On the local base of a primitive and nonpowerful sign pattern

Guanglong Yu\textsuperscript{a,b}, Zhengke Miao\textsuperscript{b,\*}

\textsuperscript{a} Department of Mathematics, Yancheng Teachers University, Yancheng, 224002, China
\textsuperscript{b} Department of Mathematics, Jiangsu Normal University, Xuzhou, 221116, China

\textbf{ARTICLE INFO}

Article history:
Received 14 January 2014
Received in revised form 17 February 2015
Accepted 25 February 2015

Keywords:
Bound
Sign pattern
Primitive and nonpowerful
Local base

\textbf{ABSTRACT}

Denote by $l_A(k)$ the $k$th smallest local base of a primitive and nonpowerful sign pattern $A$. In this paper, more "gaps" for the $k$th local base are shown, and the primitive and nonpowerful sign patterns with the $k$th local base in $[2n^2 - 8n + 9 + k, 2n^2 - 4n + k]$ are completely characterized.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

We adopt the standard conventions, notations and definitions for sign patterns and generalized sign patterns, their entries, arithmetics and powers. The reader who is not familiar with these matters is referred to [3,6–9].

In this paper, we permit loops but no multiple arcs in a digraph. We denote by $V(S)$ the vertex set and denoted by $E(S)$ the arc set for a digraph $S$. A digraph is called a signed digraph if each edge is assigned one of the signs $-1$ and 1. For an unsigned general digraph, we apply the sign 1 to each edge. Consequently, every digraph can be viewed as a signed digraph. In a signed digraph, the sign of a directed walk $W = v_0 e_1 v_1 e_2 \cdots e_k v_k$ ($e_i = (v_{i-1}, v_i)$, $1 \leq i \leq k$), denoted by $\text{sgn}(W)$, is $\prod_{i=1}^{k} \text{sgn}(e_i)$.

\textbf{Definition 1.1.} Let $A$ be a square sign pattern matrix of order $n$. The associated digraph of $A$, denoted by $D(A)$, has vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, j) | a_{ij} \neq 0\}$. The associated signed digraph of $A$, denoted by $S_A$, is obtained from $D(A)$ by assigning the sign of $a_{ij}$ to the arc $(i, j)$ for all $a_{ij} \neq 0$.

For a signed digraph $S$, the associated sign pattern matrix of $S$, denoted by $A_S$, is a square sign pattern matrix $A$ that $S_A = S$. The underlying graph of a signed digraph $S$, denote by $|S|$, is obtained by replacing the sign of each negative edge with sign 1. Clearly, for a signed digraph $S$, $|S| = D(A_S)$. Noting the relation between the sign patterns and the signed digraphs, from [3,7] or [9], we know that the signed digraphs can be used to study the bases of the sign patterns.

For a sign pattern matrix $A = (a_{ij})$, we denote by $|A|$ the nonnegative matrix (which is a $(0, 1)$-Boolean matrix) obtained from $A$ by replacing $a_{ij}$ with $|a_{ij}|$. Note that a $(0, 1)$-Boolean matrix is a special sign pattern. Denoted by $\text{exp}(S)$ the primitive
nonpowerful sign pattern. More "gaps" for the bases distribute in \[ k \] path and a directed cycle is abbreviated into a walk, a path and a cycle, respectively. We denote by \( l(S) \) the base of a primitive and nonpowerful signed digraph \( S \). If a signed digraph \( S \) is primitive and nonpowerful, for any \( u, v \in V(S) \), there exists an integer \( k \) such that for any integer \( t \geq k \), there is a directed walk of length \( t \) from \( u \) to \( v \). The least such integer \( k \) is called the local primitive index from \( v \) to \( v \), denoted by \( \exp_2(u, v) \). \( \exp_2(u, v) = \max_{s \in V(S)} \exp_2(u, v) \) is called the local primitive index at \( v \).

Let \( n \) be a positive integer, \( S \) be a primitive digraph of order \( n \) and \( V(S) = \{1, 2, \ldots , n \} \). The vertices can be ordered so that \( \exp_2(1) \leq \exp_2(2) \leq \cdots \leq \exp_2(n) \). We call \( \exp_2(k) \) the \( k \)th smallest primitive index of \( S \). From \[6\], we know that \( \exp_2(n) = \exp_2(k) \).

**Definition 1.2.** Assume that \( W_1, W_2 \) are two directed walks in a signed digraph \( S \). They are called a pair of SSD walks if they have the same initial vertex, the same terminal vertex and the same length, but they have different signs.

Denoted by \( l(S) \) the base of a primitive and nonpowerful signed digraph \( S \). If a signed digraph \( S \) is primitive and nonpowerful, for any \( u, v \in V(S) \), there exists an integer \( k \) such that there is a pair of SSD walks of length \( t \) from \( u \) to \( v \) for any integer \( t \geq k \). The least such \( k \) is called the local base from \( u \) to \( v \), denoted by \( l_2(u, v) \). \( l_2(u, v) = \max_{s \in V(S)} l_2(u, v) \) is called the local base at vertex \( u \).

Let \( n \) be a positive integer, \( S \) be a primitive and nonpowerful signed digraph of order \( n \) and \( V(S) = \{1, 2, \ldots , n \} \). The vertices can be ordered so that \( l_2(1) \leq l_2(2) \leq \cdots \leq l_2(n) \). We call \( l_2(k) \) the \( k \)th smallest base of \( S \). From \[6\], we know that \( l(S) = l_2(n) \). Denoted by \( l(A) \) the base of a primitive and nonpowerful sign pattern \( A \). From \[3,7,9\], we know that a sign pattern \( A \) is primitive and nonpowerful if and only if \( S_A \) is primitive and nonpowerful and \( l(A) = l(S_A) \). In \[6\], for a primitive and nonpowerful sign pattern \( A \), the local base \( l(A) \) is defined to be equal to \( l_2(k) \).

For a primitive and nonpowerful sign pattern, the local base (base) always seems to differ from its local primitive index (primitive index), and further investigation is needed (see \[3,6,7,9\]). In \[9\], the local base or base of a sign pattern was founded to be of great significance for communication science. Because the primitivity of a sign pattern is closely related to many other problems in various areas of pure and applied mathematics, it has been studied extensively (for example, see \[4,8\]).

The union of digraphs \( H \) and \( G \) is the digraph \( G \cup H \) with vertex set \( V(G) \cup V(H) \) and arc set \( E(G) \cup E(H) \). The standard \( n \)-cycle is defined to be \( C_n = (v_1, v_{n-1}, \ldots , v_2, v_1, v_n) \). Let \( D_1 = \{v \cup (v_1, v_{n-1}) \} \) and \( D_2 = D_1 \cup (v_2, v_1) \). In \[6\], Wang and Miao got the following result.

**Lemma 1.3** ([6]). Let \( A \) be a primitive and nonpowerful sign pattern of order \( n \).

(i) If \( D(A) \equiv D_1 \), then we have \( l_2(k) = 2n^2 - 4n + k + 2 \) for \( 1 \leq k \leq n \).

(ii) If \( D(A) \equiv D_2 \), then we have:

\( l_2(k) \leq \begin{cases} n^2 - 2n + 1 + k, & 1 \leq k \leq n - 1; \\ n^2 - n, & k = n. \end{cases} \)

(iii) (If the (only) two cycles of length \( n - 1 \) of \( S_A \) have the same signs, then \( l_2(k) = 2n^2 - 4n + k + 1 \) for \( 1 \leq k \leq n \).

From above narration, we see that the \( k \)th local base set of all primitive and nonpowerful sign patterns of order \( n \) consists of some positive integers. Denote by \([2n^2 - 4n + k + 2] \) the set of positive integers not more than \( 2n^2 - 4n + k + 2 \), where \( k \leq n \). From Lemma 1.3, it follows that for a primitive and nonpowerful sign pattern of order \( n \), its \( k \)th local base is an integer in \([2n^2 - 4n + k + 2] \). A natural problem is that for all primitive and nonpowerful sign patterns of order \( n \), how their \( k \)th local bases distribute in \([2n^2 - 4n + k + 2] \). From Lemma 1.3, it also follows that there is no sign pattern of order \( n \) \( \geq 2 \) such that its \( k \)th \( (1 \leq k \leq n - 1) \) local base is in \([n^2 - 2n + k + 1, 2n^2 - 4n + k + 2] \). We call such interval \([n^2 - 2n + k + 1, 2n^2 - 4n + k + 2] \) a "gap" for the \( k \)th local base, where \( 1 \leq k \leq n - 1, n \geq 2 \). In this paper, we consider the local base of a primitive and nonpowerful sign pattern. More "gaps" for the \( k \)th local base are shown, and the primitive and nonpowerful sign patterns with the \( k \)th local base in \([2n^2 - 8n + 9 + k, 2n^2 - 4n + k] \) are completely characterized.

2. Preliminaries

We first introduce some notations. Because there is no ambiguity in this paper, for convenience, a directed walk, a directed path and a directed cycle is abbreviated into a walk, a path and a cycle, respectively. We denote by \( d(v_1, v_2) \) the length of a walk, and denote by \( d(v_1, v_2) \) or \( d(v, v) \) the distance from \( v_1 \) to \( v_2 \) in a signed digraph \( S \). We denote by \( C_k \) or \( k \)-cycle a cycle with length \( k \), and denote by \( P_k \) a path of order \( k \). A cycle with even (odd) length is called an even cycle (odd cycle). The length of the shortest cycle in a digraph is called the girth of this digraph. In a signed digraph, a walk is called a positive (negative) walk if its sign is positive (negative). For a positive integer \( p \) and a cycle \( C \), we denote by \( pC \) the walk obtained by traversing through \( C \) times. If a cycle \( C \) passes through one end vertex of \( W, W \cup pC \) denotes the walk obtained by going along \( W \) and then going around the cycle \( Cp \) times; \( pC \cup W \) is similarly defined. We use the notation \( v \xrightarrow{k} u \) to denote that there exists a directed walk with length \( k \) from vertex \( v \) to \( u \). For a digraph \( S \), let \( R_k(v) = \{u | v \xrightarrow{k} u, u \in V(S)\} \).

**Definition 2.1.** Let \( \{s_1, s_2, \ldots , s_\lambda \} \) be a set of distinct positive integers with \( \gcd(s_1, s_2, \ldots , s_\lambda) = 1 \). The Frobenius number of \( \{s_1, s_2, \ldots , s_\lambda \} \) denoted by \( \phi(s_1, s_2, \ldots , s_\lambda) \), is the smallest nonnegative integer \( m \) such that for any nonnegative integers \( k \geq m \), there are nonnegative integers \( a_i (i = 1, 2, \ldots , \lambda) \) such that \( k = \sum_{i=1}^{\lambda} a_is_i \).
It is well known that if \( \gcd(s_1, s_2) = 1 \), then \( \phi(s_1, s_2) = (s_1 - 1)(s_2 - 1) \) (see [4], for example). From Definition 2.1, it is easy to see that if there exist \( s_i, s_j \in \{s_1, s_2, \ldots, s_k\} \) such that \( \gcd(s_i, s_j) = 1 \), then \( \phi(s_1, s_2, \ldots, s_i) \leq \phi(s, s_j) \). If \( \min\{s_i | 1 \leq i \leq \lambda\} = 1 \), then \( \phi(s_1, s_2, \ldots, s_k) = 0 \).

For a strongly connected digraph \( S \) with order \( n \), let \( C(S) \) denote the cycle length set.

**Lemma 2.2** ([2]). A digraph \( S \) with \( C(S) = \{p_1, p_2, \ldots, p_k\} \) is primitive if and only if \( S \) is strongly connected and \( \gcd(p_1, p_2, \ldots, p_k) = 1 \).

For a primitive digraph \( S \), suppose \( C(S) = \{p_1, p_2, \ldots, p_n\} \). Let \( d_{C(S)}(v_i, v_j) \) denote the length of the shortest walk from \( v_i \) to \( v_j \) which meets at least one \( p_i \)-cycle for each \( i, i = 1, 2, \ldots, n \). Such a shortest directed walk is called a \( C(S) \)-walk from \( v_i \) to \( v_j \). Further, \( d_{C(S)}(v_i) \) and \( d(C(S)) \) are defined as follows:

\[
\begin{align*}
&d_{C(S)}(v_i) = \max\{d_{C(S)}(v_i, v_j) | v_j \in V(S)\}, \\
&d(C(S)) = \max\{d_{C(S)}(v_i, v_j) | v_i, v_j \in V(S)\}.
\end{align*}
\]

**Lemma 2.3** ([5]). Let \( S \) be a primitive digraph of order \( n \) with \( |C(S)| \geq 3 \). Then \( \exp(k) \leq \left\lfloor \frac{1}{2}(n - 2)^2 \right\rfloor + k \) for \( 1 \leq k \leq n \).

**Lemma 2.4** ([10]). Let \( S \) be a primitive digraph of order \( n \) that has an \( s \)-cycle \( C \), \( v \in V(C) \), and \( |R_1(v)| \geq 2 \). Then \( \exp(1) \leq \exp(v) \leq 1 + s(n - 2) \).

**Lemma 2.5** ([7]). Let \( S \) be a primitive and nonpowerful signed digraph. Then \( S \) must contain a \( p_1 \)-cycle \( C_1 \) and a \( p_2 \)-cycle \( C_2 \) satisfying one of the following two conditions:

1. \( p_1 \) is odd, \( p_2 \) is even and \( \text{sgn}C_2 = -1 \);
2. \( p_1 \) and \( p_2 \) are both odd and \( \text{sgn}C_1 = -\text{sgn}C_2 \).

\( C_1, C_2 \) satisfying condition (1) or (2) are always called a distinguished cycle pair. From Lemma 2.5, it follows that in a primitive and nonpowerful signed digraph, if \( p_1 \)-cycle \( C_1 \) and \( p_2 \)-cycle \( C_2 \) form a distinguished cycle pair, then \( \text{sgn}C_1 \) is \( (\text{sgn}C_2)^2 = -\text{sgn}C_2 \), and then the walks \( W_1 = p_2C_1 \) and \( W_2 = p_1C_2 \) have the same length \( p_1p_2 \) but differ in signs.

**Lemma 2.6** ([9]). Let \( S \) be a primitive signed digraph. Then \( S \) is nonpowerful if and only if \( S \) contains a distinguished cycle pair.

**Lemma 2.7** ([1]). Let \( S \) be a primitive digraph of order \( n \) and \( C(S) = \{p_1, p_2, \ldots, p_n\} \). Then \( \exp(v_i, v_j) \leq d_{C(S)}(v_i, v_j) + \phi(p_1, p_2, \ldots, p_n) \) for \( v_i, v_j \in V(S) \). Furthermore, we have \( \exp(S) \leq d(C(S)) + \phi(p_1, p_2, \ldots, p_n) \).

**Lemma 2.8** ([9]). Let \( S \) be a primitive and nonpowerful signed digraph of order \( n \) with \( C(S) = \{p_1, p_2, \ldots, p_n\} \). If all cycles in \( S \) have the same length, and if every pairing of a \( p_1 \)-cycle with a \( p_2 \)-cycle form a distinguished cycle pair, then

(i) \( l_S(v_i, v_j) \leq d_{C(S)}(v_i, v_j) + \phi(p_1, p_2, \ldots, p_m) + p_1p_2 \), \( v_i, v_j \in V(S) \).

(ii) \( l_S(v_i) \leq d_{C(S)}(v_i) + \phi(p_1, p_2, \ldots, p_m) + p_1p_2 \).

**Lemma 2.9** ([6]). Let \( S \) be a primitive and nonpowerful signed digraph of order \( n \).

(i) For \( u \in V(S) \), if there exists a pair of \( SSSD \) walks with length \( r \) from \( u \) to \( u \), then \( l_S(u) \leq \exp_5(u) + r \).

(ii) For \( S \), we have \( l_S(k) \leq l_S((k - 1) + 1 \) for \( 2 \leq k \leq n \).

**Lemma 2.10** ([11]). Let \( S \) be a primitive and nonpowerful signed digraph of order \( n \geq 6 \). If \( C(S) = \{p, q\} \) \((p < q \leq n, p + q > n)\) and if all cycles with the same length have the same sign in \( S \), then \( p(2q - 1) \leq l(S) \leq 2p(q - 1) + n \).

3. Bounds of the local bases

**Theorem 3.1.** Let \( S \) be a primitive and nonpowerful signed digraph of order \( n \geq 6 \). If \( |C(S)| \geq 3 \), then \( l_S(k) \leq \frac{1}{2}n^2 - 3n + k + 3 \) for \( 1 \leq k \leq n \).

**Proof.** By Lemma 2.5, there exists a distinguished cycle pair \( p_1 \)-cycle \( C_1 \) and \( p_2 \)-cycle \( C_2 \) in \( S \).

**Case 1** \( C_1, C_2 \) have no common vertex. Then \( p_1 + p_2 \leq n \). Suppose \( p_1 \leq \frac{n}{2} \), \( Q_1 \) is a shortest walk with length \( q_1 \) from \( C_1 \) to \( C_2 \), \( \{v_1\} = V(Q_1) \cap V(C_1) \), \( \{v_2\} = V(Q_2) \cap V(C_2) \) and \( Q_2 \) is a shortest walk with length \( q_2 \) from \( v_2 \) to \( v_1 \). Then \( q_1 \leq n - p_1 - p_2 + 1, q_2 \leq n - 1, p_2C_1 \cup Q_1 \cup Q_2 \) and \( Q_1 \cup p_1C_2 \cup Q_2 \) are a pair of \( SSSD \) walks with length \( p_1p_2 + q_1 + q_2 \) from \( v_1 \) to \( v_1 \).

Note that

\[
\begin{align*}
p_1p_2 + q_1 + q_2 &\leq p_1p_2 + 2n - p_1 - p_2 = (p_1 - 1)(p_2 - 1) + 2n - 1 \\
&\leq \left[ \frac{1}{2}(p_1 + p_2 - 2) \right]^2 + 2n - 1 \\
&\leq \left[ \frac{1}{2}(n - 2) \right]^2 + 2n - 1 = \frac{n^2}{4} + n.
\end{align*}
\]
By Lemma 2.4, \( \exp(v_1) \leq p_1(n - 2) + 1 \). Then
\[
l_5(1) \leq l_5(v_1) \leq \exp_2(v_1) + p_1p_2 + q_1 + q_2 \leq \frac{n}{2}(n - 2) + 1 + \frac{n^2}{4} + n = \frac{3n^2}{4} + 1.
\]

By Lemma 2.9, for \( 1 \leq k \leq n \), \( l_5(k) \leq l_5(1) + k - 1 \leq \frac{3n^2}{4} + k \).

**Case 2** \( C_1, C_2 \) have common vertices.

**Subcase 2.1** \( p_1 = p_2 \). It is easy to see that \( p_1 \) is odd.

1° \( p_1 = n \). Suppose \( \exp_2(u) = \exp_2(1) \) in \( S \). Note that \( |C(S)| \) \( \geq 3 \). By Lemma 2.3, \( \exp_2(u) = \frac{1}{2}(n - 2)^2 + 1 \). Note that \( C_1 \) and \( C_2 \) form a pair of SSSD walks from \( u \) to itself now. By Lemma 2.9, \( l_5(1) \leq l_5(u) \leq n + \frac{1}{2}(n - 2)^2 + 1 = \frac{1}{2}n^2 - n + 3 \), and for \( 1 \leq k \leq n \), \( l_5(k) \leq l_5(1) + k - 1 \leq \frac{1}{2}n^2 - n + k + 2 \).

2° \( p_1 \neq n - 1 \). Suppose \( u \in V(C_1) \cap V(C_2) \) and \( |R_1(u)| \) \( \geq 2 \). By Lemmas 2.4 and 2.9, it follows that
\[
\exp_3(u) \leq p_1(n - 2) + 1 \leq n^2 - 3n + 3, \quad l_5(1) \leq l_5(u) \leq p_1 + \exp_3(u) \leq n^2 - 2n + 2,
\]
and for \( 1 \leq k \leq n \), \( l_5(k) \leq l_5(1) + k - 1 \leq n^2 - 2n + k + 1 \).

**Subcase 2.2** \( \min(p_1, p_2) = p_1 \leq n - 2 \). Suppose \( V(C_1) \cap V(C_2) = \{ v_1, v_2, \cdots, v_t \} \) and suppose \( \exp_3(u) = \exp_3(1) \) in \( S \). Note that \( |C(S)| \) \( \geq 3 \). By Lemma 2.3, \( \exp_2(u) \leq \lfloor \frac{1}{2}(n - 2)^2 \rfloor + 1 \). Let \( q_i = d(u, v_i) \), \( 1 \leq i \leq t \). Suppose \( q_1 = \min_{1 \leq i \leq t} \{ q_i \} \). Then \( q_1 \leq n - (p_1 + p_2 - t) + p_2 - t = n - p_1 \). Note that there exists a pair of SSSD walks with length \( q_1 + d(v_1, u) + p_1p_2 \) from \( u \) to \( u \) and
\[
d(v_1, u) \leq n - 1, \quad q_1 + d(v_1, u) + p_1p_2 \leq 2n - 1 + p_1(p_2 - 1) \leq 2n - 1 + (n - 2)(n - 1) \leq n^2 - n + 1.
\]

By Lemma 2.9, it follows that
\[
l_5(1) \leq l_5(u) \leq q_1 + d(v_1, u) + p_1p_2 + \exp_3(u) \leq \left[ \frac{1}{2}(n - 2)^2 \right] + 1 + n^2 + n - 1 \leq \frac{3n^2}{2} - 3n + 4,
\]
and for \( 1 \leq k \leq n \), \( l_5(k) \leq l_5(1) + k - 1 \leq n^2 - 3n + k + 3 \).

**Subcase 2.3** \( \{ p_1, p_2 \} = \{ n - 1, n \} \). Let \( C_1 = C_{n-1} \), \( C_2 = C_n \). Suppose \( \exp_3(u) = \exp_3(1) \). By Lemma 2.3, \( \exp_2(u) \leq \frac{1}{2}(n - 2)^2 + 2 \). Note that if \( u \in V(C_1) \), there exists a pair of SSSD walks with length \( n(n - 1) \) from \( u \) to \( u \). By Lemma 2.9, it follows that
\[
l_5(1) \leq l_5(u) \leq \frac{(n - 2)^2}{2} + 1 + n(n - 1) = \frac{3n^2 - 6n + 2}{2} + 4,
\]
and for \( 1 \leq k \leq n \), \( l_5(k) \leq \frac{3n^2 - 6n + 2}{2} + k + 3 \).

**Corollary 3.2.** Let \( S \) be a primitive and nonpowerful signed digraph with order \( n \geq 6 \). If there exists some \( k (1 \leq k \leq n) \) such that \( l_5(k) \geq \frac{3}{4}n^2 - 3n + k + 4 \), then \( |C(S)| = 2 \).

**Theorem 3.3.** Let \( S \) be a primitive and nonpowerful signed digraph with order \( n \geq 6 \). Let cycle \( C_1 \) with length \( p_1 \) and cycle \( C_2 \) with length \( p_2 \) form a distinguished cycle pair \( (p_1, \leq p_2) \). If \( p_1 + p_2 \leq n \), then \( l_5(k) \leq \frac{3}{4}n^2 + k \) for \( 1 \leq k \leq n \).

**Proof.** **Case 1** \( C_1 \) and \( C_2 \) have no common vertex. Similarly as proved in case 1 of Theorem 3.1, \( l_5(k) \leq \frac{3}{4}n^2 + k \) is obtained.

**Case 2** \( C_1 \) and \( C_2 \) have common vertex. Similarly as proved in case 1 of Theorem 3.1, \( l_5(k) \leq \frac{3}{4}n^2 + k \) is obtained.

**Subcase 2.1** If \( p_1 = p_2 \), then \( p_1 \leq \frac{n}{4} \). Let \( v_1 \in V(C_1) \cap V(C_2) \) satisfy that \( |R_1(v_1)| \geq 2 \). By Lemmas 2.4 and 2.9, it follows that
\[
\exp_3(v_1) \leq p_1(n - 2) + 1 \leq \frac{1}{2}n^2 - n + 1, \quad l_5(1) \leq l_5(v_1) \leq p_1 + \exp_3(v_1) \leq \frac{1}{2}n^2 - \frac{1}{2}n + 1,
\]
and for \( 1 \leq k \leq n \), \( l_5(k) \leq l_5(1) + k - 1 \leq \frac{1}{2}n^2 - \frac{1}{2}n + k \).

**Subcase 2.2** If \( p_1 < p_2 \), then \( p_1 < \frac{n}{4} \). Let \( v_1 \in V(C_1) \cap V(C_2) \) satisfy \( |R_1(v_1)| \geq 2 \). Similar to Subcase 2.1, noting that there is a pair of SSSD walks with length \( p_1p_2 \) from \( v_1 \) to itself and \( p_1p_2 \leq \left( \frac{p_1 + p_2}{2} \right)^2 \), we get
\[
l_5(v_1) \leq p_1p_2 + \exp_3(v_1) \leq \frac{3}{4}n^2 - n + 1, \quad l_5(k) \leq \frac{3}{4}n^2 - n + k (1 \leq k \leq n).
\]
Corollary 3.4. Let $S$ be a primitive and nonpowerful signed digraph of order $n \geq 6$. Let cycle $C_1$ with length $p_1$ and cycle $C_2$ with length $p_2$ form a distinguished cycle pair $(p_1 \leq p_2)$. If there exists some $k (1 \leq k \leq n)$ such that $l_5(k) \geq \frac{3}{2}n^2 + k + 1$, then $p_1 + p_2 > n$.

Theorem 3.5. Let $S$ be a primitive and nonpowerful signed digraph with order $n \geq 3$. If there exist two cycles having the same length but different signs, then $l_5(k) \leq n^2 - n + k$ for $1 \leq k \leq n$.

Proof. Suppose two cycles $C_1$ and $C_2$ satisfying $|l(C_1)| = p = l(C_2)$ but $\text{sgn}(C_1) = -\text{sgn}(C_2)$.

Case 1 $C_1$ and $C_2$ have common vertex. Let $v_1 \in V(C_1) \cap V(C_2)$ satisfy that $|R(t(v_1))| \geq 2$. Noting that $p \leq n$ and there is a pair of SSSD walks with length $p$ from $v_1$ to itself, similar to Subcase 2.1 in the proof of Theorem 3.3, we get $l_5(v_1) \leq n^2 - n + 1$ and $l_5(k) \leq n^2 - n + k$ for $1 \leq k \leq n$.

Case 2 $C_1$ and $C_2$ have no common vertex. Then $p \leq \frac{n}{2}$. Let $Q_1$ be a shortest walk with length $q_1$ from $C_1$ to $C_2$, $\{v_1\} = V(Q_1) \cap V(C_1)$, $\{v_2\} = V(Q_1) \cap V(C_2)$, and let $Q_2$ be a shortest walk with length $q_2$ from $v_2$ to $v_1$. Then $q_1 \leq n - 2p + 1$, $q_2 \leq n - 1$. $C_1 + Q_1 + Q_2$ and $Q_1 + C_2 + Q_2$ are a pair of SSSD walks with length $p + q_1 + q_2$ from $v_1$ to $v_1$. Similar to Subcase 2.1 in the proof of Theorem 3.3, we get $\exp_5(v_1) \leq p(n - 2) + 1$, $l_5(v_1) \leq \frac{n^2}{2} + \frac{n}{2} + 1$, $l_5(k) \leq \frac{n^2}{2} + \frac{n}{2} + k (1 \leq k \leq n)$. \hfill \Box

Corollary 3.6. Let $S$ be a primitive and nonpowerful signed digraph of order $n \geq 6$. If there exists some $k (1 \leq k \leq n)$ such that $l_5(k) \geq n^2 - n + 1$, then any two cycles with the same length have the same sign.

Theorem 3.7. Let $A$ be a primitive and nonpowerful sign pattern with order $n \geq 6$. If there exists some $k (1 \leq k \leq n)$ such that $l_4(k) \geq \frac{3}{2}n^2 - 3n + k + 4$, then $|C(S_0)| = 2$. Suppose $C(S_0) = \{p_1, p_2\} (p_1 < p_2)$. Then

(i) $\gcd(p_1, p_2) = 1, p_1 + p_2 > n$;
(ii) in $S_0$, all $p_1$-cycles have the same sign, all $p_2$-cycles have the same sign, and every $p_1$-cycle with every $p_2$-cycle form a distinguished cycle pair.

Proof. This theorem follows from Corollaries 3.2, 3.4 and 3.6. \hfill \Box

Let $n, n - k$ be two positive integers satisfying $\gcd(n, n - k) = 1$, and let $D_{k,i} = \bigvee_{l=1}^{\min(k+1, n-k-1)}(v_l, v_{n-k+l-1})$ (see Fig. 3.1). Let $S_{k,i}$ be a primitive and nonpowerful signed digraph with underlying digraph $D_{k,i} (1 \leq i \leq \min(k+1, n-k-1))$ in which all $(n - k)$-cycles have the same sign.

Lemma 3.8. (i) For $1 \leq m \leq n, \exp_{D_{k,i}}(m) = \exp_{D_{k,i}}(v_m) = (n - 2)(n - k) + 1 - i + m$;
(ii) For $1 \leq m \leq n, l_{k,i}(m) = l_{k,i}(v_m) = (2n - 2)(n - k) + 1 - i + m$.

Proof. (i) It is easy to see $D_{k,i}$ is primitive by Lemma 2.2. Also, it is not difficult to check that

$$R_{n-k-(i-2)}(v_1) \supseteq \begin{cases} \{v_n, v_{n-k}, v_k\}, & i = 1; \\ \{v_{n-i}, v_{n-k+i-1}, v_{k+i-1}\}, & 2 \leq i \leq \min(k+1, n-k-1). \end{cases}$$

We claim that if $|\bigcup_{l=1}^{i-1} R_{n-k-(i-2)}(v_1)| < n$, then $|\bigcup_{l=1}^{i-1} R_{n-k-(i-2)}(v_1) \setminus \bigcup_{l=1}^{i-1} R_{n-k-(i-2)}(v_1)| \geq 1$. Otherwise, $|\bigcup_{l=1}^{\infty} R_{n-k-(i-2)}(v_1)| < n$, which contradicts that $D_{k,i}$ is primitive. Because $|R_{n-k-(i-2)}(v_1)| \geq 3$, we get $|\bigcup_{l=1}^{\infty} R_{n-k-(i-2)}(v_1)| = n$. As a result, $\exp_{D_{k,i}}(v_1) \leq (n - 2)(n - k) + 2 - i$.

Case 1 $i - 1 < k$. Then $n - k + i - 1 < n$ and $d(C(D_{k,i})) = d_{C(D_{k,i})}(v_n, v_{n-k+i}) = n + k - i$. By Lemma 2.7, we get $\exp(D_{k,i}) \leq d(C(D_{k,i})) + \phi(n, n-k) = d_{C(D_{k,i})}(v_n, v_{n-k+i}) + (n-1)(n-k-1)$

$$= n + k - i + (n-1)(n-k-1).$$
We claim that there is no directed walk of length $n + k - i + \phi(n, n - k) - 1$ from $v_i$ to $v_{n-k+i}$. Otherwise, suppose $W$ is a directed walk of length $n + k - i + \phi(n, n - k) - 1$ from $v_i$ to $v_{n-k+i}$. Let $\mathcal{P}$ denote the path from $v_i$ to $v_{n-k+i}$ on cycle $\mathcal{C}$. Then $\mathcal{P} \cup \mathcal{C}$ meet only $n$-cycle not any $(n-k)$-cycle. $W$ must contain $\mathcal{P} \cup \mathcal{C}$, some $(n-k)$-cycles and some $n$-cycles, that is,

$$n + k - i + \phi(n, n - k) - 1 = k - i + n + a_1n + a_2(n - k) \geq 0, \quad j = 1, 2$$

and $\phi(n, n - k) - 1 = a_1n + a_2(n - k)$, which contradicts the definition of $\phi(n, n - k)$. Then our claim holds. Moreover, we have

$$\exp(D_{k,i}) = \exp(D_{k,i}(v_n) = \exp(D_{k,i}(v_n, v_{n-k+i})) = n + k - i + (n - 1)(n - k - 1).$$

Note that for $1 \leq m \leq n$, $\exp(D_{k,i}(v_m) = \exp(D_{k,i}(v_1)) + m - 1$, and note that

$$n + k - i + (n - 1)(n - k - 1) - ((n - 2)(n - k) + 2 - i) = n - 1.$$

Hence $\exp(D_{k,i}(v_1)) \geq \exp(D_{k,i}(v_n)) = (n - 1)(n - k) - 1 - (n - 2)(n - k) + 2 - i.\] Combining with above conclusion that $\exp(D_{k,i}(v_1)) \leq \exp(D_{k,i}(v_n))$, we have $\exp(D_{k,i}(v_1)) = (n - 2)(n - k) + 2 - i$, and then we have $\exp(D_{k,i}(m)) = \exp(D_{k,i}(v_1)) = (n - 2)(n - k) + 1 + i + m$ for $1 \leq m \leq n$.

**Case 2** $k = i - 1$. Then $n - k + (i - 1) = n$ and $d(C(D_{k,i})) = d(C(D_{k,i})) = n - 1$. Analogous to Case 1, we get

$$\exp(D_{k,i}) = \exp(D_{k,i}(v_n) = \exp(D_{k,i}(v_1) = (n - 1)(n - k),$$

and $\exp(D_{k,i}(m)) = \exp(D_{k,i}(v_m)) = (n - 2)(n - k) + 1 + i + m$ for $1 \leq m \leq n$.

(ii) Because $S_{k,i}$ is a primitive and nonpowerful signed digraph, every pair of $(n-k)$-cycle and $n$-cycle form a distinguished cycle pair. By Lemma 2.9 and (i),

$$l_{k,i}(v_1) \leq \exp(D_{k,i}(v_1) + (n - n) = (2n - 2)(n - k) + 2 - i.$$
Lemma 3.9. (1) For $1 \leq k \leq n$, $\exp_{\mathcal{S}}(k) = \exp_{\mathcal{S}}(v_k) = n^2 - 4n + 2 + k$.

(2) For $G \in \{F_0, F_2\}$, $\exp_{\mathcal{C}}(k) = \exp_{\mathcal{C}}(v_k) = \begin{cases} n^2 - 5n + 7 + k, & \text{if } 1 \leq k \leq n - 2; \\ n^2 - 5n + 6 + k, & \text{if } n - 1 \leq k \leq n. \end{cases}$

(3) For $1 \leq k \leq n$, $\exp_{\mathcal{F}_1}(k) = \exp_{\mathcal{F}_1}(v_k) = n^2 - 5n + 6 + k$.

(4) For $G \in \{F_3, F_5\}$, $\exp_{\mathcal{C}}(k) = \exp_{\mathcal{C}}(v_k) = \begin{cases} n^2 - 5n + 6 + k, & \text{if } 1 \leq k \leq n - 1; \\ n^2 - 4n + 5, & \text{if } k = n. \end{cases}$

(5) For $2 \leq i \leq n - 3$, $\exp_{\mathcal{F}_i}(k) = \exp_{\mathcal{F}_i}(v_k) = \begin{cases} \exp_{\mathcal{F}_i}(v_k) = n^2 - 5n + 6 + k, & \text{if } 1 \leq k \leq i; \\ \exp_{\mathcal{F}_i}(v_{k-1}) = n^2 - 5n + 5 + k, & \text{if } i + 2 \leq k \leq n. \end{cases}$

(6) For $G \in \{F_4, F_6\}$, $\exp_{\mathcal{C}}(k) = \exp_{\mathcal{C}}(v_k) = n^2 - 5n + 5 + k$ for $1 \leq k \leq n$.

(7) $\exp_{\mathcal{F}_0}(k) = \exp_{\mathcal{F}_0}(v_k) = n^2 - 5n + 5 + k$ for $1 \leq k \leq n$.

(8) $\exp_{\mathcal{B}_1}(k) = \exp_{\mathcal{B}_1}(v_k) = n^2 - 5n + 4 + k$ for $1 \leq k \leq n$.

(9) For $G \in \{\mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4\}$, $\exp_{\mathcal{C}}(v_k) = n^2 - 5n + 3 + k$ for $1 \leq k \leq n$.

Proof. (1) It is not difficult to check that $R_{n-3}(v_1) = \{v_2, v_4, v_6\}$. Similar to the proof of Lemma 3.8, we can prove $|\bigcup_{t=0}^{n-3} R_{t(n-2)+n-3}(v_1)| = n$, $\exp(v_1) = (n - 1)(n - 3)$ and

$$\exp(k) = \exp(v_k) = (n - 1)(n - 3) + k - 1 (1 \leq k \leq n).$$

(2) For $F_0$, it is not difficult to check that $R_{t(n-2)+2}(v_1) = \begin{cases} \{v_{n-1}, v_{n-3}\}, & t = 0; \\ \{v_{n-2}, v_{n-3}\}, & t = 1. \end{cases}$

Noting that $|R_2(v_1) \cup R_{n-2}+2(v_1)| = 4$, similar to the proof of Lemma 3.8, we can prove that $|\bigcup_{t=0}^{n-3} R_{t(n-2)+2}(v_1)| = n$, and get that $\exp_{\mathcal{F}_0}(v_1) = (n - 3)(n - 2) + 2$ and

$$\exp_{\mathcal{F}_0}(m) = \exp_{\mathcal{F}_0}(v_m) = \begin{cases} n^2 - 5n + 7 + m, & \text{if } 1 \leq m \leq n - 2; \\ n^2 - 5n + 6 + m, & \text{if } n - 1 \leq m \leq n. \end{cases}$$

In a same way, we can prove that (2) holds for $F_2$ and prove that (3)–(9) hold. □

Lemma 3.10. (1) For $1 \leq k \leq n$, $l_{\mathcal{S}}(k) = l_{\mathcal{S}}(v_k) = 2n^2 - 6n + k + 2$.

(2) For $S \in \{\mathcal{S}_0, \mathcal{S}_2\}$, $l_S(k) = l_S(v_k) = \begin{cases} 2n^2 - 8n + 9 + k, & 1 \leq k \leq n - 2; \\ 2n^2 - 8n + 8 + k, & n - 1 \leq k \leq n. \end{cases}$

(3) $l_{\mathcal{S}_1}(k) = l_{\mathcal{S}_1}(v_k) = 2n^2 - 8n + 8 + k (1 \leq k \leq n)$.

(4) For $S \in \{\mathcal{S}_3, \mathcal{S}_5\}$, $l_S(k) = l_S(v_k) = \begin{cases} 2n^2 - 8n + 8 + k, & 1 \leq k \leq n - 1; \\ 2n^2 - 7n + 7, & k = n. \end{cases}$

(5) For $S \in \{\mathcal{S}_4, \mathcal{S}_6\}$, $l_S(k) = \begin{cases} l_S(v_k) = 2n^2 - 8n + 8 + k, & 1 \leq k \leq n - 2; \\ l_S(v_n) = 2n^2 - 7n + 6, & k = n - 1; \\ l_S(v_{n-1}) = 2n^2 - 7n + 7, & k = n. \end{cases}$

(6) For $1 \leq k \leq n$, $l_{\mathcal{S}_2}(k) = l_{\mathcal{S}_2}(v_k) = 2n^2 - 8n + 8 + k$.

(7) For $2 \leq i \leq n - 3$, $l_{\mathcal{S}_i}(k) = \begin{cases} l_{\mathcal{S}_i}(v_k) = 2n^2 - 8n + 8 + k, & 1 \leq k \leq i; \\ l_{\mathcal{S}_i}(v_{k-1}) = 2n^2 - 8n + 7 + k, & i + 2 \leq k \leq n. \end{cases}$

(8) For $1 \leq k \leq n$, $l_{\mathcal{S}_1}(k) = l_{\mathcal{S}_1}(v_k) = 2n^2 - 8n + 4 + k$.

(9) For $S \in \{\mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4\}$, $l_S(k) = l_S(v_k) = 2n^2 - 8n + 3 + k (1 \leq k \leq n)$.

Proof. (1) By Lemma 3.9, we get $l_{\mathcal{S}}(v_1) \leq (n - 1)(n - 3) + n(n - 2)$ and $l_{\mathcal{S}}(v_n) \leq l_{\mathcal{S}}(v_1) + d(v_n, v_1) \leq (2n - 1)(n - 2)$. As Case 1 in Lemma 3.8, we can prove

$$l_{\mathcal{S}}(v_n) = l_{\mathcal{S}}(v_1, v_1) = (2n - 1)(n - 2), \quad l_{\mathcal{S}}(v_1) = (n - 1)(n - 3) + n(n - 2)$$

and $l_{\mathcal{S}}(k) = l_{\mathcal{S}}(v_k) = 2n(n - 3) + k + 2 (1 \leq k \leq n)$. In a same way, we can prove (2)–(14) □

Theorem 3.11. Let $S$ be a primitive and nonpowerful signed digraph with order $n \geq 14$.

(1) For any positive integer $n$, for $1 \leq k \leq n$, there is no $S$ such that $l_S(k) \in [2n^2 - 9n + 13, 2n^2 - 8n + 2 + k]$.

(2) If $n$ is even, then there is no $S$ such that $l_S(k) \in [2n^2 - 8n + 10 + k, 2n^2 - 4n + k]$ for $1 \leq k \leq n - 2$, and no $S$ such that $l_S(k) \in [2n^2 - 8n + 9 + k, 2n^2 - 4n + k]$ for $n - 1 \leq k \leq n$. 

(3) If \( n \) is odd, for \( 1 \leq k \leq n \), then there is no \( S \) such that \( l_5(k) \in [2n^2 - 6n + 5 + k, 2n^2 - 4n + k] \); for \( 1 \leq k \leq n - 2 \), there is no \( S \) such that \( l_5(k) \in [2n^2 - 8n + 10 + k, 2n^2 - 6n + k + 1] \); for \( n - 1 \leq k \leq n \), there is no \( S \) such that \( l_5(k) \in [2n^2 - 8n + 9 + k, 2n^2 - 6n + k + 1] \); and further, for \( 1 \leq k \leq n \), \( l_5(k) = 2n^2 - 6n + 4 + k \) if and only if \( |S| \cong D_{2,1}, l_5(k) = 2n^2 - 6n + 3 + k \) if and only if \( |S| \cong D_{2,2} \), in which all cycles with the same length have the same sign; \( l_5(k) = 2n^2 - 6n + 2 + k \) if and only if \( |S| \cong D_{2,3} \) or \( |S| \cong \mathcal{L} \), in which all cycles with the same length have the same sign.

(4) For any positive integer \( n \), for \( 1 \leq k \leq n - 2 \), \( l_5(k) = 2n^2 - 8n + 9 + k \) if and only if \( |S| \cong F_0 \) or \( |S| \cong F_2 \), in which all cycles with the same length have the same sign; for \( 1 \leq k \leq n \), \( l_5(k) = 2n^2 - 8n + 8 + k \) if and only if \( |S| \cong F_1 \), in which all cycles with the same length have the same sign; for \( 1 \leq k \leq n - 2 \), \( l_5(k) = 2n^2 - 8n + 8 + k \) if and only if \( |S| \) is isomorphic to one in \( \{F_1, F_2, F_3\} \), in which all cycles with the same length have the same sign; for \( 1 \leq k \leq n - 1 \), \( l_5(k) = 2n^2 - 8n + 8 + k \) if and only if \( |S| \) is isomorphic to one in \( \{F_1, F_3, F_5\} \), in which all cycles with the same length have the same sign; for \( n - 1 \leq k \leq n \), \( l_5(k) = 2n^2 - 8n + 7 + k \) if and only if \( |S| \) is isomorphic to one in \( \{F_0, F_1, F_2\} \), in which all cycles with the same length have the same sign; for \( n - 1 \leq k \leq n \), \( l_5(k) = 2n^2 - 8n + 7 + k \) if and only if \( |S| \) is isomorphic to one in \( \{F_3, F_4, F_5, F_6, F_7\} \), in which all cycles with the same length have the same sign; for \( n - 1 \leq k \leq n \), \( l_5(k) = 2n^2 - 8n + 7 + k \) if and only if \( |S| \) is isomorphic to one in \( \{F_3, F_4, F_5, F_6, F_7\} \), in which all cycles with the same length have the same sign; for \( n - 1 \leq k \leq n \), \( l_5(k) = 2n^2 - 8n + 7 + k \) if and only if \( |S| \) is isomorphic to one in \( \{F_3, F_4, F_5, F_6, F_7\} \), in which all cycles with the same length have the same sign; for \( n - 1 \leq k \leq n \), \( l_5(k) = 2n^2 - 8n + 7 + k \) if and only if \( |S| \) is isomorphic to one in \( \{F_3, F_4, F_5, F_6, F_7\} \), in which all cycles with the same length have the same sign.

(5) For any positive integer \( n \), for \( 1 \leq k \leq n \), \( l_5(k) = 2n^2 - 8n + 7 + k \) if and only if \( |S| \cong F_7 \), in which all cycles with the same length have the same sign; \( 2n^2 - 8n + 6 + k \) if and only if \( |S| \cong D_{3,1} \); \( 2n^2 - 8n + 5 + k \) if and only if \( |S| \cong D_{3,2} \), in which all cycles with the same length have the same sign; \( 2n^2 - 8n + 4 + k \) if and only if \( |S| \cong D_{3,3} \) or \( |S| \cong \mathcal{R} \), in which all cycles with the same length have the same sign; \( 2n^2 - 8n + 3 + k \) if and only if \( |S| \) is isomorphic to one of \( \{D_{3,4} \cup \mathcal{B}, \mathcal{R}, \mathcal{R}_4, \mathcal{R}_4\} \), in which all cycles with the same length have the same sign.

**Proof.** Note that \( n \geq 14 \). Then \( 2n^2 - 9n + 12 \geq \frac{3}{2}n^2 - 3n + k + 4 \). By Theorem 3.7, then \( C(S) = \{p_1, p_2\}, p_1 < p_2, p_1 + p_2 \geq n \), all the \( p_1 \)-cycles have the same sign, all the \( p_2 \)-cycles have the same sign in \( S \). By Lemma 2.10, we know that for \( 1 \leq k \leq n \),

\[
l_5(k) \leq l(S) \leq \begin{cases} 
2n^2 - 9n + 8, & p_2 = n, p_1 \leq n - 4; \\
2n^2 - 9n + 12, & p_1 \leq n - 3, p_2 \leq n - 1.
\end{cases}
\]

So, if \( l_5(k) \geq 2n^2 - 9n + 13 \), then there are just the following cases:

1. \( p_2 = n, p_1 = n - 1 \);
2. \( p_2 = n, p_1 = n - 2 \);
3. \( p_2 = n, p_1 = n - 3 \);
4. \( p_2 = n - 1, p_1 = n - 2 \).

Then the theorem follows from Lemmas 3.8 and 3.10.

**Acknowledgments**

We offer many thanks to the referees for their kind comments. Their valuable suggestions are greatly helpful for the improvement of this paper.

**References**