

\section{Introduction}

In a program to associate quantum vertex algebras to quantum affine algebras, a theory of \(\phi\)-coordinated (quasi) modules for quantum vertex algebras was developed in [25], where \(\phi\) is what was called an associate of the one-dimensional additive formal group (law) \(F_\lambda(x, y) = x + y\).
Since the very beginning, it had been recognized that the theory of vertex algebras and their modules was governed by the formal group $F_a$. This can be seen from the definition of a vertex algebra $V$ and more generally the definition of a module $(W,Y_W)$ for a given vertex algebra $V$, where the weak associativity axiom for a $V$-module $(W,Y_W)$ states that for any $u,v \in V$, $w \in W$, there exists a nonnegative integer $l$ such that

$$(x+y)^l Y_W(u,x+y)Y_W(v,y)w = (x+y)^l Y_W(Y(u,x)v,y)w.$$  \hspace{1cm} (1.1)

The notion of associate of the formal group $F_a$ was designed in [25] to be an analog of that of $G$-set of a group $G$. By definition, an associate of $F_a$ is a formal series $\phi(x,z) \in \mathbb{C}((x))[[z]]$ such that

$$\phi(x,0) = x \text{ and } \phi(x,\phi(y,z)) = \phi(x,y+z).$$

Interestingly, it was proved therein that for any $p(x) \in \mathbb{C}((x))$, $\phi_p(x,z) := e^{zp(x)d/dx}x$ is an associate of $F_a$ and every associate of $F_a$ is of this form. When $p(x) = 1$, we get the formal group itself, whereas when $p(x) = x$, we get $\phi_p(x,z) = xe^z$.

Let $\phi(x,z)$ be a general associate of $F_a$. The notion of $\phi$-coordinated (quasi) $V$-module for a vertex algebra (more generally for a weak quantum vertex algebra in the sense of [24]) $V$ was defined by replacing the ordinary weak associativity axiom with the property that for any $u,v \in V$, there is a nonnegative integer $k$ such that

$$(x_1 - x_2)^k Y_W(u,x_1)Y_W(v,x_2) \in \text{Hom}(W,W((x_1,x_2)))$$

and

$$((x_1 - x_2)^k Y_W(u,x_1)Y_W(v,x_2))|_{x_1=\phi(x_2,z)} = (\phi(x_2,z) - x_2)^k Y_W(Y(u,z)v,x_2).$$  \hspace{1cm} (1.2)

When $\phi(x,z) = F_a(x,z) = x + z$, this gives an equivalent definition of the ordinary notion of module (cf. [29]). In [25], the main focus is on $\phi$-coordinated (quasi) modules with $\phi(x,z) = e^{zx} \frac{d}{dx} x = xe^z$, by which weak quantum vertex algebras were canonically associated to quantum affine algebras. Among the main results, a Jacobi-type identity and a commutator formula for $\phi$-coordinated modules were obtained. Later in [28], $\phi$-coordinated quasi modules were studied furthermore, where a commutator formula, similar to that for twisted modules (see [14,27]), was obtained. Just as commutator formulas for modules and twisted modules are very important and useful in the vertex algebra theory, such commutator formulas for $\phi$-coordinated (quasi) modules were proved to be very useful.

In this current paper, we study $\phi_\epsilon$-coordinated modules for vertex algebras, where $\phi_\epsilon(x,z) = e^{zx} \frac{d^\epsilon}{dx^\epsilon} x$ with $\epsilon$ an arbitrary integer. Part of our motivation is the fact that $x^n \frac{d^\epsilon}{dx^\epsilon}$ with $n \in \mathbb{Z}$ form a basis of the Witt algebra, which plays a vital role in vertex operator algebra theory and in physics conformal field theory. Conceivably, $\phi_\epsilon$-coordinated
modules for vertex algebras will be of fundamental importance. Among the main results, we show that if $U$ is a local subset of $\text{Hom}(W, W((x)))$ with $W$ a general vector space, the nonlocal vertex algebra $\langle U \rangle_{\phi_e}$ that was obtained in [25] is a vertex algebra. We also obtain a Jacobi-type identity for $\phi_e$-coordinated modules for vertex algebras and furthermore we derive a commutator formula.

In this paper, we also study $\phi_e$-coordinated modules for some special family of vertex algebras. Note that for any Lie algebra $g$ equipped with a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$, one has an (untwisted) affine Lie algebra $\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c$. Furthermore, for every complex number $\ell$, one has a vertex algebra $V_{\hat{g}}(\ell, 0)$. This gives a very important family of vertex algebras. There is another family of vertex algebras which are associated to Novikov algebras. Recall that a (left) Novikov algebra is a non-associative algebra $\mathcal{A}$ satisfying the condition that

$$(ab)c - a(bc) = (ba)c - b(ac), \quad (ab)c = (ac)b \quad \text{for } a, b, c \in \mathcal{A}.$$ 

A result of Primc (see [36]) is that for any given (left) Novikov algebra $\mathcal{A}$ equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ such that

$$\langle ab, c \rangle = \langle a, bc \rangle, \quad \langle ab, c \rangle = \langle ba, c \rangle \quad \text{for } a, b, c \in \mathcal{A},$$

one has a Lie algebra $\tilde{L}(\mathcal{A}) = \mathcal{A} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c$, and furthermore, for every complex number $\ell$ one has a vertex algebra $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$.

In this current paper, as an application of our general results we study $\phi_e$-coordinated modules for vertex algebras $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ associated to Novikov algebras $\mathcal{A}$. To determine $\phi_e$-coordinated modules we introduce a Lie algebra $\tilde{L}^\ell(\mathcal{A})$, which has the same underlying space as that of $\tilde{L}(\mathcal{A})$. We show that a $\phi_e$-coordinated module structure on a vector space for vertex algebra $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ exactly amounts to a “restricted” module structure for the Lie algebra $\tilde{L}^\ell(\mathcal{A})$ of level $\ell$.

Throughout this paper, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of nonnegative integers and the set of integers, respectively, and all vector spaces are assumed to be over the field $\mathbb{C}$ of complex numbers. On the other hand, we shall use the standard formal variable notations and conventions (see [13,14,21]).

This paper is organized as follows: In Section 2, we recall the basic results on associates and we derive an explicit formula for $\phi_e$. In Section 3, we prove a Jacobi-type identity and a commutator formula for $\phi_e$-coordinated modules. In Section 4, we study $\phi_e$-coordinated modules for the vertex algebras associated to Novikov algebras.

2. One-dimensional additive formal group and $\phi_e$-coordinated modules for vertex algebras

2.1. We here briefly recall the notion of formal group (cf. [18]) and the notion of associate of a formal group (see [25]).
Definition 2.1. A one-dimensional formal group over \( \mathbb{C} \) is a formal power series \( F(x, y) \in \mathbb{C}[[x, y]] \) such that
\[
F(x, 0) = x, \quad F(0, y) = y, \quad F(F(x, y), z) = F(F(x, y), z).
\]
The simplest example is the one-dimensional additive formal group
\[
F_a(x, y) = x + y. \tag{2.2}
\]

The following notion of associate of a formal group was introduced in [25]:

Definition 2.3. Let \( F(x, y) \) be a one-dimensional formal group over \( \mathbb{C} \). An associate of \( F(x, y) \) is a formal series \( \phi(x, z) \in \mathbb{C}((x))[[z]] \), satisfying the condition that
\[
\phi(x, 0) = x, \quad \phi(\phi(x, x_0), x_2) = \phi(x, F(x_0, x_2)).
\]
The following is an explicit classification of associates for \( F_a(x, y) \) obtained in [25]:

Proposition 2.4. Let \( p(x) \in \mathbb{C}((x)) \). Set
\[
\phi(x, z) = e^{z(p(x)d/dx)} x = \sum_{n \geq 0} \frac{z^n}{n!} \left( p(x) \frac{d}{dx} \right)^n x \in \mathbb{C}((x))[[z]].
\]
Then \( \phi(x, z) \) is an associate of \( F_a(x, y) \). Furthermore, every associate of \( F_a(x, y) \) is of this form with \( p(x) \) uniquely determined.

For any integer \( \epsilon \), set
\[
\phi_{\epsilon}(x, z) = e^{z(x^\epsilon d/dx)} x, \tag{2.5}
\]
an associate of \( F_a(x, y) \). For the rest of this paper, we shall be only concerned about the 1-dimensional additive formal group \( F_a(x, y) \) and its associates \( \phi_{\epsilon}(x, z) \).

As special cases, we have (see [25])
\[
\phi_0(x, z) = x + z, \quad \phi_1(x, z) = xe^z, \quad \phi_2(x, z) = \frac{x}{1 - zx}. \tag{2.6}
\]

For the general case, from definition we have
\[
\phi_{\epsilon+1}(x, z) = x + \sum_{k \geq 1} \frac{z^k}{k!} x^{k+1} \prod_{j=0}^{k-1} (1 + j\epsilon). \tag{2.7}
\]
Assume \( \epsilon \neq 0 \). For \( k \geq 1 \), we have
\[
\frac{1}{k!} \prod_{j=0}^{k-1} (1 + j\epsilon) = (-\epsilon)^k \frac{1}{k!} \prod_{j=0}^{k-1} \left( -\frac{1}{\epsilon} - j \right) = (-\epsilon)^k \left( \frac{1}{k} \right).
\]

Then
\[
\phi_{\epsilon+1}(x, z) = x + x \sum_{k \geq 1} (-\epsilon)^k \left( \frac{-1}{k} \right) (zx^\epsilon)^k = x(1 - \epsilon x^\epsilon)^{-\frac{1}{\epsilon}}. \tag{2.8}
\]

Notice that
\[
\lim_{\epsilon \to 0} \phi_{\epsilon+1}(x, z) = xe^z = \phi_1(x, z). \tag{2.9}
\]

The following are some simple facts we shall use:

**Lemma 2.10.** We have
\[
\phi_\epsilon(x, z) - x = zh(x, z), \tag{2.11}
\]
\[
\phi_\epsilon(x, x_1) - \phi_\epsilon(x, x_2) = (x_1 - x_2)g(x, x_1, x_2), \tag{2.12}
\]
where \(h(x, z)\) is a unit in \(\mathbb{C}((x))[z]\) and \(g(x, x_1, x_2)\) is a unit in \(\mathbb{C}((x))[x_1, x_2]\).

**Proof.** By definition we have
\[
\phi_\epsilon(x, z) - x = z \sum_{j \geq 1} \frac{1}{j!} z^{j-1} \left( x^\epsilon \frac{d}{dx} \right)^j x.
\]

Set \(h(x, z) = \sum_{j \geq 1} \frac{1}{j!} z^{j-1} (x^\epsilon \frac{d}{dx})^j x \in \mathbb{C}((x))[z]\). As \(h(x, 0) = 1\), \(h(x, z)\) is a unit in \(\mathbb{C}((x))[z]\). On the other hand, by definition we have
\[
\phi_\epsilon(x, x_1) - \phi_\epsilon(x, x_2) = (e^{x_1 x^\epsilon \frac{d}{dx}} - e^{x_2 x^\epsilon \frac{d}{dx}}) x = (e^{(x_1 - x_2)x^\epsilon \frac{d}{dx}} - 1) e^{x_2 x^\epsilon \frac{d}{dx}} \cdot x
\]
\[
= (x_1 - x_2) \sum_{j \geq 1} \frac{1}{j!} (x_1 - x_2)^{j-1} \left( x^\epsilon \frac{d}{dx} \right)^j \phi_\epsilon(x, x_2).
\]

Set
\[
g(x, x_1, x_2) = \sum_{j \geq 1} \frac{1}{j!} (x_1 - x_2)^{j-1} \left( x^\epsilon \frac{d}{dx} \right)^j \phi_\epsilon(x, x_2).
\]

We have \(g(x, x_1, x_2) \in \mathbb{C}((x))[x_1, x_2]\) and \(g(x, x_2, x_2) = \phi_\epsilon(x, x_2)\). Note that \(\phi_\epsilon(x, x_2)\) is a unit in \(\mathbb{C}((x))[x_2]\) as \(\phi_\epsilon(x, 0) = x\) (nonzero in \(\mathbb{C}((x))\)). It then follows that \(g(x, x_1, x_2)\) is a unit in \((\mathbb{C}((x))[x_2])[[x_1]] = \mathbb{C}((x))[x_1, x_2]\). \(\square\)
2.2. Next we recall the definition of a (nonlocal) vertex algebra and the definitions of a module and a $\phi$-coordinated module for a (nonlocal) vertex algebra. The notion of nonlocal vertex algebra was studied in [23] (under the name “axiomatic G1-vertex algebra”) and in [24], and it was also independently studied in [4] (under the name “field algebra”). The theory of $\phi$-coordinated modules for quantum vertex algebras was developed in [26], whereas $\phi_1$-coordinated modules were the main focus therein.

Definition 2.13. A nonlocal vertex algebra is a vector space $V$ equipped with a linear map

$$Y(\cdot, x) : V \rightarrow \text{Hom}(V, V((x))) \subset (\text{End} V)[[x, x^{-1}]]$$

$$v \mapsto Y(v, x) = \sum v_n x^{-n-1} \quad \text{(where } v_n \in \text{End} V)$$

and with a distinguished vector $1 \in V$, satisfying the conditions that

$$Y(1, x) = 1, \quad Y(v, x)1 \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)1 = v \quad \text{for } v \in V$$

and that for any $u, v, w \in V$, there exists a nonnegative integer $l$ such that

$$(x_0 + x_2)^l Y(u, x_0 + x_2)Y(v, x_2)w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2)w. \quad (2.14)$$

Furthermore, a vertex algebra is a nonlocal vertex algebra $V$ satisfying the condition that for any $u, v \in V$, there exists a nonnegative integer $k$ such that

$$(x_1 - x_2)^k Y(u, x_1)Y(v, x_2) = (x_1 - x_2)^k Y(v, x_2)Y(u, x_1). \quad (2.15)$$

For each nonlocal vertex algebra $V$, there is a canonical operator $D$ on $V$, which is defined by

$$D(v) = v_{-2}1 = \left( \frac{d}{dx} Y(v, x)1 \right) \bigg|_{x=0} \quad \text{for } v \in V.$$

This operator $D$ satisfies the following property:

$$[D, Y(v, x)] = Y(Dv, x) = \frac{d}{dx} Y(v, x). \quad (2.16)$$

Definition 2.17. Let $V$ be a vertex algebra. A $V$-module is a vector space $W$ equipped with a linear map

$$Y_W(\cdot, x) : V \rightarrow \text{Hom}(W, W((x))) \subset (\text{End} W)[[x, x^{-1}]],$$

$$v \mapsto Y_W(v, x),$$
satisfying the conditions that $Y_W(1, x) = 1_W$ (the identity operator on $W$) and that for $u, v \in V$, $w \in W$, there exists $l \in \mathbb{N}$ such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) = (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2)w. \quad (2.18)$$

**Remark 2.19.** It was shown in [29] (Lemma 2.9) that the weak associativity axiom in the definition of a $V$-module can be equivalently replaced by the condition that for any $u, v \in V$, there exists $k \in \mathbb{N}$ such that

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))), \quad (2.20)$$

$$x_0^k Y_W(Y(u, x_0)v, x_2) = ((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2))_{|x_1=x_2+x_0}. \quad (2.21)$$

**Definition 2.22.** Let $V$ be a nonlocal vertex algebra and let $\phi$ be an associate of the one-dimensional additive formal group $F_a$. A $\phi$-coordinated $V$-module is a vector space $W$ equipped with a linear map $Y_W(\cdot, x)$ as in Definition 2.17, satisfying the conditions that $Y_W(1, x) = 1_W$ and that for $u, v \in V$, there exists $k \in \mathbb{N}$ such that

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))), \quad (2.23)$$

$$(\phi(x_2, x_0) - x_2)^k Y_W(Y(u, x_0)v, x_2)$$

$$= ((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2))_{|x_1=\phi(x_2, x_0)}. \quad (2.24)$$

It is clear that the notion of $\phi_0$-coordinated $V$-module is equivalent to that of $V$-module.

**3. Jacobi-type identity for $\phi_\epsilon$-coordinated modules**

In this section, we shall present some axiomatic results on $\phi_\epsilon$-coordinated modules for vertex algebras. In particular, we establish a Jacobi-type identity and a commutator formula.

Let $W$ be a vector space. Set

$$\mathcal{E}(W) = \text{Hom}(W, W((x)) \subset (\text{End} W)[[x, x^{-1}]]. \quad (3.1)$$

A subset $U$ of $\mathcal{E}(W)$ is said to be local if for any $a(x), b(x) \in U$ there exists a nonnegative integer $k$ such that

$$(x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k b(x_2)a(x_1). \quad (3.2)$$

A pair $(a(x), b(x))$ in $\mathcal{E}(W)$ is said to be compatible if there exists $k \in \mathbb{N}$ such that

$$(x_1 - x_2)^k a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2))). \quad (3.3)$$

Note that (3.2) implies (3.3). Thus each pair in a local subset is always compatible.
Fix an integer $\epsilon$ throughout this section. Let $(a(x), b(x))$ be any compatible pair in $\mathcal{E}(W)$ with $k \in \mathbb{N}$ such that (3.3) holds. We define $a(x)^\epsilon_n b(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ in terms of generating function

$$Y_\epsilon^* (a(x), z)b(x) = \sum_{n \in \mathbb{Z}} a(x)^\epsilon_n b(x)z^{-n-1}$$

by

$$Y_\epsilon^* (a(x), z)b(x) = (\phi_\epsilon(x, z) - x)^{-k}( (x_1 - x)^k a(x_1)b(x)) |_{x_1 = \phi_\epsilon(x, z)}, \quad (3.4)$$

where $(\phi_\epsilon(x, z) - x)^{-k}$ is viewed as an element of $\mathbb{C}((x))((z))$ (recalling Lemma 2.10).

**Remark 3.5.** A notion of compatible subset of $\mathcal{E}(W)$ was introduced in [23] and it was proved therein that each compatible subset generates a nonlocal vertex algebra in a certain canonical way. It was proved in [25] (Theorem 4.10) that any compatible subset $U$ of $\mathcal{E}(W)$ generates a nonlocal vertex algebra $\langle U \rangle_{\phi_\epsilon}$ with $W$ as a canonical $\phi_\epsilon$-coordinated module. On the other hand, it was proved in [23] that every local subset of $\mathcal{E}(W)$ is compatible. Thus each local subset $U$ of $\mathcal{E}(W)$ generates a nonlocal vertex algebra $\langle U \rangle_{\phi_\epsilon}$. It was proved in [25] that $\langle U \rangle_{\phi_1}$ is a vertex algebra, whereas it was proved in [22] that $\langle U \rangle_{\phi_0}$ is a vertex algebra.

In the following, we shall prove that for every integer $\epsilon$, $\langle U \rangle_{\phi_\epsilon}$ is a vertex algebra, generalizing the corresponding results of [22,25].

**Proposition 3.6.** Let $W$ be a vector space and let $V$ be a local subspace of $\mathcal{E}(W)$, which is $Y_\epsilon^*$-closed in the sense that

$$u(x)^\epsilon_n v(x) \in V \quad \text{for } u(x), v(x) \in V, \quad n \in \mathbb{Z}.$$ 

Let $a(x), b(x) \in V$. Suppose

$$(x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k b(x_2)a(x_1) \quad (3.7)$$

for some nonnegative integer $k$. Then

$$(x_1 - x_2)^k Y_\epsilon^* (a(x), x_1) Y_\epsilon^* (b(x), x_2) = (x_1 - x_2)^k Y_\epsilon^* (b(x), x_2) Y_\epsilon^* (a(x), x_1). \quad (3.8)$$

**Proof.** Let $c(x) \in V$ be arbitrarily fixed. There exists $l \in \mathbb{N}$ with $l \geq k$ such that

$$(z - x)^l a(z)c(x) = (z - x)^l c(x)a(z), \quad (z - x)^l b(z)c(x) = (z - x)^l c(x)b(z).$$

Using this and (3.7) we get
\[(y - z)^{(y - x)}(z - x)^{(y - x)}a(y)b(z)c(x) \in \text{Hom}(W, W((x, y, z))).\]

By Lemma 4.7 in [25], we have
\[
(\phi_e(x, x_1) - \phi_e(x, x_2))^{(y - x)}(\phi_e(x, x_1) - x)^{(y - x)}Y_\xi^e(a(x), x_1)Y_\xi^e(b(x), x_2)c(x)
\]
\[= (y - z)^{(y - x)}(z - x)^{(y - x)}a(y)b(z)c(x)|_{y=\phi_e(x, x_1), z=\phi_e(x, x_2)}.
\]

Set
\[f(x, x_1, x_2) = (\phi_e(x, x_1) - \phi_e(x, x_2))^{(y - x)}(\phi_e(x, x_1) - x)^{(y - x)}(\phi_e(x, x_2) - x)^{(y - x)},\]
which lies in \(C((x))((x, x_2))\). Then
\[
f(x, x_1, x_2)(\phi_e(x, x_1) - \phi_e(x, x_2))^k Y_\xi^e(a(x), x_1)Y_\xi^e(b(x), x_2)c(x)
\]
\[= (y - z)^{(y - x)}(z - x)^{(y - x)}(y - z)^k a(y)b(z)c(x)|_{y=\phi_e(x, x_1), z=\phi_e(x, x_2)}
\]
\[= (y - z)^{(y - x)}(z - x)^{(y - x)}(y - z)^k a(y)b(z)c(x)|_{z=\phi_e(x, x_2), y=\phi_e(x, x_1)}
\]
\[= f(x, x_1, x_2)(\phi_e(x, x_1) - \phi_e(x, x_2))^k Y_\xi^e(b(x), x_2)Y_\xi^e(a(x), x_1)c(x).
\]

Noticing that \((\phi_e(x, x_1) - x)^{(y - x)}(\phi_e(x, x_2) - x)^{(y - x)}\) is invertible in \(C((x))((x, x_2))\), by cancellation, we get
\[
(\phi_e(x, x_1) - \phi_e(x, x_2))^l + k Y_\xi^e(a(x), x_1)Y_\xi^e(b(x), x_2)c(x)
\]
\[= (\phi_e(x, x_1) - \phi_e(x, x_2))^l + k Y_\xi^e(b(x), x_2)Y_\xi^e(a(x), x_1)c(x). \tag{3.9}
\]

By Lemma 2.10, we have
\[
(\phi_e(x, x_1) - \phi_e(x, x_2))^l + k = (x_1 - x_2)^l + k g(x, x_1, x_2)^l + k,
\]
where \(g(x, x_1, x_2)\) is a unit in \(C((x))[[x_1, x_2]]\). By cancellation we get
\[
(x_1 - x_2)^l + k Y_\xi^e(a(x), x_1)Y_\xi^e(b(x), x_2) = (x_1 - x_2)^l + k Y_\xi^e(b(x), x_2)Y_\xi^e(a(x), x_1). \tag{3.10}
\]

Combining this with the weak associativity obtained in [25] we get
\[
x_0^{-1}\delta(x_1 - x_2)x_0^{-1}\delta(x_1 - x_2) Y_\xi^e(a(x), x_1)Y_\xi^e(b(x), x_2) - x_0^{-1}\delta(x_2 - x_1)Y_\xi^e(b(x), x_2)Y_\xi^e(a(x), x_1)
\]
\[= x_2^{-1}\delta(x_1 - x_0)Y_\xi^e(a(x), x_0)b(x), x_2). \tag{3.11}
\]

From (3.7) we have
\[(x_1 - x_2)^k a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2))),\]

so that
\[
(\phi_\epsilon(x_2, x_0) - x_2)^k Y_\epsilon^x(a(x), x_0)b(x) = (x_1 - x)^k a(x_1)b(x)|_{x_1 = \phi_\epsilon(x_2, x_0)}.
\]

Multiplying both sides of (3.11) by \((\phi_\epsilon(x_2, x_0) - x_2)^k\) and then taking \(\text{Res}_{x_0}\) we get
\[
(\phi_\epsilon(x_2, x_0) - x_2)^k Y_\epsilon^x(a(x), x_1)Y_\epsilon^x(b(x), x_2) = (\phi_\epsilon(x_2, x_0) - x_2)^k Y_\epsilon^x(b(x), x_2)Y_\epsilon^x(a(x), x_1).
\]

By Lemma 2.10, we have
\[
(\phi_\epsilon(x_2, x_0) - x_2)^k = x_0^k h(x_2, x_0)^k
\]
where \(h(x_2, x_0)\) is a unit in \(\mathbb{C}(x_2)[[x_0]]\). By cancellation we obtain
\[
(x_1 - x_2)^k Y_\epsilon^x(a(x), x_1)Y_\epsilon^x(b(x), x_2) = (x_1 - x_2)^k Y_\epsilon^x(b(x), x_2)Y_\epsilon^x(a(x), x_1),
\]
as desired. \(\Box\)

Now we have:

**Theorem 3.12.** Let \(W\) be a vector space and let \(U\) be any local subset of \(\mathcal{E}(W)\). Then \(\langle U \rangle_{\phi_\epsilon}\) is a vertex algebra and \(W\) is a canonical \(\phi_\epsilon\)-coordinated \(\langle U \rangle_{\phi_\epsilon}\)-module.

**Proof.** We already knew that \(\langle U \rangle_{\phi_\epsilon}\) is a nonlocal vertex algebra and \(W\) is a \(\phi_\epsilon\)-coordinated \(\langle U \rangle_{\phi_\epsilon}\)-module with \(Y_W(a(x), z) = \alpha(z)\) for \(\alpha(x) \in \langle U \rangle_{\phi_\epsilon}\). As \(\langle U \rangle_{\phi_\epsilon}\) is the smallest \(Y_\epsilon^x\)-closed local subspace containing \(U\) and \(1_W\), we see that \(\langle U \rangle_{\phi_\epsilon}\) as a nonlocal vertex algebra is generated by \(U\). Given that \(U\) is local, by Proposition 3.6 we see that
\[
\{Y_\epsilon^x(a(x), z) \mid a(x) \in U\}
\]
is a local subset of \(\mathcal{E}(\langle U \rangle_{\phi_\epsilon})\). It follows that \(\langle U \rangle_{\phi_\epsilon}\) is a vertex algebra and \(W\) is a \(\phi_\epsilon\)-coordinated \(\langle U \rangle_{\phi_\epsilon}\)-module. \(\Box\)

We also have the following results:

**Proposition 3.13.** Let \(V\) be a vertex algebra and let \((W, Y_W)\) be a \(\phi_\epsilon\)-coordinated \(V\)-module. Suppose that for some fixed \(u, v \in V\), \(k \in \mathbb{N}\),
\[
(x_1 - x_2)^k Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))).
\]
Then
\[(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^k Y_W(v, x_2) Y_W(u, x_1). \tag{3.14}\]

**Proof.** Recall the skew-symmetry of $V$:
\[Y(u, x)v = e^{x^0} Y(v, -x)u \quad \text{for } u, v \in V.\]

From the definition, there exists $l \in \mathbb{N}$ such that
\[(x_1 - x_2)^l Y_W(u, x_1) Y_W(v, x_2), \quad (x_1 - x_2)^l Y_W(v, x_2) Y_W(u, x_1) \in \text{Hom}(W, W((x_1, x_2))).\]

Then, using Lemmas 3.6 and 3.7 in [25] we get
\[
\begin{align*}
((x_1 - x_2)^l Y_W(u, x_1) Y_W(v, x_2))|_{x_1 = \phi_e(x_2, x_0)} & = (\phi_e(x_2, x_0) - x_2)^l Y_W(Y(u, x_0) v, x_2) \\
& = (\phi_e(x_2, x_0) - x_2)^l Y_W(e^{x_0} Y(v, -x_0) u, x_2) \\
& = (\phi_e(x_2, x_0) - x_2)^l Y_W(Y(-x_0) u, \phi_e(x_2, x_0)).
\end{align*}
\]

On the other hand, we have
\[
((x_1 - x_2)^l Y_W(v, x_2) Y_W(u, x_1))|_{x_2 = \phi_e(x_1, -x_0)} = (x_1 - \phi_e(x_1, -x_0))^l Y_W(Y(-x_0) u, x_1).
\]

Hence
\[
\begin{align*}
((x_1 - x_2)^l Y_W(u, x_1) Y_W(v, x_2))|_{x_1 = \phi_e(x_2, x_0)} & = (((x_1 - x_2)^l Y_W(v, x_2) Y_W(u, x_1))|_{x_2 = \phi_e(x_1, -x_0)})|_{x_1 = \phi_e(x_2, x_0)} \\
& = ((x_1 - x_2)^l Y_W(v, x_2) Y_W(u, x_1))|_{x_1 = \phi_e(x_2, x_0)}.
\end{align*}
\]

It follows that
\[(x_1 - x_2)^l Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^l Y_W(v, x_2) Y_W(u, x_1),\]

which implies
\[(x_1 - x_2)^l (x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^l (x_1 - x_2)^k Y_W(v, x_2) Y_W(u, x_1).\]

Noticing that $(x_1 - x_2)^k Y_W(v, x_2) Y_W(u, x_1) \in \text{Hom}(W, W((x_2))((x_1)))$ and
\[(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))) \subset \text{Hom}(W, W((x_2))((x_1)))\]

by assumption, we can multiply both sides by the inverse of $(x_1 - x_2)^l$ in $\mathbb{C}((x_2))((x_1))$ to obtain the desired relation (3.14). \qed
**Proposition 3.15.** Let $V$ be a vertex algebra and let $(W, Y_W)$ be a $\phi_\epsilon$-coordinated $V$-module. Set $V_W = \{ Y_W(v, x) | v \in V \}$. Then $V_W$ is a local subspace of $E(W)$, $(V_W, Y^\epsilon_\xi, 1_W)$ is a vertex algebra, and $Y_W$ is a homomorphism of vertex algebras.

**Proof.** By Definition 2.22 and Proposition 3.13, $V_W$ is a local subspace of $E(W)$. Let $u, v \in V$. By definition, there exists $k \in \mathbb{N}$ such that

$$(x_1 - x)^k Y_W(u, x_1) Y_W(v, x) \in \text{Hom}(W, W((x_1, x)))$$

and

$$(\phi_\epsilon(x_2, z) - x_2)^k Y^\epsilon_\xi(Y(u, x), z) Y_W(v, x) = (x_1 - x)^k Y_W(u, x_1) Y_W(v, x)|_{x_1 = \phi_\epsilon(x_2, z)}.$$ 

On the other hand, from the definition of $Y^\epsilon_\xi(\cdot, x)$ we have

$$(\phi_\epsilon(x_2, z) - x_2)^k Y^\epsilon_\xi(Y_W(u, x), z) Y_W(v, x) = (x_1 - x)^k Y_W(u, x_1) Y_W(v, x)|_{x_1 = \phi_\epsilon(x_2, z)}.$$ 

It follows that

$$(\phi_\epsilon(x_2, z) - x_2)^k Y^\epsilon_\xi(Y(u, x), z) Y_W(v, x) = (\phi_\epsilon(x_2, z) - x_2)^k Y^\epsilon_\xi(Y_W(u, x), z) Y_W(v, x).$$ 

Since both $Y^\epsilon_\xi(Y(u, x), z) Y_W(v, x)$ and $Y^\epsilon_\xi(Y_W(u, x), z) Y_W(v, x)$ involve only finitely many negative powers of $z$, and since $(\phi_\epsilon(x_2, z) - x_2)^k$ is a unit in $C((x_2))(z)$ (by Lemma 2.10), by cancellation we get

$$Y^\epsilon_\xi(Y(u, x), z) Y_W(v, x) = Y^\epsilon_\xi(Y_W(u, x), z) Y_W(v, x).$$

Then $(V_W, Y^\epsilon_\xi, 1_W)$ is a vertex algebra, and $Y_W$ is a homomorphism of vertex algebras. □

We also have the following result generalizing the corresponding results of [23,25]:

**Lemma 3.16.** Let $W$ be a vector space and let

$$A(x_1, x_2) \in \text{Hom}(W, W((x_1))((x_2))), \quad B(x_1, x_2) \in \text{Hom}(W, W((x_2))((x_1))),$$

$$C(x_0, x_2) \in \left( \text{Hom}(W, W((x_2))) \right)((x_0)).$$

If there exists a nonnegative integer $k$ such that

$$(x_1 - x_2)^k A(x_1, x_2) = (x_1 - x_2)^k B(x_1, x_2),$$

$$((x_1 - x_2)^k A(x_1, x_2))|_{x_1 = \phi_\epsilon(x_2, x_0)} = (\phi_\epsilon(x_2, x_0) - x_2)^k C(x_0, x_2),$$
then
\[
(x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) A(x_1, x_2) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) B(x_1, x_2)
\]
\[
= x_1^{-1} \delta \left( \frac{x_2 (1 + z)}{x_1} \right) C(f_\epsilon(x_2, z), x_2),
\]
where
\[
f_\epsilon(x_2, z) = \begin{cases} 
  x_2^{1-\epsilon} \cdot \frac{(1+z)^{1-\epsilon-1}}{1-\epsilon}, & \text{for } \epsilon \neq 1, \\
  \log(1 + z), & \text{for } \epsilon = 1.
\end{cases}
\]

**Proof.** We start with the standard delta-function identity
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right).
\]
Substituting \(x_0 = x_2 z\) with \(z\) a new formal variable, we have
\[
(x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) = x_1^{-1} \delta \left( \frac{x_2 (1 + z)}{x_1} \right).
\]

Then we get
\[
(x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) (x_2 z)^k A(x_1, x_2) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) (x_2 z)^k B(x_1, x_2)
\]
\[
= (x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) (x_1 - x_2)^k A(x_1, x_2) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) (x_1 - x_2)^k B(x_1, x_2)
\]
\[
= x_1^{-1} \delta \left( \frac{x_2 (1 + z)}{x_1} \right) ((x_1 - x_2)^k A(x_1, x_2))
\]
\[
= x_1^{-1} \delta \left( \frac{x_2 (1 + z)}{x_1} \right) ((x_1 - x_2)^k A(x_1, x_2))|_{x_1 = x_2 (1 + z)}
\]
\[
= x_1^{-1} \delta \left( \frac{x_2 (1 + z)}{x_1} \right) (\phi_\epsilon(x_2, x_0))|_{x_0 = f_\epsilon(x_2, z)}
\]
\[
= x_1^{-1} \delta \left( \frac{x_2 (1 + z)}{x_1} \right) (x_2 z)^k C(f_\epsilon(x_2, z), x_2),
\]
noticing that
\[
\phi_\epsilon(x, f_\epsilon(x, z)) = x(1 + z).
\]

Then (3.17) follows. \(\square\)
As the main result of this section we have:

**Theorem 3.19.** Let $V$ be a vertex algebra and let $(W, Y_W)$ be a $\phi_\epsilon$-coordinated module. Then

\[
(x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) Y_W(u, x_1) Y_W(v, x_2) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) Y_W(v, x_2) Y_W(u, x_1)
\]

\[
= x_1^{-1} \delta \left( \frac{x_2(1 + z)}{x_1} \right) Y_W(Y(u, f_\epsilon(x_2, z)) v, x_2)
\]

(3.20)

for $u, v \in V$, where

\[
f_\epsilon(x, z) = \begin{cases} x^{1-\epsilon} \cdot \frac{(1+z)^{1-\epsilon}-1}{1-\epsilon} & \text{for } \epsilon \neq 1, \\ \log(1 + z), & \text{for } \epsilon = 1. \end{cases}
\]

Furthermore, we have

\[
[Y_W(u, x_1), Y_W(v, x_2)] = \sum_{j \geq 0} \frac{1}{j!} \left( x_2 \partial \frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) Y_W(u_j v, x_2).
\]

(3.21)

**Proof.** From definition, there exists $k \in \mathbb{N}$ such that

\[
(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2)))
\]

and

\[
(\phi_\epsilon(x_2, x_0) - x_2)^k Y_W(Y(u, x_0) v, x_2) = ((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2))|_{x_1 = \phi_\epsilon(x_2, x_0)}.
\]

On the other hand, by **Proposition 3.13** we also have

\[
(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^k Y_W(v, x_2) Y_W(u, x_1).
\]

(3.22)

Then the first assertion follows immediately from **Lemma 3.16**. Furthermore, applying $\text{Res}_{x_2} x_2$ we get

\[
[Y_W(u, x_1), Y_W(v, x_2)]
\]

\[
= \text{Res}_{x_2} x_2^{-1} \delta \left( \frac{x_2(1 + z)}{x_1} \right) x_2 Y_W(Y(u, f_\epsilon(x_2, z)) v, x_2)
\]

\[
= \text{Res}_{x_0} x_2^{-1} \delta \left( \frac{\phi_\epsilon(x_2, x_0)}{x_1} \right) x_2 \frac{\partial}{\partial x_0} \left( \frac{\phi_\epsilon(x_2, x_0)}{x_2} - 1 \right) Y_W(Y(u, x_0) v, x_2)
\]

\[
= \text{Res}_{x_0} x_1^{-1} \delta \left( \frac{\phi_\epsilon(x_2, x_0)}{x_1} \right) \phi_\epsilon(x_2, x_0)^r Y_W(Y(u, x_0) v, x_2)
\]
= \text{Res}_{x_0} x_1^{\epsilon-1} \delta \left( \frac{\phi_\epsilon(x_2, x_0)}{x_1} \right) Y_W(Y(u, x_0)v, x_2)

= \sum_{j \geq 0} \frac{1}{j!} \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^j x_1^{\epsilon-1} \delta \left( \frac{x_2}{x_1} \right) Y_W(u_j v, x_2),

noticing that

\frac{\partial}{\partial x_0} \phi_\epsilon(x_2, x_0) = e^{x_0 x_2^\epsilon} \frac{\partial}{\partial x_2} \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right) (x_2) = e^{x_0 x_2^\epsilon} \frac{\partial}{\partial x_2} (x_2) = (e^{x_0 x_2^\epsilon} \frac{\partial}{\partial x_2} x_2)^\epsilon = \phi_\epsilon(x_2, x_0)^\epsilon.

This proves the second assertion. \qed

Remark 3.23. We here collect some basic facts that we shall use. We have

(3.24)

\left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^j x_1^{\epsilon-1} \delta \left( \frac{x_2}{x_1} \right) = - \left( x_1^\epsilon \frac{\partial}{\partial x_1} \right) x_2^{\epsilon-1} \delta \left( \frac{x_1}{x_2} \right),

(3.25)

(x_1 - x_2)^n \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) = 0

for any nonnegative integers \( m \) and \( n \) with \( m > n \), and

(3.26)

(x_1 - x_2)^n \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) = x_2^{n\epsilon} (x_1 - x_2)^n \left( \frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left( \frac{x_2}{x_1} \right)

for any nonnegative integer \( n \). Furthermore, we have

(3.27)

\text{Res}_{x_1} x_1^{-\epsilon} (x_1 - x_2)^n \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) = \frac{x_2^{n\epsilon}}{n!}.

These facts can be proved by using the special case with \( \epsilon = 0 \) (cf. [22]) and the facts that for any positive integer \( n \), there exists polynomials \( f_1(x), \ldots, f_n(x) \) such that

(3.28)

\left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^n = x_2^{n\epsilon} \left( \frac{\partial}{\partial x_2} \right)^n + f_1(x_2) \left( \frac{\partial}{\partial x_2} \right)^{n-1} + \cdots + f_n(x_2).

The following, which is a generalization of a result in [22], follows immediately from Theorem 3.19 and the basic facts in Remark 3.23:

Lemma 3.29. Let \( V \) be a vertex algebra and let \( (W, Y_W) \) be a faithful \( \phi_\epsilon \)-coordinated \( V \)-module. Suppose that

(3.30)

\[ [Y_W(u, x_1), Y_W(v, x_2)] = \sum_{j \geq 0} \frac{1}{j!} \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^j x_1^{\epsilon-1} \delta \left( \frac{x_2}{x_1} \right) Y_W(A^j, x_2), \]

where \( u, v, A^0, A^1, \ldots \) are fixed vectors in \( V \). Then \( A^j = u_j v \) for all \( j \geq 0 \).
4. Vertex algebras arising from Novikov algebras

In this section, we study $\phi$-coordinated modules for the vertex algebras associated to Novikov algebras by Primc.

We first recall the definition of a Novikov algebra (see [6,15,32,41]).

**Definition 4.1.** A (left) Novikov algebra is a non-associative algebra $A$ satisfying

\begin{align*}
(ab)c - a(bc) &= (ba)c - b(ac), \quad (4.2) \\
(ab)c &= (ac)b \quad (4.3)
\end{align*}

for $a, b, c \in A$.

Note that any commutative and associative algebra is a Novikov algebra.

**Remark 4.4.** We here recall the Gelfand construction of Novikov algebras due to S. Gelfand (see [15]). Let $A$ be a commutative associative algebra with a derivation $\partial$. Define a new operation $\circ$ on $A$ by $a \circ b = a\partial b$ for $a, b \in A$. Then $(A, \circ)$ is a (left) Novikov algebra.

The following result was due to Balinsky and Novikov (see [6]):

**Proposition 4.5.** Let $A$ be a non-associative algebra. Set

\[ L(A) = A \otimes \mathbb{C}[t, t^{-1}], \quad \partial = \frac{d}{dt}. \quad (4.6) \]

Define a bilinear operation $[\cdot, \cdot]$ on $L(A)$ by

\[ [a \otimes f, b \otimes g] = ab \otimes (\partial f)g - ba \otimes (\partial g)f \quad (4.7) \]

for $a, b \in A, f, g \in \mathbb{C}[t, t^{-1}]$. Then $(L(A), [\cdot, \cdot])$ is a Lie algebra if and only if $A$ is a Novikov algebra.

The following refinement was due to Primc (see [36], Example 3; cf. [6]):

**Proposition 4.8.** Let $\mathcal{A}$ be a non-associative algebra equipped with a bilinear form $\langle \cdot, \cdot \rangle$. Set

\[ \tilde{L}(\mathcal{A}) = L(\mathcal{A}) \oplus \mathbb{C}c, \]

where $c$ is a distinguished nonzero element. For $a \in \mathcal{A}, m \in \mathbb{Z}$, set $L(a, m) = a \otimes t^{m+1}$. Define a bilinear operation $[\cdot, \cdot]$ on $\tilde{L}(A)$ by
\[ [L(a, m), L(b, n)] = (m + 1)L(ab, m + n) - (n + 1)L(ba, m + n) \]
\[ + \frac{1}{12}(m^3 - m)\langle a, b \rangle \delta_{m+n,0}c, \] (4.9)
\[ [c, \tilde{L}(A)] = 0 = [\tilde{L}(A), c] \] (4.10)

for \( a, b \in A, m, n \in \mathbb{Z} \). Then \( \tilde{L}(A), [\cdot, \cdot] \) is a Lie algebra if and only if \( A \) is a Novikov algebra and \( \langle \cdot, \cdot \rangle \) is a symmetric form satisfying

\[ \langle ab, c \rangle = \langle a, bc \rangle, \quad \langle ab, c \rangle = \langle ba, c \rangle \] for \( a, b, c \in A \). (4.11)

**Remark 4.12.** Note that a unital Novikov algebra \( A \) with a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) satisfying (4.11) amounts to a Frobenius algebra, i.e., a unital commutative and associative algebra with a nondegenerate symmetric and associative form. For a Frobenius algebra \( A \), the corresponding Lie algebra \( \tilde{L}(A) \) is isomorphic to the map Virasoro algebra (see [5,20,31]). In particular, if \( A = \mathbb{C}e \) is 1-dimensional with \( e \cdot e = e \) and \( \langle e, e \rangle = \frac{1}{12} \), then \( \tilde{L}(A) \) is isomorphic to the Virasoro algebra. More examples can be found in [33–35].

**Example 4.13.** Let \( A = \mathbb{C}[x, x^{-1}] \) and let \( p(x) \in \mathbb{C}[x, x^{-1}] \). By the Gelfand construction, one has a Novikov algebra \( (A, o_p(x)) \), where

\[ x^i o_p(x) x^j = x^i \left( p(x) \frac{d}{dx} \right) x^j = jx^{i+j-1}p(x) \] for \( i, j \in \mathbb{Z} \).

In this case, the corresponding Lie algebra \( L(A) \) is a Lie algebra of Block type (cf. [7,10]). In particular, if \( p(x) = 1 \), \( L(A) \) is isomorphic to the Poisson algebra \( \mathbb{C}[x, x^{-1}, y, y^{-1}] \) with bracket relation

\[ [f, g] = \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} \] for \( f, g \in \mathbb{C}[x, x^{-1}, y, y^{-1}] \).

For this special case, taking a basis \( L^i_m = x^{i+1+m}y^{m+1} \) for \( i, m \in \mathbb{Z} \), we have

\[ [L^i_m, L^j_n] = (j(m + 1) - i(n + 1))L^{i+j}_{m+n} \] for \( i, j, m, n \in \mathbb{Z} \). (4.14)

The structure and representation theory of this Lie algebra and its subalgebras have been extensively studied in [3,37–39].

Let \( A \) be a Novikov algebra equipped with a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) satisfying (4.11). FollowingPrimc [36] we associate vertex algebras to \( A \). For \( a \in A \), set

\[ L(a, x) = \sum_{n \in \mathbb{Z}} L(a, n)x^{-n-2} \in \tilde{L}(A)[[x, x^{-1}]]. \] (4.15)

In terms of generating functions the relation (4.9) can be rewritten as
for $a, b \in A$. Set

$$\tilde{L}(A)_+ = A \otimes \mathbb{C}[t] \oplus \mathbb{C}c, \quad \tilde{L}(A)_- = A \otimes t^{-1} \mathbb{C}[t^{-1}].$$

Note that $\tilde{L}(A)_\pm$ are Lie subalgebras and $\tilde{L}(A) = \tilde{L}(A)_+ \oplus \tilde{L}(A)_-$ as a vector space. Let $\ell \in \mathbb{C}$ and denote by $\mathbb{C}_\ell$ the one-dimensional $\tilde{L}(A)_+$-module with $c$ acting as scalar $\ell$ and with $A \otimes \mathbb{C}[t]$ acting trivially. Form an induced module

$$V_{\tilde{L}(A)}(\ell, 0) = U(\tilde{L}(A)) \bigotimes_{U(\tilde{L}(A)_+)} \mathbb{C}_\ell. \quad (4.17)$$

Set $1 = 1 \otimes 1 \in V_{\tilde{L}(A)}(\ell, 0)$ and identify $A$ as a subspace of $V_{\tilde{L}(A)}(\ell, 0)$ through the linear map

$$a \mapsto L(a, -2)1 \quad \text{for} \ a \in A.$$ From [36] (cf. [11,40]), there exists a vertex algebra structure on $V_{\tilde{L}(A)}(\ell, 0)$, which is uniquely determined by the condition that $1$ is the vacuum vector and

$$Y(a, x) = L(a, x) = \sum_{n \in \mathbb{Z}} L(a, n)x^{-n-2} \quad \text{for} \ a \in A.$$ Furthermore, $A$ is a generating subspace of vertex algebra $V_{\tilde{L}(A)}(\ell, 0)$ with

$$a_0 b = D(ba), \quad a_1 b = ab + ba, \quad a_3 b = \frac{1}{2} \ell \langle a, b \rangle 1, \quad a_2 b = 0 = a_k b \quad (4.18)$$

for $a, b \in A$ and for $k \geq 4$.

Next, we discuss a $\mathbb{Z}$-graded vertex algebra structure on $V_{\tilde{L}(A)}(\ell, 0)$.

**Definition 4.19.** A $\mathbb{Z}$-graded vertex algebra is a vertex algebra $V$ equipped with a $\mathbb{Z}$-grading $V = \bigoplus_{n \in \mathbb{Z}} V_n$ such that $1 \in V_0$ and

$$u_k v \in V_{m+n-k-1} \quad \text{for} \ u \in V_m, \ v \in V_n, \ m, n, k \in \mathbb{Z}. \quad (4.20)$$

Let $(A, \langle \cdot, \cdot \rangle)$ be given as before. It can be readily seen that $\tilde{L}(A)$ is a $\mathbb{Z}$-graded Lie algebra with $\deg c = 0$ and

$$\deg(a \otimes t^m) = \deg(L(a, m - 1)) = -m + 1 \quad \text{for} \ a \in A, \ m \in \mathbb{Z}. \quad (4.21)$$
As $\tilde{L}(A)_+$ is a graded subalgebra, $V_{\tilde{L}(A)}(\ell, 0)$ is naturally a $\mathbb{Z}$-graded $\tilde{L}(A)$-module with deg $1 = 0$ and with $V_{\tilde{L}(A)}(\ell, 0)_2 = A$. Furthermore, by Lemma A (in Appendix A) $V_{\tilde{L}(A)}(\ell, 0)$ equipped with this $\mathbb{Z}$-grading is a $\mathbb{Z}$-graded vertex algebra. In view of the P-B-W Theorem, $D_a = a - 2^{1} \neq 0$ for any nonzero $a \in A$ and $V_{\tilde{L}(A)}(\ell, 0)$ is linearly spanned by the vectors

$$a^{(1)}_{-m_1} \cdots a^{(r)}_{-m_r} 1$$

for $r \geq 0$, $a^{(i)} \in A$, $m_i \geq 1$.

On the other hand, we have (see also [2]):

**Proposition 4.22.** Let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ be a $\mathbb{Z}$-graded vertex algebra with the following properties:

1. $V_n = 0$ for $n < 0$, $V_0 = \mathbb{C} 1$, $V_1 = 0$;
2. $(\text{Ker } D) \cap V_2 = 0$, where $D$ is the linear operator on $V$ defined by $Dv = v_{-2} 1$;
3. $V = \text{span}\{a^{(1)}_{-m_1} \cdots a^{(r)}_{-m_r} 1 | r \geq 0, a^{(i)} \in V_2, m_i \geq 1\}$.

Then there exist a bilinear operation $*$ on $V_2$, uniquely determined by

$$b_0 a = D(a * b) \text{ for } a, b \in V_2,$$

and a bilinear form $\langle \cdot, \cdot \rangle$ on $V_2$, uniquely determined by

$$a_3 b = \frac{1}{2} \langle a, b \rangle 1 \text{ for } a, b \in V_2.$$

Furthermore, $(V_2, *)$ is a Novikov algebra and $\langle \cdot, \cdot \rangle$ is a symmetric form satisfying (4.11).

**Proof.** As $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is a $\mathbb{Z}$-graded vertex algebra, we have

$$a_0 b \in V_3, \quad a_1 b \in V_2, \quad a_2 b \in V_1 = 0, \quad a_3 b \in V_0 = \mathbb{C} 1, \quad a_n b = 0$$

for $a, b \in V_2$ and $n \geq 4$. From the span property, we have

$$V_3 = \{a_{-2} 1 | a \in V_2\} = D V_2.$$ 

Since $(\text{Ker } D) \cap V_2 = 0$, it follows that $*$ is a well-defined operation on $V_2$. Moreover, using the skew-symmetry of $V$, we get

$$b_0 a = -a_0 b + D(b_1 a), \quad a_3 b = b_3 a \text{ for } a, b \in V_2.$$

It then follows that
\[ a \ast b + b \ast a = b_1a = a_1b, \quad \langle a, b \rangle = \langle b, a \rangle \quad \text{for } a, b \in V_2. \]  

(4.23)

Set \( \widetilde{V} = V \otimes \mathbb{C}[t, t^{-1}] \) and \( \widetilde{D} = D \otimes 1 + 1 \otimes \frac{d}{dt} \). It was known that \( \widetilde{V}/\widetilde{D}\widetilde{V} \) is a Lie algebra where for \( u, v \in V, \ m, n \in \mathbb{Z}, \)

\[
[u \otimes t^m, v \otimes t^n] = \sum_{j \geq 0} \binom{m}{j} u_jv \otimes t^{m+n-j}. \]

For \( a, b \in V_2, \ m, n \in \mathbb{Z}, \) we have

\[
[a_m, b_n] = \sum_{i \geq 0} \binom{m}{i} (a_i b)_{m+n-i} \\
= (a_0b)_{m+n} + m(a_1b)_{m+n-1} + \frac{1}{6} m(m-1)(m-2)(a_3b)_{m+n-3} \\
= (D(b \ast a))_{m+n} + m(a \ast b + b \ast a)_{m+n-1} + \frac{1}{6} m(m-1)(m-2)(a_3b)_{m+n-3} \\
= -(m+n)(b \ast a)_{m+n-1} + m(a \ast b + b \ast a)_{m+n-1} + \frac{1}{6} m(m-1)(m-2)(a_3b)_{m+n-3} \\
= m(a \ast b)_{m+n-1} - n(b \ast a)_{m+n-1} + \frac{1}{12} m(m-1)(m-2)\langle a, b \rangle \mathbf{1}_{m+n-3} \\
= m(a \ast b)_{m+n-1} - n(b \ast a)_{m+n-1} + \frac{1}{12} m(m-1)(m-2)\langle a, b \rangle \delta_{m+n,2}. \\
\]

From this and the assumption (2), we see that the non-associative algebra \( \tilde{L}(V_2, \ast) \) defined in Proposition 4.8 is a subalgebra of \( \widetilde{V}/\widetilde{D}\widetilde{V} \). In view of Proposition 4.8, \( (V_2, \ast) \) is a Novikov algebra and \( \langle \cdot, \cdot \rangle \) is a symmetric bilinear form satisfying (4.11). \( \square \)

**Remark 4.24.** Let \( V \) be a vertex algebra. Suppose \( v \in \text{Ker} \ D \subset V \). Then

\[
\frac{d}{dx} Y(v, x) = Y(Dv, x) = 0, 
\]

which implies that \( v_n = 0 \) for all \( n \neq -1 \). Consequently, \( v \) lies in the center of \( V \) and \( [D, v_{-1}] = 0 \). It then follows that \( v_{-1}V \) is an ideal of \( V \). If \( V \) is \( \mathbb{N} \)-graded and if \( v \in (\text{Ker} \ D) \cap V_n \) with \( n \geq 1 \), then \( v_{-1}V \) is a proper ideal. Thus, if \( V \) is a graded simple \( \mathbb{N} \)-graded vertex algebra, we have \( (\text{Ker} \ D) \cap V_n = 0 \) for all \( n \geq 1 \).

**Remark 4.25.** The bilinear operation \( \ast \) on \( V_2 \) was used by Dijkgraaf in [9] in his study on the genus one partition function, which is controlled by a contact term pre-Lie algebra (see [8]) given in terms of the operator product expansion.

We next discuss quasi vertex operator algebras, or namely Möbius vertex algebras. Fix a basis \( \{L(1), L(0), L(-1)\} \) for \( sl(2, \mathbb{C}) \) such that
\[ [L(0), L(\pm 1)] = \mp L(\pm 1), \quad [L(1), L(-1)] = 2L(0). \]

A Möbius vertex algebra (see [13]) is a \( \mathbb{Z} \)-graded vertex algebra \( V = \bigoplus_{n \in \mathbb{Z}} V_n \), equipped with a representation of \( sl(2, \mathbb{C}) \) on \( V \) such that \( V_n = 0 \) for \( n \) sufficiently negative,

\[ L(0)|_{V_n} = n \quad \text{for} \quad n \in \mathbb{Z}, \]

and

\[ [L(-1), Y(v, x)] = Y(L(-1)v, x) = \frac{d}{dx} Y(v, x), \quad (4.26) \]
\[ [L(1), Y(v, x)] = Y(L(1)v, x) + 2xY(L(0)v, x) + x^2Y(L(-1)v, x) \quad (4.27) \]

for \( v \in V \). Note that for a Möbius vertex algebra \( V \), we have \( L(-1) = D \) on \( V \).

We have (cf. [17]):

**Proposition 4.28.** Let \( V = \bigoplus_{n \in \mathbb{Z}} V_n \) be a Möbius vertex algebra satisfying all the conditions given in Proposition 4.22. Then \( (V_2, \ast) \) is a commutative and associative algebra.

**Proof.** For \( v \in V_2 \), as \( L(1)v \in V_1 = 0 \), we have

\[ L(1)L(-1)v = L(-1)L(1)v + 2L(0)v = 4v. \quad (4.29) \]

This implies

\[ (\ker L(-1)) \cap V_2 = 0. \]

On the other hand, for \( v \in V_2 \), from [13] we have

\[ [L(1), v_m] = (-m + 2)v_{m+1} \quad \text{for} \quad m \in \mathbb{Z}. \quad (4.30) \]

Let \( u, v \in V_2 \). Using (4.29), the definition of \( u \ast v \), and (4.30), we get

\[ 4u \ast v = L(1)L(-1)(u \ast v) = L(1)(v_0 u) = v_0 L(1)u - 2v_1 u = 2v_1 u, \]

which gives \( u \ast v = \frac{1}{2} v_1 u \). On the other hand, using skew symmetry we get

\[ u_1 v = v_1 u - L(-1)v_2 u + \frac{1}{2} L(-1)^2 v_3 u + \cdots = v_1 u. \]

Therefore \( u \ast v = v \ast u \). This proves that \( (V_2, \ast) \) is commutative. Consequently, \( (V_2, \ast) \) is commutative and associative. \( \Box \)
Furthermore, we have:

**Proposition 4.31.** Let $\mathcal{A}$ be a Novikov algebra equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ satisfying (4.11). Then for any $\ell \in \mathbb{C}$, the $\mathbb{Z}$-graded vertex algebra $V_{L(\mathcal{A})}(\ell,0)$ has a compatible Möbius vertex algebra structure if and only if $\mathcal{A}$ is commutative and associative.

**Proof.** The “only if” part follows from Proposition 4.28. For the “if” part we assume that $\mathcal{A}$ is a commutative and associative algebra with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ satisfying (4.11). First, by Lemma C, $\mathfrak{sl}_2$ acts on $\tilde{L}(\mathcal{A})$ by derivations, where $\mathfrak{sl}_2 \cdot c = 0$ and

$$L(-1 + j)(a \otimes t^n) = (j - n)(a \otimes t^{n+j-1})$$

for $j = 0, 1, 2$ and for $a \in \mathcal{A}$, $n \in \mathbb{Z}$. Then $\mathfrak{sl}_2$ acts on the universal enveloping algebra $U(\tilde{L}(\mathcal{A}))$ as a Lie algebra of derivations. We see that the action of $\mathfrak{sl}_2$ preserves the subalgebra $\tilde{L}(\mathcal{A})_+$. It follows from the construction of $V_{L(\mathcal{A})}(\ell,0)$ that $\mathfrak{sl}_2$ acts on $V_{L(\mathcal{A})}(\ell,0)$ with $\mathfrak{sl}_2 \cdot 1 = 0$. For $a \in \mathcal{A}$, we have

$$[L(-1), Y(a, x)] = \frac{d}{dx} Y(a, x),$$

$$[L(0), Y(a, x)] = x \frac{d}{dx} Y(a, x) + 2Y(a, x),$$

$$[L(1), Y(a, x)] = 4xY(a, x) + x^2 \frac{d}{dx} Y(a, x).$$

Then by Lemma B (in Appendix A) $V_{L(\mathcal{A})}(\ell,0)$ is a Möbius vertex algebra.  

**Remark 4.32.** Let $\mathcal{A} = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ with a multiplicative operation $\circ$ given by

$$e_1 \circ e_1 = e_1 + e_2, \quad e_2 \circ e_1 = e_2, \quad e_1 \circ e_2 = e_2 \circ e_2 = 0.$$

This is a noncommutative and nonassociative Novikov algebra. Furthermore, the bilinear form $\langle \cdot, \cdot \rangle$, defined by

$$\langle e_1, e_1 \rangle = \frac{1}{12}, \quad \langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle = \langle e_2, e_2 \rangle = 0,$$

is (degenerate) symmetric and satisfies (4.11). In view of Proposition 4.31, $V_{L(\mathcal{A})}(\ell,0)$ is not a Möbius vertex algebra. The corresponding Lie algebra $L(\mathcal{A})$ has been extensively studied in [35].

Next we study $\phi_e$-coordinated modules for vertex algebra $V_{L(\mathcal{A})}(\ell,0)$. First, we construct certain infinite-dimensional Lie algebras, generalizing Lie algebra $\tilde{L}(\mathcal{A})$. 

Lemma 4.33. Let \( \mathcal{A} \) be a Novikov algebra and let \( \mathcal{K} \) be a commutative and associative algebra with a derivation \( \partial \). Define a bilinear operation \([\cdot, \cdot]\) on \( \mathcal{A} \otimes \mathcal{K} \) by
\[
[a \otimes f, b \otimes g] = ab \otimes (\partial f)g - ba \otimes (\partial g)f
\]
for \( a, b \in \mathcal{A}, \ f, g \in \mathcal{K} \). Then \( (\mathcal{A} \otimes \mathcal{K}, [\cdot, \cdot]) \) is a Lie algebra.

Proof. It is straightforward. Alternatively, it follows from a general result in algebraic operad theory (see [30] and [42]) as follows: First, define a new operation \( \ast \) on \( \mathcal{K} \) by
\[
f \ast g = (\partial f)g \quad \text{for } f, g \in \mathcal{K}.
\]
Then \( (\mathcal{K}, \ast) \) is a right Novikov algebra. Note that from [12] (Theorem 1.3), left Novikov algebras and right Novikov algebras are algebras over binary quadratic operads dual to each other. It follows from [16] (Theorem 2.2.6 (b)) that \( (\mathcal{A} \otimes \mathcal{K}, [\cdot, \cdot]) \) is a Lie algebra. \( \square \)

In view of Lemma 4.33, for any Novikov algebra \( \mathcal{A} \) and for any integer \( \epsilon \) we have a Lie algebra \( L^{\epsilon}(\mathcal{A}) \), where
\[
L^{\epsilon}(\mathcal{A}) = \mathcal{A} \otimes \mathbb{C}[t, t^{-1}]
\]
as a vector space and the bilinear operation \([\cdot, \cdot]\) is given by
\[
[a \otimes f, b \otimes g] = ab \otimes \left( t^\epsilon \frac{d}{dt} f \right)g - ba \otimes \left( t^\epsilon \frac{d}{dt} g \right)f
\]
for \( a, b \in \mathcal{A}, \ f, g \in \mathbb{C}[t, t^{-1}] \).

Remark 4.37. Assume that \( \mathcal{A} \) is a commutative Novikov algebra. It can be readily seen that the linear map \( \theta : L(\mathcal{A}) \to L^{\epsilon}(\mathcal{A}) \) defined by \( \theta(L(a, m)) = L^{\epsilon}(a, m) \) for \( a \in \mathcal{A}, \ m \in \mathbb{Z} \) is an isomorphism of Lie algebras. Furthermore, if \( \mathcal{A} \) is unital, that is, \( \mathcal{A} \) is a unital commutative and associative algebra, one can show that \( \tilde{L}(\mathcal{A}) \simeq \tilde{L}^{\epsilon}(\mathcal{A}) \).

Remark 4.38. Let \( \mathcal{A} = \mathbb{C}[z, z^{-1}] \) with a derivation \( z \frac{d}{dz} \). Define
\[
z^i \circ z^j = z^i \left( z \frac{d}{dz} \right) (z^j) = jz^{i+j} \quad \text{for } i, j \in \mathbb{Z}.
\]
Then we have a Novikov algebra \( (\mathcal{A}, \circ) \). Furthermore, for any \( \epsilon \in \mathbb{Z} \) we have a Lie algebra \( L^{\epsilon}(\mathcal{A}) \). Denote \( L^{\epsilon}(i, m) = L^{\epsilon}(z^i, m) \in L^{\epsilon}(\mathcal{A}) \) for \( i, m \in \mathbb{Z} \). Then
\[
[L^{\epsilon}(i, m), L^{\epsilon}(j, n)] = (j(m + 1 - \epsilon) - i(n + 1 - \epsilon)) L^{\epsilon}(i + j, m + n) \quad \text{for } i, j, m, n \in \mathbb{Z}.
\]
Note that \( L^0(\mathcal{A}) \) is isomorphic to the Poisson Lie algebra defined as in (4.14). On the other hand, \( L^1(\mathcal{A}) \) is isomorphic to the Lie algebra of area-preserving diffeomorphisms of
the two-torus investigated by V. Arnold in [1], which is generated by \( L_m^i \) with \( m, i \in \mathbb{Z} \), subject to relations

\[
[L_m^i, L_n^j] = (jm - in)L_{m+n}^{i+j} \quad \text{for } i, j, m, n \in \mathbb{Z}. \tag{4.39}
\]

It was also called the Virasoro-like algebra in [19]. Note that from [10], Lie algebra \( L^1(\mathcal{A}) \) is not isomorphic to \( L(\mathcal{A}) \).

**Proposition 4.40.** Let \( \mathcal{A} \) be a Novikov algebra with a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) satisfying (4.11). Set

\[
\tilde{L}^\epsilon(\mathcal{A}) = L^\epsilon(\mathcal{A}) \oplus \mathbb{C}c_\epsilon, \tag{4.41}
\]

where \( c_\epsilon \) is a nonzero element. For \( a \in \mathcal{A}, m \in \mathbb{Z} \), denote \( L^\epsilon(a, m) = a \otimes t^{m+1-\epsilon} \). Then \( \tilde{L}^\epsilon(\mathcal{A}) \) is a Lie algebra with

\[
[L^\epsilon(a, m), L^\epsilon(b, n)] = (m + 1 - \epsilon)L^\epsilon(ab, m + n) - (n + 1 - \epsilon)L^\epsilon(ba, m + n)
+ \frac{1}{12}(m + 1 - \epsilon)m(m - 1 + \epsilon)(a, b)\delta_{m+n,0}c_\epsilon \tag{4.42}
\]

for \( a, b \in \mathcal{A}, m, n \in \mathbb{Z} \), and with \( c_\epsilon \) central.

**Proof.** Define a bilinear form \( (\cdot, \cdot) \) on the Lie algebra \( L^\epsilon(\mathcal{A}) \) by

\[
(L^\epsilon(a, m), L^\epsilon(b, n)) = \frac{1}{12}(m + 1 - \epsilon)m(m - 1 + \epsilon)(a, b)\delta_{m+n,0}
\]

for \( a, b \in A, m, n \in \mathbb{Z} \). Notice that

\[
(m + 1 - \epsilon)m(m - 1 + \epsilon) = m(m^2 - (1 - \epsilon)^2),
\]

which is an odd function of \( m \). As \( (\cdot, \cdot) \) is symmetric, \( (\cdot, \cdot) \) is skew symmetric.

For cocycle condition, let \( a, b, c \in \mathcal{A}, m, n, k \in \mathbb{Z} \). We have

\[
\left( [L^\epsilon(a, m), L^\epsilon(b, n)], L^\epsilon(c, k) \right) = \frac{1}{12}(m + 1 - \epsilon)(m + n)((m + n)^2 - (1 - \epsilon)^2)(a, b)\delta_{m+n+k,0}
- \frac{1}{12}(n + 1 - \epsilon)(m + n)((m + n)^2 - (1 - \epsilon)^2)(b, a)\delta_{m+n+k,0}
= \frac{1}{12}(m - n)(m + n)((m + n)^2 - (1 - \epsilon)^2)(a, b)\delta_{m+n+k,0}
= \frac{1}{12}(m^2 - n^2)(k^2 - (1 - \epsilon)^2)(a, b)\delta_{m+n+k,0},
\]
where we used the property $\langle ab, c \rangle = \langle ba, c \rangle$. Furthermore, we have

$$\langle bc, a \rangle = \langle a, bc \rangle = \langle ab, c \rangle, \quad \langle ca, b \rangle = \langle c, ab \rangle = \langle ab, c \rangle,$$

and

$$(m^2 - n^2)(k^2 - (1 - \epsilon)^2) + (n^2 - k^2)(m^2 - (1 - \epsilon)^2) + (k^2 - m^2)(n^2 - (1 - \epsilon)^2) = 0.$$ 

Then the cocycle condition follows immediately. Therefore, $\tilde{L}^\epsilon(\mathcal{A})$ is a Lie algebra. \qed

Note that

$$\tilde{L}^0(\mathcal{A}) = \tilde{L}(\mathcal{A}).$$

For $a \in \mathcal{A}$, set

$$L^\epsilon(a, x) = \sum_{n \in \mathbb{Z}} L^\epsilon(a, n)x^{-n-2+2\epsilon} \in \tilde{L}^\epsilon(\mathcal{A})[[x, x^{-1}]]. \quad (4.43)$$

In terms of generating functions the relation (4.42) can be written as

$$[L^\epsilon(a, x_1), L^\epsilon(b, x_2)]$$

$$= \left(x_2^\epsilon \frac{\partial}{\partial x_2} L^\epsilon(ba, x_2)\right) x_1^{-1+\epsilon} \delta \left(\frac{x_2}{x_1}\right)$$

$$+ \left(L^\epsilon(ab, x_2) + L^\epsilon(ba, x_2)\right) \left(x_2^\epsilon \frac{\partial}{\partial x_2}\right) x_1^{-1+\epsilon} \delta \left(\frac{x_2}{x_1}\right)$$

$$+ \frac{1}{12} \langle a, b \rangle c_\epsilon \left(x_2^\epsilon \frac{\partial}{\partial x_2}\right)^3 x_1^{-1+\epsilon} \delta \left(\frac{x_2}{x_1}\right) \quad (4.44)$$

for $a, b \in \mathcal{A}$.

**Definition 4.45.** An $\tilde{L}^\epsilon(\mathcal{A})$-module on which $c_\epsilon$ acts as a scalar $\ell \in \mathbb{C}$ is said to be of level $\ell$. An $\tilde{L}^\epsilon(\mathcal{A})$-module $W$ is said to be restricted if $L^\epsilon(a, x)w \in W((x))$ for every $a \in \mathcal{A}$, $w \in W$. Denote by $L^\epsilon_W(a, x)$ the corresponding element of $\mathcal{E}(W)$.

As the main result of this section we have:

**Theorem 4.46.** Let $\ell \in \mathbb{C}$ and let $W$ be a restricted $\tilde{L}^\epsilon(\mathcal{A})$-module of level $\ell$. Then there exists a $\phi_\epsilon$-coordinated $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$-module structure on $W$, which is uniquely determined by $Y_W(a, x) = L^\epsilon_W(a, x)$ for $a \in \mathcal{A}$. On the other hand, let $(W, Y_W)$ be a $\phi_\epsilon$-coordinated $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$-module. Then $W$ is a restricted $\tilde{L}^\epsilon(\mathcal{A})$-module of level $\ell$, which is given by $L^\epsilon_W(a, x) = Y_W(a, x)$ for $a \in \mathcal{A}$. 

**Proof.** Assume that $W$ is a restricted $\tilde{L}^\ell(\mathcal{A})$-module of level $\ell$. Set

$$U_W = \text{span}\{L_W^a(a, x) \mid a \in \mathcal{A}\} \subset \mathcal{E}(W).$$

For $a, b \in \mathcal{A}$, from (4.16) we have

$$(x_1 - x_2)^4[L_W^a(a, x_1), L_W^b(x_2)] = 0.$$ 

Thus $U_W$ is local. By Theorem 3.12, $U_W$ generates a vertex algebra $\langle U_W \rangle_{\phi_x}$ and $W$ is a faithful $\phi_x$-coordinated $\langle U_W \rangle_{\phi_x}$-module with

$$Y_W(\alpha(x), z) = \alpha(z) \quad \text{for} \ \alpha(x) \in \langle U_W \rangle_{\phi_x}.$$ 

Using the commutation relation of $\tilde{L}^\ell(\mathcal{A})$ we have

$$[Y_W(L_W^a(a, x_1), x_1), Y_W(L_W^b(x_2), x_2)]$$

$$= [L_W^a(a, x_1), L_W^b(x_2)]$$

$$= (x_2^\epsilon \frac{\partial}{\partial x_2} L_W^a(ba, x_2)) x_1^{-1+\ell} \delta(x_2/x_1)$$

$$+ (L_W^a(ab + ba, x_2)) (x_2^\epsilon \frac{\partial}{\partial x_2}) x_1^{-1+\ell} \delta(x_2/x_1)$$

$$+ \frac{1}{12} \langle a, b \rangle \ell (x_2^\epsilon \frac{\partial}{\partial x_2}) x_1^2 \delta(x_2/x_1)$$

$$= (x_2^\epsilon \frac{\partial}{\partial x_2} Y_W(L_W^a(ba, x_2), x_2)) x_1^{-1+\ell} \delta(x_2/x_1)$$

$$+ Y_W(L_W^a(ab + ba, x_2), x_2) (x_2^\epsilon \frac{\partial}{\partial x_2}) x_1^{-1+\ell} \delta(x_2/x_1)$$

$$+ \frac{1}{12} \langle a, b \rangle \ell (x_2^\epsilon \frac{\partial}{\partial x_2}) x_1^2 \delta(x_2/x_1),$$  \hspace{1cm} (4.47)

In view of Lemma 3.29, we have

$$L_W^a(a, x) L_W^b(b, x) = D L_W^a(ba, x), \quad L_W^a(a, x) L_W^b(b, x) = L_W^a(ab + ba, x),$$

$$L_W^a(a, x) L_W^b(b, x) = \frac{1}{2} \ell(a, b) 1_W, \quad L_W^a(a, x) L_W^b(b, x) = 0$$

for $j = 2$ and for $j \geq 4$. Then by Theorem 3.19 we have

$$[Y^\ell_{\mathcal{E}}(L_W^a(a, x), x_1), Y^\ell_{\mathcal{E}}(L_W^b(x_2), x_2)]$$

$$= Y^\ell_{\mathcal{E}}(D L_W^a(ba, x), x_2) x_1^{-1} \delta(x_2/x_1) + Y^\ell_{\mathcal{E}}(L_W^a(ab + ba, x), x_2) \left( \frac{\partial}{\partial x_2} \right) x_1^{-1} \delta(x_2/x_1)$$

$$+ \frac{1}{12} \langle a, b \rangle \ell \left( \frac{\partial}{\partial x_2} \right) x_1^2 \delta(x_2/x_1)$$
for $a, b \in A$. This shows that $\langle U_W \rangle_{\phi}$ is an $\tilde{L}(A)$-module of level $\ell$ with $L(a, x_1)$ acting as $Y^\ell(A)(a, x_1)$ for $a \in A$ and

$$L(a, n)W = L^\ell_{\tilde{L}}(a, x_1)^nW = 0 \quad \text{for } a \in A, \ n \in \mathbb{N}.$$ 

From the construction of $V^\ell_{\tilde{L}}(A)\ell, 0)$, there exists an $\tilde{L}(A)$-module homomorphism $\rho$ from $V^\ell_{\tilde{L}}(A)\ell, 0)$ to $\langle U_W \rangle_{\phi}$, with $\rho(1) = 1_W$. That is,

$$\rho(Y(a, x_1)v) = Y^\ell(A)(a, x_1)\rho(v) \quad \text{for } a \in A, \ v \in V^\ell_{\tilde{L}}(A)\ell, 0).$$ 

Since the vertex algebra $\langle U_W \rangle_{\phi}$ is generated by $L^\ell_{\tilde{L}}(a, x)$ for $a \in A$, it follows that $\rho$ is a homomorphism of vertex algebras. As $W$ is a $\phi_\epsilon$-coordinated module for $\langle U_W \rangle_{\phi}$, $W$ is a $\phi_\epsilon$-coordinated $V^\ell_{\tilde{L}}(A)\ell, 0)$-module through homomorphism $\rho$. Therefore $W$ is a $\phi_\epsilon$-coordinated $V^\ell_{\tilde{L}}(A)\ell, 0)$-module.

On the other hand, let $(W, Y_W)$ be a $\phi_\epsilon$-coordinated $V^\ell_{\tilde{L}}(A)\ell, 0)$-module. Using the relations (4.18) and the commutator formula (3.21) for $\phi_\epsilon$-coordinated modules for vertex algebras in Theorem 3.19, we have

$$[Y_W(a, x_1), Y_W(b, x_2)] = Y_W(D(ba), x_2) x_1^{1+\epsilon} \delta(x_2/x_1) + Y_W(ab + ba, x_2) x_1^{1+\epsilon} \delta(x_2/x_1)$$

$$+ \frac{\ell}{12} \langle a, b \rangle x_2^\epsilon \frac{\partial}{\partial x_2} x_1^{1+\epsilon} \delta(x_2/x_1)$$

$$= \left( x_1^\epsilon \frac{\partial}{\partial x_1} \right) Y_W(ba, x_2) x_1^{1+\epsilon} \delta(x_2/x_1) + Y_W(ab + ba, x_2) x_1^{1+\epsilon} \delta(x_2/x_1)$$

$$+ \frac{\ell}{12} \langle a, b \rangle x_2^\epsilon \frac{\partial}{\partial x_2} x_1^{1+\epsilon} \delta(x_2/x_1)$$

for $a, b \in A$, where we use the fact (see [25])

$$Y_W(Dv, x) = \left( x^\epsilon \frac{\partial}{\partial x} \right) Y_W(v, x) \quad \text{for } v \in V^\ell_{\tilde{L}}(A)\ell, 0).$$

This proves that $W$ is a restricted module for $\tilde{L}(A)$ of level $\ell$. □

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Appendix A

We here establish some basic results we needed in the main body of the paper.

Lemma A. Let $V$ be a vertex algebra equipped with a $\mathbb{Z}$-grading $V = \bigoplus_{n \in \mathbb{Z}} V_n$. Suppose that $U$ is a graded subspace such that $V$ as a vertex algebra is generated by $U$ and

$$u_r V_n \subset V_{m + n + r - 1} \text{ for } u \in U \cap V_m, \ m, n, r \in \mathbb{Z}. \quad (A.1)$$

Then $V$ is a $\mathbb{Z}$-graded vertex algebra.

Proof. From the definition we need to prove that for every $v \in V_m$ with $m \in \mathbb{Z}$ and for every $n \in \mathbb{Z}$, $v_n$ is a homogeneous operator of degree $m - n - 1$. Let $K$ be the linear span of homogeneous vectors $v \in V$ such that

$$\deg v_n = \deg v - n - 1 \quad \text{for all } n \in \mathbb{Z}.$$ 

Now we must prove $K = V$. By assumption we have $U \subset K$ and it is clear that $1 \in K$. Recall the iterate formula: For $a, b \in V$, $m, n \in \mathbb{Z},$

$$(a_m b)_n = \sum_{i \geq 0} \binom{m}{i} (-1)^i (a_{m-i} b_{n+i} - (-1)^m b_{m+n-i} a_i). \quad (A.2)$$

It follows from this formula that $K$ is a graded vertex subalgebra. As $U$ generates $V$, we must have $K = V$. \qed

Lemma B. Let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ be a $\mathbb{Z}$-graded vertex algebra and an $sl_2$-module such that $sl_2 \cdot 1 = 0$. Suppose that $U$ is a graded subspace such that $V$ as a vertex algebra is generated by $U$ and

$$[L(-1), u_n] = -nu_{n-1}, \quad [L(0), u_n] = (\deg u - n - 1)u_n,$$

$$[L(1), Y(u, x)] = Y((L(1) + 2xL(0) + x^2L(-1))u, x)$$

for homogeneous $u \in U$ and for every integer $n$. Then $V$ is a Möbius vertex algebra.

Proof. Just as in the proof of Lemma A, using (A.2) we get $\deg v_n = \deg v - n - 1$ for every homogeneous vector $v \in V$ and for every integer $n$. It follows that $L(0)|_{V_m} = m$ for $m \in \mathbb{Z}$ as

$$L(0)v = L(0)v_{-1}1 = v_{-1}L(0)1 + mv_{-1}1 = mv \quad \text{for } v \in V_m.$$ 

Note that by assumption, we have $[L(-1), Y(u, x)] = \frac{d}{dx}Y(u, x)$ for $u \in U$. Assume
\[
[L(-1), Y(a, x)] = \frac{d}{dx} Y(a, x), \quad [L(-1), Y(b, x)] = \frac{d}{dx} Y(b, x)
\]
for some \( a, b \in V \). Using the vertex-operator form of (A.2), as in [24] (Lemma 3.1.8) we get

\[
[L(-1), Y(Y(a, x_0)b, x)] = \frac{\partial}{\partial x} Y(Y(a, x_0)b, x).
\]

Then it follows that \( [L(-1), Y(v, x)] = \frac{d}{dx} Y(v, x) \) for all \( v \in V \).

Next, we consider \( L(1) \). Set \( A(x) = L(1) + 2xL(0) + x^2L(-1) \). Suppose

\[
[L(1), Y(a, x)] = Y(A(x)a, x), \quad [L(1), Y(b, x)] = Y(A(x)b, x)
\]
for some \( a, b \in V \). Then

\[
[L(1), Y(Y(a, x_0)b, x_2)] = \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) [L(1), Y(a, x_1)Y(b, x_2)]
\]

\[
- \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) [L(1), Y(b, x_2)Y(a, x_1)]
\]

\[
= \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \left( Y(A(x_1)a, x_1)Y(b, x_2) + Y(a, x_1)Y(A(x_2)b, x_2) \right)
\]

\[
- \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \left( Y(A(x_2)b, x_2)Y(a, x_1) + Y(b, x_2)Y(A(x_1)a, x_1) \right)
\]

\[
= \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \left[ Y(Y(A(x_1)a, x_0)b, x_2) + Y(Y(a, x_0)A(x_2)b, x_2) \right]
\]

\[
= Y(Y(A(x_2 + x_0)a, x_0)b, x_2) + Y(Y(a, x_0)A(x_2)b, x_2).
\]

On the other hand, we have

\[
Y\left( (L(1) + 2x_2L(0) + x_2^2L(-1))Y(a, x_0)b, x_2 \right)
\]

\[
= Y(Y(a, x_0)L(1)b, x_2) + Y\left( (L(1) + 2x_0L(0) + x_0^2L(-1))a, x_0\right)b, x_2
\]

\[
+ 2x_2Y\left( Y(a, x_0)L(0)b, x_2 \right) + 2x_2Y\left( (L(0) + x_0L(-1))a, x_0 \right)b, x_2
\]

\[
+ x_2^2Y\left( Y(a, x_0)L(-1)b, x_2 \right) + x_2^2Y\left( Y(L(-1)a, x_0)b, x_2 \right)
\]

\[
= Y\left( (L(1) + 2(x_2 + x_0)L(0) + (x_2 + x_0)^2L(-1))a, x_0 \right)b, x_2
\]

\[
+ Y\left( Y(a, x_0)(L(1) + 2x_2L(0) + x_2^2L(-1))b, x_2 \right)
\]

\[
= Y(Y(A(x_2 + x_0)a, x_0)b, x_2) + Y(Y(a, x_0)A(x_2)b, x_2).
\]

Thus
\[ [L(1), Y(Y(a, x_0)b, x_2)] = Y(A(x_2)Y(a, x_0)b, x_2). \]

Then it follows that \([L(1), Y(v, x)] = Y(A(x)v, x)\) for all \(v \in V\). Therefore, \(V\) is a Möbius vertex algebra. \(\Box\)

**Lemma C.** Let \(\mathcal{A}\) be a commutative and associative algebra equipped with a symmetric associative bilinear form \(\langle \cdot, \cdot \rangle\) and let \(\tilde{L}(\mathcal{A}) = \mathcal{A} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c\) be the corresponding Lie algebra with \(c\) central and with

\[
[a \otimes t^m, b \otimes t^n] = (m-n)(ab \otimes t^{m+n-1}) + \frac{1}{12}m(m-1)(m-2)(a, b)\delta_{m+n-2,0}c
\]

for \(a, b \in \mathcal{A}, m, n \in \mathbb{Z}\). Then \(sl_2\) acts on \(\tilde{L}(\mathcal{A})\) as a Lie algebra of derivations with

\[
L(-1) \cdot (a \otimes t^n) = -\frac{d}{dt}(a \otimes t^n) = -n(a \otimes t^{n-1}),
\]

\[
L(0) \cdot (a \otimes t^n) = \left(1 - t \frac{d}{dt}\right)(a \otimes t^n) = (1-n)(a \otimes t^n),
\]

\[
L(1) \cdot (a \otimes t^n) = \left(2t - t^2 \frac{d}{dt}\right)(a \otimes t^n) = (2-n)(a \otimes t^{n+1})
\]

for \(a \in \mathcal{A}, n \in \mathbb{Z}\) and with \(sl_2 \cdot c = 0\).

**Proof.** First, we have the required commutation relations

\[
\begin{align*}
2t - t^2 \frac{d}{dt} - \frac{d}{dt} &= \left[ \frac{d}{dt} t, 2t - t^2 \frac{d}{dt}\right] = 2 - 2t \frac{d}{dt}, \\
1 - t \frac{d}{dt} - \frac{d}{dt} &= \left[ \frac{d}{dt} t, 1 - t \frac{d}{dt}\right] = -\frac{d}{dt}, \\
1 - t \frac{d}{dt} t - t^2 \frac{d}{dt} &= -\left[ \frac{d}{dt} t, t \left(2 - t \frac{d}{dt}\right)\right] = -2t \left(2 - t \frac{d}{dt}\right) = -\left(2t - t^2 \frac{d}{dt}\right).
\end{align*}
\]

For \(a, b \in \mathcal{A}, m, n \in \mathbb{Z}\), we have

\[
\begin{align*}
[L(-1)(a \otimes t^m), b \otimes t^n] + [a \otimes t^m, L(-1)(b \otimes t^n)]
&= -m[a \otimes t^{m-1}, b \otimes t^n] - n[a \otimes t^m, b \otimes t^{n-1}] \\
&= -m(m-1-n)(ab \otimes t^{m+n-2}) - \frac{1}{12}m(m-1)(m-2)(m-3)\delta_{m+n-3,0}(a, b)c \\
&\quad - n(m-n+1)(ab \otimes t^{m+n-2}) - \frac{1}{12}nm(m-1)(m-2)\delta_{m+n-3,0}(a, b)c \\
&= (m-n)(1-m-n)(ab \otimes t^{m+n-2}) \\
&= L(-1)(a \otimes t^m, b \otimes t^n),
\end{align*}
\]
\[
L(1)(a \otimes t^m), b \otimes t^n] + [a \otimes t^m, L(1)(b \otimes t^n)]
\]
\[
= (2 - m)[a \otimes t^{m+1}, b \otimes t^n] + (2 - n)[a \otimes t^m, b \otimes t^{n+1}]
\]
\[
= (2 - m)(m + 1 - n)(ab \otimes t^{m+n}) + \frac{1}{12}(2 - m)(m^3 - m)\delta_{m+n-1,0}(a, b)c
\]
\[
+ (2 - n)(m - n - 1)(ab \otimes t^{m+n}) + \frac{1}{12}(2 - n)m(m - 1)(m - 2)\delta_{m+n-1,0}(a, b)c
\]
\[
= (m - n)(3 - m - n)(ab \otimes t^{m+n})
\]
\[
= L(1)[a \otimes t^m, b \otimes t^n].
\]

This proves that \(L(-1)\) and \(L(1)\) act as derivations. As \([L(1), L(-1)] = 2L(0)\), \(L(0)\) also acts as a derivation. \(\Box\)

References


