for DM \( d_k \), which satisfies \( v_k \geq 0 \) for \( k = 1, 2, \ldots, m \) and \( \sum_{k=1}^{m} v_k = 1 \). Assume that \( \hat{W}^{(k)} = (\hat{w}_{ij}^{(k)})_{n \times n} = (\tilde{w}_{ij}^{(k)} / \tilde{w}_{ij}^{(k)})_{n \times n} \) is the continuous characteristic preference relation of \( \tilde{A}^{(k)}, k = 1, 2, \ldots, m \), where \( \tilde{w}_{ij}^{(k)} = (\tilde{w}_{ij}^{(k)})_1, \ldots, (\tilde{w}_{ij}^{(k)})_n \) is the priority vector of \( \tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n \times n} = (\tilde{g}_i(\tilde{a}_{ij}^{(k)}))_{n \times n} \), which satisfies \( \tilde{w}_{ij}^{(k)} > 0 \) for \( i = 1, 2, \ldots, n \) and \( \sum_{i=1}^{n} \hat{w}_{ij}^{(k)} = 1 \). Let \( \tilde{A} = (\tilde{a}_{ij})_{n \times n} \) be the synthetic preference relation of \( \tilde{A}^{(k)} \), and let \( \hat{W} = (\hat{w}_{ij})_{n \times n} \) be the continuous characteristic preference relation of \( \tilde{A}^{(k)} \), where \( \hat{w}_{ij} = \prod_{k=1}^{n} (\hat{w}_{ij}^{(k)})^{v_k} \) for \( i, j = 1, 2, \ldots, n \).

Then, we call

\[
C - LCI(\tilde{A}^{(k)}, \hat{W}^{(k)}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\log g_a(\tilde{a}_{ij}^{(k)}) - \log g_a(\hat{w}_{ij}^{(k)}))^2
\]  

the individual continuous logarithm compatibility measure of DM \( d_k \), and we call

\[
C - LCI(\tilde{A}, \hat{W}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\log g_a(\tilde{a}_{ij}) - \log g_a(\hat{w}_{ij}))^2
\]

the group continuous logarithm compatibility measure.

According to **Theorem 7**, when there is less compatibility between \( \tilde{A} \) and \( \hat{W} \), there is more consistency of \( \hat{A} \) as given by the DMs. Therefore, the weights of the DMs may depend on the continuous compatibility measure of \( \tilde{A} \) and \( \hat{W} \). To determine the weights of the DMs in a GDM problem with interval multiplicative preference relations, we can minimize the group continuous logarithm compatibility measure of \( \tilde{A} \) and \( \hat{W} \). From **Definition 8**, it follows that

\[
C - LCI(\tilde{A}, \hat{W}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\log g_a(\tilde{a}_{ij}) - \log g_a(\hat{w}_{ij}))^2
\]

Let \( E_m = (e_{k_1k_2})_{m \times m} \), where

\[
e_{k_1k_2} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\log g_a(\tilde{a}_{ij}) - \log (\hat{w}_{ij}^{(k_1)})) (\log g_a(\tilde{a}_{ij}) - \log (\hat{w}_{ij}^{(k_2)})) + (\log (\hat{w}_{ij}^{(k_1)})) (\log g_a(\tilde{a}_{ij}) - \log (\hat{w}_{ij}^{(k_2)})) k_1, k_2
\]

\[
e_{k_1k_2} = 1, 2, \ldots, m.
\]
Then, $E_m$ is called the continuous logarithm compatibility information matrix. In particular, if $k_1 - k_2 = k$, then $e_{k_1,k_2}$ reduces to the continuous logarithm compatibility measure of DM $d_k$. Therefore, we can obtain the optimal model of the continuous compatibility measure for interval multiplicative preference relations as follows:

$$
\min C - L\text{CI}(\hat{A}, \hat{W}) = v^T E_m v
$$

s.t. \( R_m^T v = 1 \)

$$
\begin{align*}
\nu & \geq 0
\end{align*}
$$

where $R_m = (1, 1, \ldots, 1)_m$. For simplicity, we call model (20) the Group Continuous Compatibility (GCC) Model, and we let $C_m(v) = C - L(\hat{A}_{m,n}, \hat{W}_{m,n})$. If we do not consider the constraint that $\nu \geq 0$ in Eq. (20), then we obtain Eq. (21) as follows:

$$
\min C_m(v) = v^T E_m v
$$

s.t. $R_m^T v = 1$.

**Theorem 8.** If not all of the interval multiplicative preference relations $\hat{A}^{(k)} = (\hat{a}_{ij}^{(k)})_{n \times n}$ can satisfy the requirement of consistency, then an unique optimal solution to Eq. (21) exists:

$$
\nu^* = \frac{E_m^{-1} R_m}{R_m^T E_m^{-1} R_m}.
$$

**Proof.** From Eq. (19), it follows that $E_m$ is a symmetrical matrix. If not all $\hat{A}^{(k)} = (\hat{a}_{ij}^{(k)})_{n \times n}$ are consistent, then by Eq. (2), for any $k$, there exist some $i_0, j_0 \in \{1, 2, \ldots, n\}$ such that $i_0 \neq j_0$ that satisfy $g_{i,j}(\hat{a}_{i_0j_0}) = \hat{w}_{i_0j_0}$, i.e., $(\log (g_{i,j}(\hat{a}_{i_0j_0})) - \log (\hat{w}_{i_0j_0}))^2 > 0$. Therefore,

$$
C_m(v) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\log g_{i,j}(\hat{a}_{ij}) - \log g_{i,j}(\hat{w}_{ij}))^2 > 0,
$$

which means that $E_m$ is a positive-definite and invertible matrix and that $E_m^{-1}$ is also a positive-definite matrix. Based on Eq. (21), we can construct the Lagrange function as follows:

$$
\begin{align*}
L(v, \theta) &= v^T E_m v + \theta(R_m^T v - 1),
\end{align*}
$$

where $\theta$ is the Lagrange multiplier. Based on the necessary conditions for the existence of an extremum, by setting the partial derivatives with respect to $\nu$ and $\theta$ equal to zero, we can obtain the following formulae:

$$
\begin{align*}
2E_m v + \theta R_m &= 0 \\
R_m^T v - 1 &= 0.
\end{align*}
$$

By solving Eq. (24), we find that $\nu^* = E_m^{-1} R_m / R_m^T E_m^{-1} R_m$ and $\theta^* = -2 / R_m^T E_m^{-1} R_m$. Moreover, if $\partial^2 L(v, \theta) / \partial \nu^2 = E_m$ is a positive-definite matrix, then $L(v, \theta)$ is a strictly convex function. Thus, by the sufficient conditions for the existence of an extremum, $\nu^*$ is the unique optimal solution to Eq. (21).

**Theorem 9.** If $\nu^*$, the optimal solution to Eq. (21), is an interior point or boundary point of the feasible region of Eq. (20), then $\nu^*$ is also the optimal solution to Eq. (20).

**Proof.** Let $\hat{v}$ be the optimal solution to Eq. (20). Assume that $D_1$ and $D_2$ are the feasible regions of models (20) and (21), respectively. Then, $D_1 \subset D_2$. We will complete the proof as a proof of contradiction. Let $\hat{v} \neq \nu^*$; then, $\forall \nu \in D_1, \nu \neq \hat{v}$, we have $L(\hat{v}, \theta) \leq L(\nu, \theta)$. Because $\nu^*$ is an interior point or boundary point of $D_1$, $\nu^* \in D_1$. It follows that

$$
L(\hat{v}, \theta) \leq L(\nu^*, \theta).
$$

Additionally, $\hat{v} \in D_1 \subset D_2$, which means that $\hat{v}$ is a feasible solution to Eq. (21). Thus, $L(\nu^*, \theta) < L(\hat{v}, \theta)$, which is contrary to Eq. (25).

**Theorem 9** shows that if $\nu^* = E_m^{-1} R_m / R_m^T E_m^{-1} R_m$ is nonnegative, then we can obtain the optimal solution from Eq. (22) or by solving Eq. (20). Otherwise, we can transform Eq. (20) into a linear programming problem by applying the Kuhn–Tucker condition because Eq. (20) is a quadratic programming problem.

Furthermore, if $\nu^* = E_m^{-1} R_m / R_m^T E_m^{-1} R_m$ is the optimal solution to the GCC Model, then the objective function can be expressed using the following formula:

$$
C_m(\nu^*) = \nu^T E_m v^* = 1 / R_m^T E_m^{-1} R_m,
$$

which is called the group optimal continuous logarithm compatibility.
3.3. The effectiveness of the GCC model

In a GDM problem, if all DMs have the same importance, then the weighting vector of the DMs is \( v_A = (1/m, 1/m, \ldots, 1/m)^T \), which means that the synthetic preference relation \( \hat{A} \) and the continuous synthetic characteristic preference relation \( \hat{W} \) are the simple geometric averages of \( \hat{A}^{(k)} \) and \( \hat{W}^{(k)} \) respectively. Generally, \( \nu_A \) is not the optimal solution to the GCC Model, but it can be converted into the optimal solution to the GCC Model under certain conditions.

**Theorem 10.** Let \( E_m \) be the continuous logarithm compatibility information matrix. Then, \( \nu_A = (1/m, 1/m, \ldots, 1/m)^T \) is the optimal solution to the GCC Model if and only if the sum of all elements in each line of \( E_m \) is constant, i.e.,

\[
r_i = \sum_{j=1}^{m} e_{ij} = r, \quad i = 1, 2, \ldots, m.
\]

**Proof.** Sufficiency. Eq. (27) can be expressed in matrix form as follows: \( E_m R_m = R_m r \), that is, \( E_m^{-1} R_m = R_m r \). Therefore,

\[
\nu^* = \frac{E_m^{-1} R_m}{R_m^T E_m^{-1} R_m} = \frac{R_m r}{R_m^T r} = \frac{R_m^T r}{m} = v_A.
\]

**Necessity.** Assume that \( \nu_A = (1/m, 1/m, \ldots, 1/m)^T \) is the optimal solution to the GCC Model. Then, \( \nu_A = R_m^T / R_m \), which means that

\[
E_m R_m = m R_m^T E_m^{-1} R_m.
\]

Let \( r = m / R_m^T E_m^{-1} R_m \); then, we have \( E_m R_m = r R_m \). It follows that \( \sum_{j=1}^{m} e_{ij} = r \) for all \( i \).

**Corollary 2.** Let \( E_m = (e_{ij})_{m \times m} \) be the continuous logarithm compatibility information matrix and let \( \nu_A = (1/m, 1/m, \ldots, 1/m)^T \). If \( \sum_{j=1}^{m} e_{ij} = r \) for all \( i \), then \( C_m(\nu_A) = r / m \).

**Proof.** \( C_m(\nu_A) = \sum_{k=1}^{m} \sum_{j=1}^{m} e_{kk} = \frac{1}{m^2} \sum_{j=1}^{m} e_{kk} = \frac{1}{m} r = \frac{r}{m} \).

**Definition 15.** Let \( C_m = \min \{ e_{kk} : k = 1, 2, \ldots, m \} \) be the minimum individual continuous logarithm compatibility measure of \( d(k = 1, 2, \ldots, m) \), and let \( C_m = \max \{ e_{kk} : k = 1, 2, \ldots, m \} \) be the maximum individual continuous logarithm compatibility measure of \( d(k = 1, 2, \ldots, m) \). Assume that \( v = (v_1, v_2, \ldots, v_m)^T \) is a feasible solution to the GCC Model. If \( C_m(\nu) = C_m \), then we call the GCC Model a non-inferior GCC Model, \( v \) a non-inferior solution to the GCC Model, and \( C_m(\nu) \) a non-inferior group continuous logarithm compatibility. If \( C_m(\nu) = C_m \), then we call the GCC Model a superior GCC Model, \( v \) a superior solution to the GCC Model, and \( C_m(\nu) \) a superior group continuous logarithm compatibility.

**Lemma 1.** Let \( C_m(\nu) \) be the objective function of the GCC Model, and let \( \nu^* = (v_1^*, v_2^*, \ldots, v_m^*)^T \) be the optimal solution to the GCC Model. Then,

\[
C_m(\nu^*) \leq e_{kk}, \quad k = 1, 2, \ldots, m,
\]

where \( e_{kk} \) is the individual continuous logarithm compatibility measure of \( d_k \).

**Proof.** For any feasible solution to the GCC Model \( \nu = (v_1, v_2, \ldots, v_m)^T \),

\[
C_m(\nu) = \nu^T E_m \nu \leq \nu^T E_m v.
\]

Assume that \( v = (0, \ldots, 0, 1, 0, \ldots, 0)^T \), where \( v_k = 1 \) and \( v_i = 0 \) for \( i \neq k \). Then, for any \( k \), it follows that

\[
C_m(\nu^*) \leq (0, \ldots, 0, 1, 0, \ldots, 0) \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1m} \\ e_{21} & e_{22} & \cdots & e_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ e_{m1} & e_{m2} & \cdots & e_{mm} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = e_{kk}.
\]

**Theorem 11.** The optimal solution to the GCC Model is at least a non-inferior solution.

**Proof.** Let \( \nu^* = (v_1^*, v_2^*, \ldots, v_m^*)^T \) be the optimal solution to the GCC Model. Then, by **Lemma 1**, \( C_m(\nu^*) \leq e_{kk} \) for \( k = 1, 2, \ldots, m \). Thus, \( C_m(\nu^*) \leq C_m \), which completes the proof of the Theorem.

**Lemma 2.** Let \( E_m = (e_{ij})_{m \times m} \) be the continuous logarithm compatibility information matrix. Then,

\[
E_m^{-1} = \left( \frac{E_m^{-1} - e_0 e_0^T E_m^{-1}}{e_0 e_0^T E_m^{-1}} \right),
\]

where \( E_m = \left( \frac{E_m e_0 e_0^T E_m}{e_0 e_0^T E_m} \right), e_0 = (e_{1m}, e_{2m}, \ldots, e_{m-1,m})^T, \) and \( p = e_{mm} - e_0^T E_m^{-1} e_0 > 0 \).
Proof. The lemma can be proved immediately through the elementary transformation of the partitioned matrix.

It is worth mentioning that if we calculate the determinant of $E_m$, then we obtain

$$|E_m| = \begin{vmatrix} E_{m-1}^{-1} & e_0 \\ e_0^T & e_{mn} \end{vmatrix} = |E_{m-1}| |e_{mm} - e_0^T E_{m-1}^{-1} e_0| = |E_{m-1}| \cdot p.$$ 

Because $|E_m| > 0$ and $|E_{m-1}| > 0$, $p > 0$.

**Theorem 12.** Let $v^* = (v_1^*, v_2^*, \ldots, v_m^*)^T$ be the optimal solution to the GCC Model, and let $C_m(v^*)$ be the optimal group continuous logarithm compatibility. If the $t$th individual continuous logarithm compatibility measure $e_{tt}$ is the minimum $e_{kl}$ for $k = 1, 2, \ldots, m$, i.e., $e_{tt} = C_{\min}$, then $C_m(v^*) = e_{tt}$ if and only if $e_{tt} = e_{2t} = \cdots = e_{mt}$.

**Proof.** 

**Necessity.** Without loss of generality, we assume that $e_{1m} = e_{2m} = \cdots = e_{mm}$. Then, $e_0 = (e_{1m}, e_{2m}, \ldots, e_{m-1,m})^T = (e_{mm}, e_{mm}, \ldots, e_{mm})^T = e_{mm} R_{m-1}$. From Lemma 2, it follows that $p = e_{mm} - e_0^T E_{m-1}^{-1} e_0 = e_{mm} - e_{mm} R_{m-1} E_{m-1}^{-1} R_{m-1}$. Thus,

$$E_m^{-1} R_m = \begin{pmatrix} E_m^{-1} + E_{m-1}^{-1} e_0 e_0^T E_{m-1}^{-1} & E_{m-1}^{-1} e_0 \\ e_0^T E_{m-1}^{-1} & 1 \end{pmatrix} \begin{pmatrix} R_{m-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0_{m-1} \\ 1/e_{mm} \end{pmatrix},$$

where $0_{m-1}$ is the $(m-1)$th zero vector. We also obtain $R_m^T E_m^{-1} R_m = 1/e_{mm}$. By Eqs. (22) and (26), we have

$$v^* = \frac{E_m^{-1} R_m}{R_m^T E_m^{-1} R_m} = \frac{(0_{m-1}, 1/e_{mm})^T}{1/e_{mm}} = \frac{0_{m-1}}{1} = \frac{1}{1} = e_{mm} = e_{mm} e_{mm}.$$

which means that the optimal group continuous logarithm compatibility is equal to the individual continuous logarithm compatibility measure of $d_m$ and that none of the other individual continuous logarithm compatibility measures of $d_k (k \neq m)$ contributes.

By Theorem 8, $E_m = (e_{ij})_{m \times m}$ is a positive-definite matrix. Then, for any $t \neq m$, $e_{tt} e_{mm} e_{mm}$ is also a positive-definite matrix. It follows that

$$e_{tt} e_{mm} e_{mm} = e_{tt} e_{mm} - e_{mm}^2 = e_{mm} (e_{tt} - e_{mm}) > 0.$$

Because $e_{mm} > 0$, $e_{tt} > e_{mm}$. Therefore, $e_{mm} = \min_t \{e_{tt}\} = C_m(v^*)$.

In general, if $e_{tt} = e_{2t} = \cdots = e_{mt}$, then we can move $e_{tt}(i = 1, 2, \ldots, m)$ from the $t$th column to the $m$th column, but the properties of matrix $E_m$ remain unchanged, and the results described above still hold.

**Necessity.** Let $e_{mm} = \min_t \{e_{tt}\} = C_m(v^*)$. Assume that $v_0 = (0, 0, \ldots, 0, 1)^T$. Then, we have $v_0^T E_m v_0 = e_{mm}$, i.e., $C_m(v^*) = v_0^T E_m v_0$, which means that $v_0$ is the optimal solution to the GCC Model. By Eq. (22), we have $v_0 = E_m^{-1} R_m / R_m^T E_m^{-1} R_m$. Then, $E_m v_0 = R_m / R_m^T E_m^{-1} R_m$. Moreover, by Eq. (26), we obtain $C_m(v^*) = 1/R_m^T E_m^{-1} R_m = e_{mm}$. It follows that $E_m v_0 = e_{mm} R_m$, i.e., $(e_{1m}, e_{2m}, \ldots, e_{mm})^T = (e_{mm}, e_{mm}, \ldots, e_{mm})^T$. Therefore, $e_{1m} = e_{2m} = \cdots = e_{mm}$.

As seen from Theorem 12, the optimal group continuous logarithm compatibility does not necessarily imply all of the individual continuous logarithm compatibility measures. In extreme cases, it is equal to the minimum of all of the individual continuous logarithm compatibility measures.

**Theorem 13.** Let $E_m = (e_{ij})_{m \times m}$ be the continuous logarithm compatibility information matrix. Then, the GCC Model is a superior GCC Model if and only if the elements in each column of $E_m$ are not fully equal.

**Proof.** This result follows immediately from Theorem 12 and Definition 15.

**Definition 16.** Let $v^* = (v_1^*, v_2^*, \ldots, v_m^*)^T$ be the optimal solution to the GCC Model. If a weighting coefficient $v_{k0}^* = 0$ exists in $v^*$, then $A_{(k0)}$ is called a redundant preference relation.

It is apparent from Definition 16 that if $A_{(k0)}$ is a redundant preference relation, then it can be deleted or reconstructed in the GDM process.
**Definition 17.** Let $\nu^* = (v_1^*, v_2^*, \ldots, v_m^*)^T$ be the optimal solution to the GCC Model. If there exists an $m_0$th weighting coefficient $v_{k_0}^* = 0$ for $i = 1, 2, \ldots, m_0$ and if $v_j^* \neq 0$ for $j \neq k_i$ in $\nu^*$, then the ratio $q = m_0/m$ is called the degree of redundancy of the GCC Model.

It is obvious that $0 \leq q \leq (m - 1)/m$. In particular, if $q = (m - 1)/m$, then there is only one effective preference relation in the GCC Model, and the remaining preference relations are all redundant.

**Theorem 14.** Let $E_m = (e_{ij})_{m \times m}$ be the continuous logarithm compatibility information matrix. If the coefficients of the $t$th column of $E_m$ satisfy $e_{1t} = e_{2t} = \cdots = e_{mt}$, then the degree of redundancy of the GCC Model is $(m - 1)/m$.

**Proof.** This theorem is immediately proven by Theorem 12 and Definition 17.

### 3.4. The efficiency of the GCC model

In a GDM problem with interval multiplicative preference relations, if a DM were to be added to the group of DMs, there would then be $m + 1$ interval multiplicative preference relations, and the continuous logarithm compatibility information matrix $E_m$ would change to

$$E_{m+1} = (e_{ij})_{(m+1) \times (m+1)} = \left(\begin{array}{c} E_m u \\ uT E_{m+1} \end{array}\right), \quad u = (e_{1,m+1}, e_{2,m+1}, \ldots, e_{m,m+1})^T.$$

However, the optimal group continuous logarithm compatibility is monotonically non-increasing, as expressed by the following theorem.

**Theorem 15.** Let $C_m$ be the optimal group continuous logarithm compatibility, and let $\nu^*$ be the optimal solution to the GCC Model. Then, $C_m$ is monotonically non-increasing with respect to the number of DMs, i.e., $C_{m+1} \leq C_m$. $C_{m+1} = C_m$ if and only if $\nu^T u = C_m$.

**Proof.** Based on Eq. (26), we find that

$$C_m = 1/R_m E_m^{-1} R_m \quad \text{and} \quad C_{m+1} = 1/R_{m+1} E_{m+1}^{-1} R_{m+1},$$

where $R_{m+1} = (1, 1, \ldots, 1)^T_{(m+1) \times 1} = (R_m^T, 1)^T$. From Lemma 2, it follows that

$$R_{m+1} E_{m+1}^{-1} R_{m+1} = (R_m^T, 1) \left( E_m^{-1} + E_m^{-1} uu^T E_m^{-1} / p' - E_m^{-1} u / p' \right) E_m^{-1} (R_m^T, 1)$$

$$= R_m^T E_m^{-1} R_m + (R_m^T E_m^{-1} u - 1)/p' \geq R_m^T E_m^{-1} R_m,$$

where $p' = e_{m+1,m+1} - u^T E_m^{-1} u > 0$. Thus, $C_{m+1} \leq C_m$. Moreover, by Eq. (29), $C_{m+1} = C_m$ if and only if $R_m^T E_m^{-1} u = 1$, which means that

$$R_m^T E_m^{-1} u / R_m^T E_m^{-1} R_m = 1 / R_m^T E_m^{-1} R_m.$$

As seen from Theorem 15, the vector $u$ completely determines whether the optimal group continuous logarithm compatibility will be reduced when a preference relation is added to the GDM problem.

**Theorem 16.** Let $C_m$ be the optimal group continuous logarithm compatibility, and let $\nu^*$ be the optimal solution to the GCC Model. The $(m + 1)$th interval multiplicative preference relation added to the GCC Model is a redundant preference relation if and only if $\nu^T u = C_m$.

**Proof.** **Sufficiency.** Assume that $\nu^{**}$ is the optimal solution to the GCC Model with $m + 1$ interval multiplicative preference relations. Then, from Theorem 8 and Lemma 2, we obtain

$$\nu^{**} = \frac{E_{m+1}^{-1} R_m}{R_{m+1}^T E_{m+1}^{-1} R_{m+1}} = \left(\begin{array}{c} E_m^{-1} + E_m^{-1} uu^T E_m^{-1} / p' - E_m^{-1} u / p' \\ -u^T E_m^{-1} / p' \end{array}\right) E_m^{-1} (R_m^T, 1)$$

$$= \frac{(E_m^{-1} R_m + E_m^{-1} u (R_m^T E_m^{-1} u - 1)/p')}{(1 - R_m^T E_m^{-1} u)/p'}.$$

Thus, by Theorem 15, $\nu^{**} u = C_m$ if and only if $R_m^T E_m^{-1} u = 1$, which can be substituted into Eq. (31). Therefore,

$$\nu^{**} = \left(\begin{array}{c} E_m^{-1} R_m \\ R_m^T E_m^{-1} R_m \end{array}\right) = \left(\begin{array}{c} \nu^* \\ 0 \end{array}\right).$$
indicating that the \((m + 1)\)th interval multiplicative preference relation added to the GCC Model is a redundant preference relation.

**Necessity.** If the \((m + 1)\)th interval multiplicative preference relation added to the GCC Model is a redundant preference relation, then we have \(v^* = (v^{*'} , 0)^T\). By Eq. (30), \((1 - R_m^T E_m^{-1} u)/p' = 0\), which means that \(R_m^T E_m^{-1} u = 1\), i.e., \(v^* u = C_m\).

**Theorem 17.** Let \(C_m\) be the optimal group continuous logarithm compatibility, and let \(v^* u\) be the optimal solution to the GCC Model. If \(v^* u = 0\), then \(C_{m+1} < C_m\).

**Proof.** If \(v^* u = 0\), then \(R_m^T E_m^{-1} u / R_m^T E_m^{-1} R_m = 0\), which means that \(R_m^T E_m^{-1} u = 0\). From Eq. (30), it follows that

\[
R_{m+1}^T E_{m+1}^{-1} R_{m+1} = R_m^T E_m^{-1} R_m + (R_m^T E_m^{-1} u - 1)^2/p' = R_m^T E_m^{-1} R_m + 1/p'.
\]

Thus, \(R_{m+1}^T E_{m+1}^{-1} R_{m+1} < R_m^T E_m^{-1} R_m\), i.e., \(C_{m+1} < C_m\).

**Theorem 18.** Let \(C_m\) be the optimal group continuous logarithm compatibility, and let \(v^* u\) be the optimal solution to the GCC Model. If \(u = (u_0, u_0, \ldots, u_0)^T\) for \(u_0 < 0\), then \(C_{m+1} < C_m\).

**Proof.** If \(u = (u_0, u_0, \ldots, u_0)^T = u_0 R_m\), then \(R_m^T E_m^{-1} u = u_0 R_m^T E_m^{-1} R_m < 0\), which means that \(R_{m+1}^T E_{m+1}^{-1} R_{m+1} = R_m^T E_m^{-1} R_m + (R_m^T E_m^{-1} u - 1)^2/p' < R_m^T E_m^{-1} R_m\). Thus, \(C_{m+1} < C_m\).

**Theorem 19.** Let \(C_m\) be the optimal group continuous logarithm compatibility, and let \(v^* u\) be the optimal solution to the GCC Model. If \(u = u_0 v^* u\) for \(u_0 < 0\), then \(C_{m+1} < C_m\).

**Proof.** If \(u = u_0 v^* u\) for \(u_0 < 0\), then

\[
R_m^T E_m^{-1} u = R_m^T E_m^{-1} u_0 \cdot E_m^{-1} R_m \cdot \frac{E_m^{-1} R_m}{R_m^T E_m^{-1} R_m} = u_0 \cdot \frac{(E_m^{-1} R_m)^T (E_m^{-1} R_m)}{R_m^T E_m^{-1} R_m} \leq 0,
\]

which means that \(R_{m+1}^T E_{m+1}^{-1} R_{m+1} = R_m^T E_m^{-1} R_m + (R_m^T E_m^{-1} u - 1)^2/p' < R_m^T E_m^{-1} R_m\). Thus, \(C_{m+1} < C_m\).

It is apparent that Theorems 17–19 are sufficient conditions for the \((m + 1)\)th interval multiplicative preference relation added to the GCC Model to be effective.

4. Illustrative examples

In this section, we develop a methodology for the use of the continuous logarithm compatibility measure for interval multiplicative preference relations, and we then compare previous approaches to the compatibility of interval multiplicative preference relations with this new approach.

**Example 1.** We study a bid-inviting process in which an investor is attempting to identify the optimal bidding scheme (adapted from [34]). To keep pace with the development of the modern iron and steel industry and to improve the urban environment, Steel and Iron Works wishes to construct a pelletizing plant in its primary iron-ore-producing area, which has a production capacity of 1.20 million tons per year.

Based on the characteristics of the construction project, the construction plan is divided into four bid packages, including the construction project, the installation project, etc. The construction project is the principal component of this civil works project. Considering the regulations relevant to the project, the investor will invite bidding and select from among five bidders based on the quotation, construction period, quality, construction technology, and business reputation.

Suppose that five construction organizations, \(x_1, x_2, x_3, x_4, x_5\), are selected as possible alternatives to be evaluated by three DMs, denoted by \(d_k (k = 1, 2, 3)\), using interval multiplicative preference relations. Each DM \(d_k (k = 1, 2, 3)\) compares the five construction organizations and formulates his/her own interval multiplicative preference relation \(\tilde{A}^{(k)} (k = 1, 2, 3)\), which are
Based on Eq. (22) or the GCC Model, we obtain the optimal weights of the DMs as follows:

\[ \nu_1^* = 0.6916, \quad \nu_2^* = 0.1842, \quad \nu_3^* = 0.1242. \]
From Eq. (6), we have the expected multiplicative preference relation corresponding to the synthetic preference relation:

\[
\hat{A} = \begin{pmatrix}
1.0000 & 6.4807 & 0.5023 & 2.6562 & 0.4514 \\
0.1543 & 1.0000 & 6.0162 & 6.4807 & 0.2454 \\
1.9909 & 0.1662 & 1.0000 & 6.3536 & 7.4833 \\
0.3765 & 0.1543 & 0.1674 & 1.0000 & 1.1552 \\
2.2155 & 4.0749 & 0.1336 & 0.8657 & 1.0000
\end{pmatrix}.
\]

We calculate the expected value \(\hat{a}_i\) of the preference degree of bidder \(x_i\) for all bidders using the following formula:

\[
\hat{a}_i = \left( \prod_{j=1}^{5} \hat{a}_{ij} \right)^{1/5}, \quad i = 1, 2, \ldots, 5.
\]

Then, we have \(\hat{a}_1 = 1.3130, \hat{a}_2 = 1.0810, \hat{a}_3 = 1.7353, \hat{a}_4 = 0.4025,\) and \(\hat{a}_5 = 1.0087.\)

The arguments \(\hat{a}_i (i = 1, 2, 3, 4, 5)\) are ranked in descending order as follows: \(\hat{a}_3 > \hat{a}_1 > \hat{a}_2 > \hat{a}_5 > \hat{a}_4.\)

Then, the bidders \(x_i (i = 1, 2, 3, 4, 5)\) are ranked in accordance with their collective overall interval preference values \(\hat{a}_i (i = 1, 2, 3, 4, 5)\) as follows: \(x_3 > x_1 > x_2 > x_5 > x_4.\) Note that “\(\succ\)” means “preferred over”.

Thus, the first organization is the best alternative in this GDM problem. The individual continuous logarithm compatibility measure and the group continuous logarithm compatibility measure are as follows:

\[
\begin{align*}
C - \text{LCI}(\hat{A}^{(1)}, \hat{W}^{(1)}) &= 1.2641; \\
C - \text{LCI}(\hat{A}^{(2)}, \hat{W}^{(2)}) &= 1.3067; \\
C - \text{LCI}(\hat{A}^{(3)}, \hat{W}^{(3)}) &= 1.3073; \\
C - \text{LCI}(\hat{A}, \hat{W}) &= 1.2535.
\end{align*}
\]

Let \(\alpha = 1.564;\) then, \(C - \text{LCI}(\hat{A}^{(k)}, \hat{W}^{(k)}) \leq \alpha\) for \(k = 1, 2, 3.\) We can see that the interval multiplicative preference relations \(\hat{A}^{(k)}\) and their continuous characteristic preference relations are of acceptable compatibility, as are the synthetic preference relation \(\hat{A}\) and its continuous characteristic preference relation. Moreover, \(C - \text{LCI}(\hat{A}, \hat{W}) < C - \text{LCI}(\hat{A}^{(k)}, \hat{W}^{(k)})\) for all \(k;\) therefore, by Eq. (15), \(C - \text{LCI}(\hat{A}, \hat{W})\) is a superior group continuous logarithm compatibility.

To analyze how different weights of the DMs may affect the compatibility measure in this case, we consider equal DM weights, i.e., \(v_k = 1/3\) for \(k = 1, 2, 3.\) Then, the expected multiplicative preference relation \(\hat{A} = (\hat{a}_{ij})_{5 \times 5}\) is calculated as follows:

\[
\hat{A} = \begin{pmatrix}
1.0000 & 6.4807 & 0.5023 & 4.3645 & 0.4503 \\
0.1543 & 1.0000 & 6.0986 & 6.4807 & 0.2519 \\
1.7628 & 0.1640 & 1.0000 & 6.2359 & 7.4833 \\
0.2291 & 0.1543 & 0.1604 & 1.0000 & 1.3480 \\
2.2208 & 3.9704 & 0.1336 & 0.7418 & 1.0000
\end{pmatrix}.
\]

Therefore, the group continuous logarithm compatibility measure is 1.2678, which is greater than the group continuous logarithm compatibility measure obtained for the optimal solution to the GCC Model. This shows that the optimal solution to the GCC Model is superior to that obtained with equal weight parameters.

Furthermore, to analyze the role of the attitudinal character \(\lambda\) in the aggregation results in this case, we consider different values of the \(\lambda\) parameter provided by the DMs: 0, 0.1, ..., 0.9, 1. The results for the DM weights determined using the GCC Model or Eq. (22) are shown in Fig. 1, and the results for \(\hat{a}_i (i = 1, 2, 3, 4, 5)\) are shown in Fig. 2.

It is apparent from Fig. 1 that \(v_1\) initially increases and then decreases as \(\lambda\) increases, whereas \(v_2\) monotonically increases as \(\lambda\) increases; however, the monotonicity of \(v_3\) is indistinct. Moreover, Fig. 2 indicates that the ordering of the bidders differs depending on the particular attitudinal character \(\lambda\) that is used, thus leading to different decisions. However, it seems that \(x_3\) is the best choice when \(\lambda \leq 0.75\) and that \(x_4\) is sometimes also the best.

It is also interesting to consider the continuous logarithm compatibility measures determined using different \(\lambda\) values. The results are shown in Fig. 3.

It is evident from Fig. 3 that \(C - \text{LCI}(\hat{A}, \hat{W}) \leq C - \text{LCI}(\hat{A}^{(k)}, \hat{W}^{(k)})\) for \(k = 1, 2, 3,\) which means that \(C - \text{LCI}(\hat{A}, \hat{W})\) is at least a non-inferior group continuous logarithm compatibility. This confirms the conclusion of Theorem 11. Furthermore, we observe that the group continuous logarithm compatibility measure is less than each individual continuous logarithm compatibility measure when \(\lambda > 0.4,\) which means that \(C - \text{LCI}(\hat{A}, \hat{W})\) is a superior group continuous logarithm compatibility and that the GCC Model is a superior GCC Model when \(\lambda > 0.4.\) For example, if \(\lambda = 0.5,\) then by Eq. (19), the continuous logarithm compatibility information matrix is

\[
E_3 = (e_{kk})_{3 \times 3} = \begin{pmatrix}
1.2641 & 1.2298 & 1.2298 \\
1.2298 & 1.3067 & 1.3063 \\
1.2298 & 1.3063 & 1.3073
\end{pmatrix}.
\]

As mentioned above, the GCC Model in this case is a superior GCC Model. Indeed, the elements in each column of \(E_3\) are not fully equal, which confirms the conclusion of Theorem 13. Additionally, for all \(\lambda \in [0, 1],\) the sum of all elements in each line of \(E_3\) is different; therefore, \(v_A = (1/3, 1/3, 1/3)^T\) is not the optimal solution to the GCC Model, which confirms the conclusion of Theorem 10.
Fig. 1. Expert weights with different $\lambda$ values.

Fig. 2. Aggregations with different $\lambda$ values.
Moreover, assume that a new interval multiplicative preference relation $\tilde{A}^{(4)}$ is added to the DM group, where

$$\tilde{A}^{(4)} = \begin{pmatrix}
\end{pmatrix}.$$ 

Then, the optimal solution to the GCC Model is as follows:

$$\hat{v}_1^* = 0.6916, \quad \hat{v}_2^* = 0.1842, \quad \hat{v}_3^* = 0.1242, \quad \hat{v}_4^* = 0.$$

This means that $\tilde{A}^{(4)}$ is a redundant interval multiplicative preference relation.

The individual continuous logarithm compatibility measures and the group continuous logarithm compatibility measure are found to be

$$C - LCI(\tilde{A}^{(1)}, \tilde{W}^{(1)}) = 1.2641; \quad C - LCI(\tilde{A}^{(2)}, \tilde{W}^{(2)}) = 1.3067; \quad C - LCI(\tilde{A}^{(3)}, \tilde{W}^{(3)}) = 1.3073; \quad C - LCI(\tilde{A}^{(4)}, \tilde{W}^{(4)}) = 1.4039,$$

$$C - LCI(\tilde{A}, \tilde{W}) = 1.2535.$$

This means that $C_4 \leq C_3$, which confirms the conclusion of Theorem 15.

In addition, the continuous logarithm compatibility information matrix $E_4$ is obtained as follows:

$$E_4 = \begin{pmatrix} E_3 & u \\ u^T e_{44} \end{pmatrix},$$

where $u = (1.2305, 1.3058, 1.3040)^T$ and $e_{44} = 1.4039$. Thus,

$$(v_1^*, v_2^*, v_3^*)u = 1.2535 = C_3,$$

which means that the 4th interval multiplicative preference relation added to the GCC Model is a redundant preference relation. This confirms the conclusion of Theorem 16.
Example 2. Consider the example concerning the selection of alternatives presented in [23,24]. Three DMs $e_k (k = 1, 2, 3)$ provide the following interval multiplicative preference relations concerning a set of four potential alternatives $X = \{x_1, x_2, x_3, x_4\}$:

$$
\tilde{A}^{(1)} = \begin{pmatrix}
[1/5, 1/2] & [1, 1] & [1, 3] & [1, 2] \\
[1/3, 1] & [1/2, 1] & [1, 2] & [1, 1]
\end{pmatrix}, \quad \tilde{A}^{(2)} = \begin{pmatrix}
[1, 1] & [1, 2] & [1, 2] & [2, 3] \\
[1/5, 1/3] & [1, 1] & [1, 1] & [6, 8] \\
\end{pmatrix},
$$

$$
\tilde{A}^{(3)} = \begin{pmatrix}
[1/5, 1/3] & [1/5, 1/2] & [1, 3] & [1, 1]
\end{pmatrix}.
$$

Wang, Chen and Zhou [23] proposed the following logarithm compatibility index for interval multiplicative preference relations $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ and $\tilde{B} = (\tilde{b}_{ij})_{n \times n}$:

$$
ILCI(\tilde{A}, \tilde{B}) = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (|\log \tilde{a}_{ij}^L - \log \tilde{b}_{ij}^L| + |\log \tilde{a}_{ij}^U - \log \tilde{b}_{ij}^U|),
$$

(32)

where $\tilde{a}_{ij} = [a_{ij}^L, a_{ij}^U]$ and $\tilde{b}_{ij} = [b_{ij}^L, b_{ij}^U]$.

Using an optimal model to minimize the logarithm compatibility indexes for these interval multiplicative preference relations and their characteristic matrices, the optimal expert weights $\hat{p}_k$ ($k = 1, 2, 3$) and the logarithm compatibility indexes for the interval multiplicative preference relations $\tilde{A}^{(k)}$ and characteristic matrices $\tilde{W}^{(k)}$ ($k = 1, 2, 3$) are found [23] to be $\hat{p}_1 = 0.5545, \hat{p}_2 = 0.1562,$ and $\hat{p}_3 = 0.2893$ and $ILCI(\tilde{A}^{(1)}, \tilde{W}^{(1)}) = 0.2256, ILCI(\tilde{A}^{(2)}, \tilde{W}^{(2)}) = 0.5571,$ and $ILCI(\tilde{A}^{(3)}, \tilde{W}^{(3)}) = 0.4203,$ respectively.

The logarithm compatibility index for synthetic preference relation $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ and its characteristic matrix $\tilde{W} = (\tilde{w}_{ij})_{n \times n}$ is found to be $ILCI(\tilde{A}, \tilde{W}) = 0.2040$, where $\tilde{a}_{ij} = \prod_{k=1}^{4} (\tilde{a}_{ij}^{(k)})^{\hat{p}_k}$ and $\tilde{w}_{ij} = \prod_{k=1}^{4} (\tilde{w}_{ij}^{(k)})^{\hat{p}_k}$ for $i, j = 1, 2, 3, 4$. If we use the GCC Model with different parameters $\lambda$, then the continuous logarithm compatibility measures $C - LCI(\tilde{A}^{(k)}, \tilde{W}^{(k)})$ ($k = 1, 2, 3$) and $C - LCI(\tilde{A}, \tilde{W})$ are obtained as shown in Fig. 4.

In a comparison of our method with the method developed in [23], we observe the following:

1. As indicated by the approach proposed in [23], the interval multiplicative preference relation $\tilde{A}^{(1)}$ and its characteristic matrix $\tilde{W}^{(1)}$ are of acceptable compatibility, but neither $\tilde{A}^{(2)}$ and $\tilde{W}^{(2)}$ nor $\tilde{A}^{(3)}$ and $\tilde{W}^{(3)}$ are of acceptable compatibility.
However, according to the new method developed in this paper, $\tilde{A}^{(1)}$ and $\tilde{W}^{(1)}$, $\tilde{A}^{(2)}$ and $\tilde{W}^{(2)}$, and $\tilde{A}^{(3)}$ and $\tilde{W}^{(3)}$ are all of acceptable compatibility. Moreover, $C - LCI(\tilde{A}, \tilde{W}) < LCI(\tilde{A}, \tilde{W})$. This means that the new approach proposed in this paper is more effective than the method presented in [23].

(2) The logarithm compatibility index for interval multiplicative preference relations proposed in [23] is based on the distance between interval arguments, and in the calculation of this distance, the left and right points have the same importance. By contrast, because it relies on a controlled parameter that accounts for the attitudes of the DMs, the continuous logarithm compatibility proposed in this paper considers each point in the interval arguments, making the continuous logarithm compatibility degree more flexible than the conventional measure.

(3) The properties of the GCC Model ensure the effectiveness of the new approach because the application of the new method is based on combining a mathematical model with scientific considerations.

**Example 3.** Consider the example presented in [26]. Three DMs $a_i(k = 1, 2, 3)$ provide the following interval multiplicative preference relations concerning a set of five potential alternatives $X = \{x_1, x_2, x_3, x_4, x_5\}$:

$$\tilde{A}^{(1)} = \begin{pmatrix}
\end{pmatrix},$$

$$\tilde{A}^{(2)} = \begin{pmatrix}
\end{pmatrix},$$

Moreover, a leading DM also provides his/her interval multiplicative preference relation:

$$P = \begin{pmatrix}
\end{pmatrix}.$$  

A compatibility degree for interval multiplicative preference relations $\tilde{A}^{(k)} = (a_{ij}^{(k)})_{n \times n}$ and $P = (p_{ij})_{n \times n}$ was proposed as follows:

$$C(\tilde{A}^{(k)}, P) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (\tilde{a}_{ij}^{(k)} + p_{ij}^{(k)}) - n^2, \quad k = 1, 2, 3, \quad \text{(33)}$$

where $\tilde{a}_{ij}^{(k)} = [a_{ij}^{(k)L}, a_{ij}^{(k)U}]$ and $p_{ij}^{(k)} = [p_{ij}^{(k)L}, p_{ij}^{(k)U}]$. By letting $Q(x) = x^{2/3}$, the following expert weights are obtained: $\tilde{v}_1 = 0.58$, $\tilde{v}_2 = 0.24$, and $\tilde{v}_3 = 0.18$. The compatibility degrees of $\tilde{A}^{(k)} (k = 1, 2, 3)$ and $P$ are found to be $C(\tilde{A}^{(1)}, P) = 1.15$, $C(\tilde{A}^{(2)}, P) = 1.425$, and $C(\tilde{A}^{(3)}, P) = 0.775$, and the compatibility degree of $\tilde{A}$ and $P$ is $C(\tilde{A}, P) = 0.1687$, where $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ such that $\tilde{a}_{ij} = \prod_{k=1}^{3} (\tilde{a}_{ij}^{(k)})^{\tilde{p}_{ij}}$ for $i, j = 1, 2, \ldots, 5$.

If we use the GCC Model with different parameters $\lambda$, then the following expert weights are obtained:

$$\tilde{v}_1^{(\lambda)} = 0, \quad \tilde{v}_2^{(\lambda)} = 1, \quad \tilde{v}_3^{(\lambda)} = 0, \quad \text{for all} \quad \lambda \in [0, 1].$$

Thus, according to Definitions 16 and 17, $\tilde{A}^{(1)}$ and $\tilde{A}^{(3)}$ are redundant preference relations, and the degree of redundancy of the GCC Model is 2/3.

In a comparison of our approach with the approach developed in [26], we observe the following:

(1) The proposed compatibility measure is superior to that developed in [26] because it is very difficult to obtain the interval multiplicative preference relation of a leading DM. In other words, it is not necessary to use a GDM model in an application if a leading DM exists; instead, his/her preference relation can be used directly.

(2) The expert weights are obtained by solving the GCC Model rather than the BUM function because objective expert weights are more reliable than subjective expert weights.

(3) The compatibility degree developed in [26] is based on the endpoints of the intervals, not each argument in the intervals; by contrast, the continuous logarithm compatibility proposed in this paper considers the attitudes of the DMs through the use of a controlled parameter.
(4) There are two redundant preference relations in this case, implying that the interval multiplicative preference relations $\tilde{A}^{(1)}$ and $\tilde{A}^{(3)}$ should be revised or deleted. The redundancy criterion guarantees the effectiveness of the newly proposed compatibility measure because it is related to the consistency of the interval multiplicative preference relations.

In summary, our experiments indicate that the new approach presented in this paper is fundamentally different from previously developed methods because the GCC Model is based on individual consistency, considers the attitudes of the DMs, and is supported by theoretical scientific considerations. Thus, the new approach has the potential to be widely applied in GDM.

5. Concluding remarks

Calculations of compatibility measures play an important role in studies of GDM with interval multiplicative preference relations in the literature. In this paper, we proposed and tested the GCC Model, in which a new compatibility measure for interval multiplicative preference relations is used. The main contributions of this paper can be summarized as follows.

First, a continuous logarithm compatibility measure for interval multiplicative preference relations was proposed in which the risk attitudes of DMs are considered using the COWGA operator. Several properties of this measure were also investigated to ensure its necessity.

Second, the GCC Model was developed to determine the expert weights in GDM problems with interval multiplicative preference relations. The sufficient and necessary conditions for the existence of an optimal solution as well as the conditions for the existence of a superior optimal solution and of redundant preference relations in the GCC Model were discussed to verify the model’s effectiveness.

Third, to verify the efficiency of the GCC Model, it was proven that the optimal group continuous logarithm compatibility in the GCC Model is monotonically non-increasing with respect to the number of DMs, and numerical examples were given to demonstrate the application of the proposed method and to confirm the theoretical findings.

Future research may be performed to investigate the possibility of extending the compatibility measure to other types of preference relations in a similar manner, including interval fuzzy preference relations [6,29], linguistic preference relations [4,5,19], hesitant fuzzy preference relations [22], and incomplete preference relations [8,21]. Group compatibility models, group consensus models [19,28], and granular fuzzy models [13] are also possible future directions of research. Additional research on the effectiveness and efficiency of the proposed models should be conducted.

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