Vector Implicit Complementarity Problems on Galerkin Cone

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Abstract: In this paper, we investigate vector implicit complementarity problems with a variable ordering relation on Galerkin cone. Under suitable assumptions, we prove existence theorems of weakly efficient solution and strong solution for vector implicit complementarity problems on Galerkin cone in Banach spaces.

Keywords: vector implicit complementarity problem; weakly efficient solution; strong solution; Galerkin cone

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0 Introduction

Throughout this paper, unless specified otherwise, we always suppose that X and Y are real Banach spaces, D ⊆ X a nonempty, closed convex subset, K ⊆ X a closed convex cone, and set-valued mapping C : K → 2Y such that, ∀x ∈ K, C(x) ⊆ Y a pointed, closed convex cone with the apex at the origin θY of Y, and intC(x) ≠ ∅. Let L(X, Y) be the space of all continuous linear operators from X into Y. Denote by ⟨ℓ, x⟩ the value of operator ℓ ∈ L(X, Y) at point x ∈ X. Let mappings T : K → L(X, Y) and g : K → K be given. We consider two types of Vector Implicit Complementarity Problems (for short, VICP).

One is to find the weakly efficient solution of (VICP), i.e., Weak Vector Implicit Complementarity Problem (for short, WVICP): find x ∈ K, such that

(WVICP) ⟨T(x), g(x)⟩ ∉ intC(x) and ⟨T(x), y⟩ ∉ −intC(x), ∀y ∈ K.

The (WVICP) was considered and studied by Yin and Xu[23]. If g = I (the identity mapping), then the (WVICP) collapses to the problem studied by Giannessi, Mastroeni and Yang[9]. If g = I, and ∀x ∈ K, C(x) = C (a fixed cone), then the (WVICP) collapses to the problem studied by Chen and Yang[3], and Yang[22].

The other is to find the strong solution of (VICP), i.e., Strong Vector Implicit Complementarity Problem (for short, SVICP): find x ∈ K, such that

(SVICP) ⟨T(x), g(x)⟩ = θY and ⟨T(x), y⟩ ∈ C(x), ∀y ∈ K.

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If \( g = I \), then the (SVICP) collapses to the problem studied by Giannessi, Mastroeni and Yang\(^9\).
If \( g = I \), and \( \forall x \in K, C(x) = C \) (a fixed cone), then the (SVICP) collapses to the problem studied by Yang\(^{22}\).
If \( \forall x \in K, C(x) = [0, +\infty) \), then the (WVICP) and the (SVICP) both collapse to the scalar implicit complementarity problem studied by Isac and Goeleven\(^{13}\).

Meanwhile, we investigate two types of relative Vector Variational Inequalities (for short, VVI).

One is to find the weakly efficient solution of (VVI), i.e., Weak Vector Variational Inequality: find \( x \in K \), such that
\[
(T(x), y - g(x)) \notin -\text{int}C(x), \quad \forall y \in K.
\]
The other is to find the strong solution of (VVI), i.e., Strong Vector Variational Inequality: find \( x \in K \), such that
\[
(T(x), y - g(x)) \in C(x), \quad \forall y \in K.
\]

It is well-known that scalar complementarity problems were introduced by the pioneer works of Lemke, Cottle and Dantzig in the mid 1960s (see [4,9,13,19]). Complementarity problem has numerous applications in such areas as: Optimization, Economics, Game theory, Mechanics and Engineering (see, for example, [12–16]). The (VVI) was first introduced and studied by Giannessi in the setting of finite-dimensional Euclidean spaces (see [8]). The (VVI) led to the introduction of Vector Complementarity Problems (for short, VCP). Since then the (VCP) and the (VVI) have been studied by many authors (see, for example, [1,3,6–7,9–11,17–18,20,22–23] and the references therein).

Galerkin’s method is an efficient approximate method in numerical analysis. Galerkin cone is inspired by Galerkin’s method, and it is a useful tool in complementarity theory. By Galerkin cones we can extend some existence theorems for complementarity problems from finite dimensional case or from locally compact cones to more general cones (see [12]).

Isac and Goeleven\(^{13}\) obtained existence theorems of scalar implicit complementarity problem on Galerkin cone. Li and Huang\(^{20}\), and Lee, Khan and Salahuddin\(^{18}\) studied vector F-implicit complementarity problems in Banach spaces, and authors\(^{[7]}\) investigated vector quasi-equilibrium problems, and obtained existence theorems of strong solutions.

Motivated by the above works, in this paper, we investigate the relationship between vector implicit complementarity problems and corresponding vector variational inequalities with a variable ordering relation. Under suitable assumptions, by using the well-known Browder fixed point theorem and Fan-KKM theorem, we establish existence theorems of weakly efficient solution and strong solution for vector variational inequalities with a variable ordering relation. Finally, we prove existence theorems of weakly efficient solution and strong solution for vector implicit complementarity problems on Galerkin cone in Banach spaces. We generalize two main results of [13], and obtain two new results of (SVICP).

1 Preliminaries

In this section, we recall some definitions and lemmas used in the sequel.
Definition 1.1 Let $X$ and $Y$ be two topological spaces, $F : X \to 2^Y$ a set-valued mapping. $F$ is said to be closed if the graph $\text{Gr}(F) = \{(x, y) \in X \times Y : x \in X, y \in F(x)\}$ of $F$ is a closed subset of $X \times Y$.

Theorem 1.1[2, Browder fixed point theorem] Let $X$ be a Hausdorff topological vector space, $D \subseteq X$ a nonempty, compact convex subset, $S : D \to 2^D$ a set-valued mapping. Assume that
(i) $\forall x \in D$, $S(x)$ a nonempty convex subset of $D$;
(ii) $\forall y \in D$, $S^{-1}(y) = \{x \in D : y \in S(x)\}$ is open in $D$.
Then $S$ has a fixed point in $D$.

Definition 1.2[5] Let $K$ be a nonempty subset of a vector space $X$. A set-valued mapping $F : K \to 2^K$ is called a KKM-mapping if, for any finite subset $\{x_1, x_2, \cdots, x_n\} \subseteq K$, $\text{Co}(x_1, x_2, \cdots, x_n) \subseteq \bigcup_{i=1}^n F(x_i)$, where $\text{Co}(A)$ denotes the convex hull of a set $A$.

Theorem 1.2[5, Fan–KKM theorem] Let $K$ be a nonempty subset of a Hausdorff topological vector space $X$. Let $F : K \to 2^K$ be a KKM-mapping such that, $\forall x \in K$, $F(x)$ is closed and at least one is compact. Then, $\bigcap_{x \in K} F(x) \neq \emptyset$.

Definition 1.3[13] Let $X$ and $Y$ be real Banach spaces. An operator $T : X \to Y$ is called strongly continuous, if for every sequence $\{x_n\} \subseteq X$, $w\text{-}\lim_{n \to \infty} x_n = x$, then $\lim_{n \to \infty} \|T(x_n) - T(x)\| = 0$, where $w\text{-}\lim$ denotes the weak limit.

Definition 1.4[12–13] Let $K$ be a convex cone of Banach space $X$. $K$ is called a Galerkin cone if there is a countable family of convex subcone $\{K_n\}_{n \in \mathbb{N}}$ ($\mathbb{N}$ denotes the set of natural numbers) of $K$ such that
(i) $\forall n \in \mathbb{N}$, $K_n$ is locally compact;
(ii) if $m < n$, then $K_m \subseteq K_n$;
(iii) $K \subseteq \bigcup_{n \in \mathbb{N}} K_n$.
Henceforth, denote by $K(K_n)_{n \in \mathbb{N}}$ a Galerkin cone.

Every closed convex cone in a separable Banach space is a Galerkin cone (see [21]). Further results on Galerkin cones can be found in [12].

Definition 1.5[12–13] Let $D$ be a nonempty, closed convex subset of Banach space $X$, $P : X \to X$ a continuous operator. $P$ is called a projection on $D$ if $P(X) = D$, and $\forall x \in D$, $P(x) = x$.

Theorem 1.3[12–13] Let $K(K_n)_{n \in \mathbb{N}}$ be a Galerkin cone. Then, $\forall n \in \mathbb{N}$, there exists a projection $P_n$ on $K_n$ such that $\forall x \in K$, $\lim_{n \to \infty} P_n(x) = x$.

2 Relationship Between (VVI) and (VICP)

Proposition 2.1 (i) If $x \in K$ is a solution of (WVVI), then it is a solution of (WVICP).
(ii) Assume that $T : K \to L(X, Y)$ is co-negative with respect to $g$, i.e., $\forall x \in K$, $\langle T(x), g(x) \rangle \in -C(x)$. If $x \in K$ is a solution of (WVICP), then it is a solution of (WVVI).

Proof (i) Let $x \in K$ be a solution of (WVVI). Then
$$\langle T(x), y - g(x) \rangle \notin -\text{int}C(x), \quad \forall y \in K.$$  \hspace{1cm} (2.1)
Setting $y = \theta_X$ (the origin of $X$) in (2.1), we have
$$\langle T(x), g(x) \rangle \notin \text{int}C(x).$$  \hspace{1cm} (2.2)
On the other hand, \( \forall z \in K \), taking \( y = z + g(x) \) in (2.1), we have

\[
\langle T(x), z \rangle \notin -\text{int}C(x), \ \forall z \in K. \quad (2.3)
\]

(2.2) and (2.3) imply that \( x \in K \) is a solution of (WVICP).

(ii) Let \( x \in K \) be a solution of (WVICP). Then (2.2) and (2.3) hold. Since \( T \) is co-negative with respect to \( g \), we have

\[
\langle T(x), g(x) \rangle \in -C(x). \quad (2.4)
\]

By (2.3) and (2.4), we have

\[
\langle T(x), z - g(x) \rangle = \langle T(x), z \rangle - \langle T(x), g(x) \rangle
\]

\[
\in Y \setminus \{-\text{int}C(x)\} + C(x)
\]

\[
\subseteq Y \setminus \{-\text{int}C(x)\}, \ \forall z \in K,
\]

i.e., \( x \in K \) is a solution of (WVVI). This completes the proof.

**Proposition 2.2** (SVVI) and (SVICP) have the same solution set.

**Proof** (I) Let \( x \in K \) be a solution of (SVVI). Then

\[
\langle T(x), y - g(x) \rangle \in C(x), \ \forall y \in K. \quad (2.5)
\]

Setting \( y = \theta_X \) in (2.5), we have

\[
\langle T(x), g(x) \rangle \in -C(x). \quad (2.6)
\]

Taking \( y = 2g(x) \) in (2.5), we have

\[
\langle T(x), g(x) \rangle \in C(x). \quad (2.7)
\]

Since \( C(x) \) is a pointed cone, by (2.6) and (2.7), we get

\[
\langle T(x), g(x) \rangle = \theta_Y. \quad (2.8)
\]

On the other hand, \( \forall z \in K \), taking \( y = z + g(x) \) in (2.5), we have

\[
\langle T(x), z \rangle \in C(x), \ \forall z \in K. \quad (2.9)
\]

(2.8) and (2.9) imply that \( x \in K \) is a solution of (SVICP).

(II) Let \( x \in K \) be a solution of (SVICP). Then (2.8) and (2.9) hold. We get

\[
\langle T(x), z - g(x) \rangle \in C(x), \ \forall z \in K.
\]

Hence, \( x \in K \) is a solution of (SVVI). This completes the proof.

### 3 Solutions of (WVVI) and (SVVI)

**Theorem 3.1** Let \( D \) be a nonempty, compact convex subset of Banach space \( X \). Assume that

(i) mappings \( T : D \to L(X, Y) \) and \( g : D \to D \) are continuous;
(ii) $\forall x \in D$, $(T(x), x) - (T(x), g(x)) \in C(x)$;

(iii) set-valued mapping $W : D \to 2^Y$, $\forall x \in D$, $W(x) = Y \setminus \{\text{int} C(x)\}$ is closed.

Then the (WVVI) is solvable in $D$.

**Proof** By contradiction, suppose that the (WVVI) is not solvable. Then $\forall x \in D$, there is some $y \in D$ such that

$$\langle T(x), y - g(x) \rangle \in -\text{int} C(x).$$

Define set-valued mapping $S : D \to 2^D$ as follows. For each $x \in D$, let

$$S(x) = \{y \in D : \langle T(x), y - g(x) \rangle \in -\text{int} C(x)\}.$$

Thus, $\forall x \in D$, $S(x) \neq \emptyset$. In addition, $\langle T(x), y - g(x) \rangle$ is linear in $y \in D$. Thus, $\forall x \in D$, $S(x)$ is a convex subset of $D$. $\forall y \in D$, let

$$S^{-1}(y) = \{x \in D : \langle T(x), y - g(x) \rangle \in -\text{int} C(x)\}.$$

We need to show that $S^{-1}(y)$ is open in $D$. It is equivalent to show that the set

$$D \setminus S^{-1}(y) = \{x \in D : \langle T(x), y - g(x) \rangle \not\in -\text{int} C(x)\}$$

$$= \{x \in D : \langle T(x), y - g(x) \rangle \in Y \setminus \{\text{int} C(x)\}\}$$

is closed in $D$.

In fact, let a sequence $\{x_n\} \subseteq D \setminus S^{-1}(y)$, $x_n \to x \in D$. Then, $\forall n$, we have

$$\langle T(x_n), y - g(x_n) \rangle \in Y \setminus \{\text{int} C(x_n)\}.$$

i.e., the sequence $\{(x_n, \langle T(x_n), y - g(x_n) \rangle)\} \subseteq \text{Gr}(W)$ (the graph of $W$). By the condition (i), it is easy to get

$$\langle T(x_n), y - g(x_n) \rangle \to \langle T(x), y - g(x) \rangle, \ n \to \infty.$$

Since $W$ has the closed graph, we have $\langle T(x), y - g(x) \rangle \in W(x)$. Therefore, $x \in D \setminus S^{-1}(y)$, i.e., $D \setminus S^{-1}(y)$ is closed in $D$.

By Browder Fixed Point Theorem, there is an $x \in D$ such that $x \in S(x)$, that is,

$$\langle T(x), x - g(x) \rangle \in -\text{int} C(x). \quad (3.1)$$

On the other hand, by the condition (ii), one has

$$\langle T(x), x - g(x) \rangle \in C(x). \quad (3.2)$$

Since $C(x)$ is a pointed cone, we have $-\text{int} C(x) \cap C(x) = \emptyset$. Therefore, (3.1) contradicts (3.2). This completes the proof.

**Theorem 3.2** Let $D$ be a nonempty, compact convex subset of Banach space $X$. Assume that

(i) set-valued mapping $C : D \to 2^Y$ is closed;

(ii) mappings $T : D \to L(X, Y)$ and $g : D \to D$ are continuous;

(iii) there is a mapping $h : D \times D \to Y$ satisfying

\begin{align*}
\langle T(x), x - g(x) \rangle &\in C(x), \quad (3.2) \\
\langle T(x), x - g(x) \rangle &\in C(x).
\end{align*}
(a) \( \forall x \in D, h(x, x) \in C(x) \);
(b) for any \( y \in D \), if there exists \( x \in D \) such that \( h(x, y) \in C(x) \), then
\[
\langle T(x), y - g(x) \rangle = h(x, y) \in C(x);
\]
(c) for any \( x \in D \), the set \( \{ y \in D : h(x, y) \notin C(x) \} \) is a convex subset of \( D \).

Then the (SVVI) is solvable in \( D \).

**Proof** Define mappings \( F_1, F_2 : D \to 2^D \) as follows. \( \forall y \in D \), let
\[
F_1(y) = \{ x \in D : \langle T(x), y - g(x) \rangle \in C(x) \},
\]
and
\[
F_2(y) = \{ x \in D : h(x, y) \in C(x) \}.
\]

We need to show that \( \bigcap_{y \in D} F_1(y) \neq \emptyset \). If \( \bar{x} \in \bigcap_{y \in D} F_1(y) \), then \( \bar{x} \) is a required solution of the (SVVI). The proof is divided into 3 steps.

(I) For any \( y \in D \), \( F_1(y) \) is closed in \( D \).

Indeed, let a sequence \( \{ x_n \} \subseteq F_1(y) \), \( x_n \to x_0 \). We shall show that \( x_0 \in F_1(y) \). Since \( \{ x_n \} \subseteq D \) and \( D \) is closed, we have \( x_0 \in D \). It follows from \( \{ x_n \} \subseteq F_1(y) \) that
\[
\langle T(x_n), y - g(x_n) \rangle \in C(x_n), \quad \forall n.
\]

This means that the sequence \( \{(x_n, \langle T(x_n), y - g(x_n) \rangle)\} \subseteq \text{Gr}(C) \). By the continuity of \( T \) and \( g \), we can get easily
\[
\langle T(x_n), y - g(x_n) \rangle \to \langle T(x_0), y - g(x_0) \rangle, \quad n \to \infty.
\]

Since \( C \) has the closed graph, we get \( \langle T(x_0), y - g(x_0) \rangle \in C(x_0) \). Thus \( x_0 \in F_1(y) \).

(II) \( F_2 \) is a KKM mapping.

In fact, if it is false, then there are \( \{ y_1, y_2, \ldots, y_m \} \subseteq D \), \( \lambda_i > 0 \), \( \sum_{i=1}^m \lambda_i = 1 \), and \( \bar{y} = \sum_{i=1}^m \lambda_i y_i \notin \bigcup_{i=1}^m F_2(y_i) \). That is,
\[
h(\bar{y}, y_i) \notin C(\bar{y}), \quad i = 1, 2, \ldots, m.
\]

By the condition (iii)(c), we have \( h(\bar{y}, y_i) \notin C(\bar{y}) \), which contradicts the condition (iii)(a). Therefore, \( F_2 \) is a KKM mapping.

(III) \( F_1 \) is a KKM mapping.

In fact, if \( x \in F_2(y) \), i.e., \( h(x, y) \in C(x) \), by the condition (iii)(b), we have \( x \in F_1(y) \). Therefore, for any \( y \in D \), \( F_2(y) \subseteq F_1(y) \). Thus, \( F_1 \) is a KKM mapping.

Since \( D \) is compact, by the Fan-KKM theorem, we have \( \bigcap_{y \in D} F_1(y) \neq \emptyset \). This completes the proof.

4 Solutions of (WVICP)

**Theorem 4.1** Let \( X \) be a Banach space, \( K \subseteq X \) a locally compact convex cone. Assume that

(i) mappings \( T : K \to L(X, Y) \) and \( g : K \to K \) are continuous;
Therefore, (i)–(iii) in Theorem 4.1 and the following condition (iv)

\[(iv), \|W\| < r, \text{ satisfying} \quad \langle T(x), v_x - g(x) \rangle \in -\text{int} C(x). \tag{4.1} \]

Then the (WVICP) has a solution \(x^* \in K\), and \(\|x^*\| < r\).

**Proof** We use the method in [13]. According to Proposition 2.1(i), we need only to show that the (WVVI) is solvable. \(\forall n \in \mathbb{N}\), let \(D_n = \{x \in K : \|x\| \leq n\}\). It follows from the local compactness of \(K\) that \(D_n\) is compact and convex. By Theorem 3.1, there is an \(x_n \in D_n\) such that

\[
\langle T(x_n), x - g(x_n) \rangle \not\in -\text{int} C(x_n), \quad \forall x \in D_n. \tag{4.2}
\]

We claim that the sequence \(\{x_n\} \subseteq K\) is bounded.

Indeed, if it is false, then \(\forall i \in \mathbb{N}, \exists n_i \in \mathbb{N}\), such that \(\|x_{n_i}\| \geq i\). Choose a natural number \(l \geq r\), then there exists a natural number \(n_l \in \mathbb{N}\) such that \(n_l \geq \|x_{n_i}\| \geq l \geq r\). For this \(x_{n_l}\), by the condition (iv) and (4.1), there exists \(v_{x_{n_l}} \in K\), \(\|v_{x_{n_l}}\| < r\) such that

\[
\langle T(x_{n_l}), v_{x_{n_l}} - g(x_{n_l}) \rangle \in -\text{int} C(x_{n_l}). \tag{4.3}
\]

On the other hand, by (4.2), one has

\[
\langle T(x_{n_l}), v_{x_{n_l}} - g(x_{n_l}) \rangle \not\in -\text{int} C(x_{n_l}),
\]

which contradicts (4.3). Therefore, \(\{x_n\}\) is bounded.

It follows from the local compactness of \(K\) that \(\{x_n\}\) has a convergent subsequence (for simplification, we still denote it by \(\{x_n\}\)). Let \(\lim_{n \to \infty} x_n = \bar{x} \in K\). We shall show that \(\bar{x}\) is a solution of the (WVVI).

Indeed, for any fixed \(z \in K\), there is a natural number \(l\) such that \(z \in D_l\). For \(n > l\), \(z \in D_l \subseteq D_n\), by (4.2), one has

\[
\langle T(x_n), z - g(x_n) \rangle \not\in -\text{int} C(x_n). \tag{4.4}
\]

(4.4) means that \(\{(x_n, \langle T(x_n), z - g(x_n) \rangle)\} \subseteq \text{Gr}(W)\). By the condition (i), one has

\[
\lim_{n \to \infty} \langle T(x_n), z - g(x_n) \rangle = \langle T(\bar{x}), z - g(\bar{x}) \rangle. \tag{4.5}
\]

Since \(W\) has the closed graph, by (4.4) and (4.5), we get

\[
\langle T(\bar{x}), z - g(\bar{x}) \rangle \not\in -\text{int} C(\bar{x}), \quad \forall z \in K.
\]

Therefore, \(\bar{x}\) is a solution of the (WVVI). Thus, \(\bar{x}\) is a solution of the (WVICP). By the condition (iv), \(\|\bar{x}\| < r\). This completes the proof.

**Corollary 4.1** Let \(X\) and \(K\) be the same as in Theorem 4.1. Assume that the conditions (i)–(iii) in Theorem 4.1 and the following condition (iv)' hold:
(iv)' there exists a real number \( r > 0 \) such that, \( \forall x \in K, \|x\| \geq r \), one has
\[
\langle T(x), g(x) \rangle \in \text{int}C(x).
\]
Then the (WVCP) has a solution \( \bar{x} \in K \), and \( \|\bar{x}\| < r \).

**Proof** In Theorem 4.1, choose \( v_x = \theta_x \). Then Theorem 4.1 yields the conclusion. This completes the proof.

**Corollary 4.2** Let \( X \) and \( K \) be the same as in Theorem 4.1. Assume that the conditions (i)--(iii) in Theorem 4.1 and the following condition (iv)' hold:

(iv)' there exist a real number \( r_0 > 0 \) and some \( v_0 \in K \) such that, \( \forall x \in K, \|x\| \geq r_0 \), one has
\[
\langle T(x), v_0 - g(x) \rangle \in -\text{int}C(x).
\]
Then the (WVCP) has a solution \( \bar{x} \in K \), and \( \|\bar{x}\| < r = 1 + \max\{r_0, \|v_0\|\} \).

**Proof** In Theorem 4.1, let \( r = 1 + \max\{r_0, \|v_0\|\} \), and for each \( x \in K, \|x\| \geq r_0 \), choose \( v_x = v_0 \). Then Theorem 4.1 yields the conclusion. This completes the proof.

**Theorem 4.2** Let \( X \) be a reflexive Banach space, and \( K(K_n), n \in \mathbb{N} \) be a Galerkin cone of \( X \). Assume that the following conditions hold:

(i) mappings \( T : K \to L(X, Y) \) and \( g : K \to K \) are strongly continuous;

(ii) \( \forall x \in K, \langle T(x), x - g(x) \rangle \in C(x) \);

(iii) set-valued mapping \( W : K \to 2^Y, \forall x \in K, W(x) = Y \setminus \{-\text{int}C(x)\} \) is closed;

(iv) there exist numbers \( n_0 \in \mathbb{N} \) and \( r > 0 \) such that, \( \forall n \geq n_0 \) and \( \forall x \in K_n, \|x\| \geq r \), there is some \( v_x \in K_n, \|v_x\| < r \) satisfying
\[
\langle T(x), v_x - g(x) \rangle \in -\text{int}C(x).
\]
Then the (WVCP) has a solution \( \bar{x} \in K \), and \( \|\bar{x}\| \leq r \).

**Proof** For each \( n \geq n_0 \), by Theorem 4.1, the (WVVI) has a solution \( x_n \in K_n \) and \( \|x_n\| < r \). Since \( X \) is reflexive and the sequence \( \{x_n\} \subseteq K \) is bounded, it has a weakly convergent subsequence (for simplification, we still denote it by \( \{x_n\} \)). Let \( \text{w-lim}_{n \to \infty} x_n = \bar{x} \in K \). We claim that \( \bar{x} \) is a solution of the (WVVI).

In fact, for any fixed \( z \in K, \forall n \in \mathbb{N} \), by Theorem 1.3, \( P_n(z) \in K_n \) and \( \text{lim}_{n \to \infty} P_n(z) = z \). For each \( n > n_0 \), since \( x_n \) is a solution of the (WVVI) in \( K_n \), we have
\[
\langle T(x_n), P_n(z) - g(x_n) \rangle \notin -\text{int}C(x_n).
\]
Thus, the sequence \( \{(x_n, \langle T(x_n), P_n(z) - g(x_n) \rangle)\} \subseteq Gr(W) \). We show that
\[
\text{lim}_{n \to \infty} (T(x_n), P_n(z) - g(x_n)) \to (T(\bar{x}), z - g(\bar{x})).
\]
Indeed, by the condition (i), \( \lim_{n \to \infty} T(x_n) = T(\bar{x}), \lim_{n \to \infty} g(x_n) = g(\bar{x}) \). Therefore, the sequences \( \{\|g(x_n)\|\} \) and \( \{\|P_n(z)\|\} \) both are bounded. We have
\[
\|\langle T(x_n), P_n(z) - g(x_n) \rangle - (T(\bar{x}), z - g(\bar{x}))\|
\leq \|T(x_n) - T(\bar{x}), P_n(z) - g(x_n)\| + \|T(\bar{x}), [P_n(z) - z] - [g(x_n) - g(\bar{x})]\|
\leq \|T(x_n) - T(\bar{x})\| \cdot (\|P_n(z)\| + \|g(x_n)\|)
+ \|T(\bar{x})\| \cdot (\|P_n(z) - z\| + \|g(x_n) - g(\bar{x})\|) \to 0, \quad n \to \infty.
\]
Since $W$ has the closed graph, we have $\langle T(\bar{x}), z - g(\bar{x}) \rangle \in Y \setminus \{-\text{int}C(\bar{x})\}.$

\[ \langle T(\bar{x}), z - g(\bar{x}) \rangle \notin -\text{int}C(\bar{x}), \forall z \in K. \]

i.e., $\bar{x}$ is a solution of the (WVVI). By Proposition 2.1(i), $\bar{x}$ is also a solution of the (WVICP).

It follows from $\|x_n\| < r$ ($\forall n \geq n_0$) that $\|\bar{x}\| \leq r.$ This completes the proof.

**Remark 4.1** If $\forall x \in K, C(x) = [0, +\infty),$ then the above Theorem 4.1 and Theorem 4.2 collapses Theorem 5 and Theorem 6 of [1], respectively.

## 5 Solutions of (SVICP)

**Theorem 5.1** Let $X$ be a Banach space, $K \subseteq X$ a locally compact convex cone. Assume that

(i) $C : K \to 2^Y$ is closed;

(ii) mappings $T : K \to L(X, Y)$ and $g : K \to K$ are continuous;

(iii) there is a mapping $h : K \times K \to Y,$ satisfying

(a) $\forall x \in K, h(x, x) \in C(x);$

(b) for any $y \in K,$ if there exists $x \in K$ such that $h(x, y) \in C(x),$ then

\[ \langle T(x), y - g(x) \rangle - h(x, y) \in C(x); \]

(c) for any $x \in K,$ the set $\{y \in K : h(x, y) \notin C(x)\}$ is a convex subset of $K;$

(iv) there exists a real number $r > 0$ such that, $\forall x \in K, \|x\| \geq r,$ there is some $v_x \in K,$ $\|v_x\| < r,$ satisfying

\[ \langle T(x), v_x - g(x) \rangle \notin C(x). \]

Then (SVICP) has a solution $\bar{x} \in K,$ and $\|\bar{x}\| < r.$

**Proof** By Proposition 2.2, we need only to show that the (SVVI) is solvable in $K.$

In fact, $\forall n \in \mathbb{N},$ let $D_n = \{x \in K : \|x\| \leq n\}.$ It follows from the local compactness of $K$ that $D_n$ is compact and convex. By Theorem 3.2, there is an $x_n \in D_n$ such that

\[ \langle T(x_n), x - g(x_n) \rangle \in C(x_n), \forall x \in D_n. \]

Applying the argument of Theorem 4.1 and making a slight modification, we can show that the sequence $\{x_n\} \subseteq K$ is bounded. Since $K$ is locally compact, $\{x_n\}$ has a convergent subsequence (for simplification, we still denote it by $\{x_n\}$). Let $\lim_{n \to \infty} x_n = \bar{x} \in K.$ By a similar argument as in the proof of Theorem 4.1, we can show that $\bar{x}$ is a solution of the (SVVI). By Proposition 2.2, it is also a solution of the (SVICP). This completes the proof.

**Theorem 5.2** Let $X$ be a reflexive Banach space, and $K(K_n)_{n \in \mathbb{N}}$ be a Galerkin cone of $X.$ Assume that the following conditions hold:

(i) $C : K \to 2^Y$ is closed;

(ii) mappings $T : K \to L(X, Y)$ and $g : K \to K$ are strongly continuous;

(iii) there is a mapping $h : K \times K \to Y$ satisfying

(a) $\forall x \in K, h(x, x) \in C(x);
(b) for any \( y \in K \), if there exists \( x \in K \) such that \( h(x, y) \in C(x) \), then

\[
\langle T(x), y - g(x) \rangle - h(x, y) \in C(x);
\]

(c) for any \( x \in K \), the set \( \{ y \in K : h(x, y) \notin C(x) \} \) is a convex subset of \( K \);

(iv) there exist numbers \( n_0 \in \mathbb{N} \) and \( r > 0 \) such that, \( \forall n \geq n_0 \) and \( \forall x \in K_n \), \( \|x\| \geq r \), there is some \( v_x \in K_n \), \( \|v_x\| < r \) satisfying

\[
\langle T(x), v_x - g(x) \rangle \notin C(x).
\]

Then the (SVICP) has a solution \( \bar{x} \in K \), and \( \|\bar{x}\| \leq r \).

**Proof** For \( n \geq n_0 \), by Theorem 5.1, the (SVVI) has a solution \( x_n \in K_n \), and \( \|x_n\| < r \). Since \( X \) is reflexive and the sequence \( \{x_n\} \subseteq K \) is bounded, it has a weakly convergent subsequence (for simplification, we still denote it by \( \{x_n\} \)). Let \( w-\lim_{n \to \infty} x_n = \bar{x} \in K \). By a similar argument as in the proof in Theorem 4.2, we can show that \( \bar{x} \) is a solution of the (SVVI). By Proposition 2.2, it is also a solution of the (SVICP). This completes the proof.

**Example 5.1** Let \( X = \mathbb{R} \), \( K = [0, +\infty) \) and \( Y = \mathbb{R}^2 \). For any \( x, y \in K \), let

\[
g(x) = x^2, \quad T(x) = ((\sqrt{x} + 1)^2, 0), \quad h(x, y) = (0, (\sqrt{y} - x) \cdot |y - x|).
\]

and

\[
C(x) = \left\{ (\xi, \eta) : \xi \geq 0, \eta \geq 0 \right\} \cup \left\{ (\xi, \eta) : \xi \leq 0, \eta \geq -\tan \left( \frac{\arctan x}{2} \right) \right\}.
\]

Then, we have

\[
\langle T(x), y - g(x) \rangle = ((\sqrt{x} + 1)^2(y - x^2), 0)
\]

and

\[
\langle T(x), y - g(x) \rangle - h(x, y) = ((\sqrt{x} + 1)^2(y - x^2), (\sqrt{y} - x) \cdot |y - x|).
\]

It is easy to check that the mappings \( C, T, g \) and \( h \) satisfy the conditions (i)–(iii) in Theorem 5.1. For the condition (iv), choose \( r = 1 \), and for any \( x \in K \), \( x \geq 1 \), set \( v_x = 0.5 \in K \). Then \( \langle T(x), v_x - g(x) \rangle \notin C(x) \). Therefore, the condition (iv) holds. Obviously, \( x = 0 \) is a solution of the (SIVCP).

**References**


**Galerkin 锥上的向量隐式互补问题**

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**摘要**：讨论 Galerkin 锥上有向量隐式互补问题。在适当的条件下，证明该问题在 Banach 空间中 Galerkin 锥上的弱有效解与强解的存在定理。

**关键词**：向量隐式互补问题；弱有效解；强解；Galerkin 锥