Bifurcations of limit cycles in equivariant quintic planar vector fields

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**Abstract**

In this paper, we obtain 23 limit cycles for a $Z_3$-equivariant near-Hamiltonian system of degree 5 which is the perturbation of a $Z_6$-equivariant quintic Hamiltonian system. The configuration of these limit cycles is new and different from the configuration obtained by H.S.Y. Chan, K.W. Chung and J. Li, where the unperturbed system is a $Z_2$-equivariant quintic Hamiltonian system. Our unperturbed system is different from the unperturbed systems studied by Y. Wu and M. Han. The limit cycles are obtained by Poincaré–Pontryagin theorem and Poincaré–Bendixson theorem.

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1. Introduction and the main result

Consider the following polynomial vector fields of degree $n$:

$$
\dot{x} = P_n(x,y), \quad \dot{y} = Q_n(x,y). \tag{1.1}
$$

It is well known that the second part of Hilbert’s 16th problem is to ask the exact upper bound of the number of limit cycles and to determine their relative positions for system (1.1). Up to now, we only know that a specific planar polynomial system always has a finite number of limit cycles, see [4] and [8]. However, it is still an open problem to find the uniform upper bounds, even for $n = 2$.

Attentions are also paid to the lower bounds of $H(n)$ and $M(n)$ for $n \geq 2$, where $H(n)$ stands for the number of all the limit cycles for system (1.1), and $M(n)$ stands for the number of small amplitude limit cycles for system (1.1). There are many references for specific $n$, for instance, see [1,2,9–15,17,18,20–31,33] etc. The lower bound of $H(n)$ for general $n$ was first considered in [3]. Recently, it was found in [6] that $H(n)$ grows at least as rapidly as $\frac{1}{2\ln 2}(n + 2)^2 \ln(n + 2)$. It was obtained that $M(5) \geq 25$ in [15] and [28].

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For general $n$, it was proved that $M(n) \geq n^2$ if $n \geq 23$ in [32]. It was also proved in [32] that $M(n)$ grows at least as rapidly as $\frac{18}{25} \cdot \frac{1}{2\ln 2} (n + 2)^2 \ln(n + 2)$ for all large $n$. One can obtain a detailed list of references with great ease via e-resources.

Let $z = x + iy$, $\bar{z} = x - iy$, then system (1.1) becomes

$$\dot{z} = F(z, \bar{z}), \quad \dot{\bar{z}} = \bar{F}(z, \bar{z}),$$

where $I^2 = -1$ and $F(z, \bar{z}) = P_n(x, y) + IQ_n(x, y)$. Let $Z_q$ be a cyclic group which is generated by a planar counterclockwise rotation through $2\pi/q$ about the origin. By Corollary 7.3 in [13], we get

**Lemma 1.1.** (See [13].) For system (1.1) with $n = 5$, all non-trivial $Z_q$-equivariant vector fields have the following forms:

(i) $q = 6$: $F(z, \bar{z}) = (A_0 + A_1|z|^2 + A_2|z|^4)z + A_3\bar{z}\bar{z}^5$;

(ii) $q = 5$: $F(z, \bar{z}) = (A_0 + A_1|z|^2 + A_2|z|^4)z + A_3\bar{z}\bar{z}^4$;

(iii) $q = 4$: $F(z, \bar{z}) = (A_0 + A_1|z|^2 + A_2|z|^4)z + (A_3 + A_4|z|^2)\bar{z}\bar{z}^3 + A_5z^5$;

(iv) $q = 3$: $F(z, \bar{z}) = (A_0 + A_1|z|^2 + A_2|z|^4)z + (A_3 + A_4|z|^2)\bar{z}\bar{z}^2 + A_5z^4 + A_6z^5$;

(v) $q = 2$: $F(z, \bar{z}) = (A_0 + A_1|z|^2 + A_2|z|^4)z + (A_3 + A_4|z|^2 + A_5|z|^4)\bar{z} + (A_6 + A_7|z|^2)\bar{z}^3 + (A_8 + A_9|z|^2)\bar{z}^2 + A_{10}z^5 + A_{11}\bar{z}\bar{z}^5$,

where $A_i = A_i + IB_i$ with $A_i, B_i \in \mathbb{R}$ for $i = 0, \cdots, 11$. The above $F(z, \bar{z})$ define $Z_q$-equivariant Hamiltonian vector fields if and only if $A_0 = A_1 = A_2 = 0$ and for $q = 4$, $A_4 = -5\bar{A}_5$; for $q = 3$, $A_4 = -5\bar{A}_5$; for $q = 2$, $A_4 = -3\bar{A}_5$, $A_5 = -2\bar{A}_7$ and $A_9 = -5\bar{A}_{10}$.

In recent years, more attentions were paid to the lower bounds and distribution of limit cycles for system (1.1) with $n = 5$, for instance, see [2,9,11,12,14,15,17,18,20,21,24–31,33] and references therein. Some interesting results are listed as follows.

(i) For $Z_2$-equivariant system: it was shown in [24] that there were $Z_2$-equivariant quintic systems with at least 25 limit cycles for each of them, and it was obtained in [25] that there was a $Z_2$-equivariant quintic planar vector field having 28 limit cycles with four different configurations.

(ii) For $Z_3$-equivariant system: it was proved in [18] that there were at least 15 limit cycles for a $Z_3$-equivariant near-Hamiltonian system of degree 5 which was a perturbation of a $Z_3$-equivariant cubic Hamiltonian system. It was shown in [2] that there were at least 23 limit cycles for a $Z_3$-equivariant near-Hamiltonian system of degree 5 which was a perturbation of a $Z_3$-equivariant quintic Hamiltonian system. It was obtained in [26] that there were at least 24 limit cycles with two different configurations for a $Z_3$-equivariant near-Hamiltonian system of degree 5 which was the perturbations of a $Z_6$-equivariant quintic Hamiltonian system.

(iii) For $Z_4$-equivariant system: it was achieved in [27] that there were at least 28 limit cycles with two different configurations.

(iv) For $Z_5$-equivariant system: it was proved dependently in [15] and [30] that there were $Z_5$-equivariant planar polynomial vector fields of degree 5 having at least 25 small limit cycles for each of them.

(v) For $Z_6$-equivariant system: it was obtained that there were $Z_6$-equivariant planar polynomial vector fields of degree 5 having at least 24 small limit cycles for each of them in [15], and there were $Z_6$-equivariant planar polynomial vector fields of degree 5 having at least 24 limit cycles for each of them in [12].

Motivated by [2,9,11,12,14,15,17,18,20,21,24–31] and [33], in this paper, we intend to study the number and the distribution of limit cycles of the following $Z_3$-equivariant quintic systems:

$$\dot{x} = H_y(x, y) + \epsilon P_5(x, y), \quad \dot{y} = -H_x(x, y) + \epsilon Q_5(x, y),$$

(1.3)
where

\[
H(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{7}{10}(x^4 + y^4) + \frac{7}{5}x^2y^2 - \frac{3}{10}x^6 + \frac{3}{5}x^4y^2 - \frac{19}{10}x^2y^4 - \frac{2}{15}y^6,
\]

\[
\mathcal{P}_5(x, y) = x(x^4 - 10x^2y^2 + 5y^4)A_6 + y(5x^4 - 10x^2 + y^4)B_6
\]

\[
+ x(x^2 + y^2)^2 A_2 - y(x^2 + y^2)^2 B_2 + (x^2 - y^2) A_3 + 2xyB_3
\]

\[
+ (x^4 - y^4) A_4 + (x^4 - 6x^2y^2 + y^4) A_5 + 2xy(x^2 + y^2)B_4 - 4xy(x^2 - y^2)B_5
\]

\[
+ x(x^2 + y^2) A_1 - y(x^2 + y^2) B_1 + xA_0 - yB_0,
\]

\[
Q_5(x, y) = -y(5x^4 - 10x^2y^2 + y^4)A_6 + x(x^4 - 10x^2y^2 + 5y^4)B_6
\]

\[
+ y(x^2 + y^2)^2 A_2 + x(x^2 + y^2)^2 B_2 - 2xyA_3 + (x^2 - y^2) B_3
\]

\[
- 2xy(x^2 + y^2) A_4 + 4xy(x^2 - y^2) A_5 + (x^4 - y^4) B_4 + (x^4 - 6x^2y^2 + y^4)B_5
\]

\[
+ y(x^2 + y^2) A_1 + x(x^2 + y^2) B_1 + yA_0 + xB_0.
\]

System (1.3) with \( \epsilon = 0 \) is \( Z_6 \)-equivariant (see Lemma 1.1). The expressions of \( \mathcal{P}_5(x, y) \) and \( Q_5(x, y) \) are obtained by Lemma 1.1(iv). System (1.3) is Hamiltonian if and only if

\[
A_0 = A_1 = A_2 = 0, \quad A_4 = -4A_5, \quad B_4 = 4B_5.
\]

Denote

\[
\Delta = (A_0, \cdots, A_6, B_0, \cdots, B_6), \quad \Delta^* = (A_0^*, \cdots, A_6^*, B_0^*, \cdots, B_6^*).
\]

Our main result is stated as follows.

**Theorem 1.1.** For \( 0 < \epsilon \ll 1 \),

(i) there exists a \( \Delta^* \) with \( A_2^* \neq 0 \) and \( A_4^* = -4A_5^* \) such that system (1.3) has at least 23 or 20 limit cycles. The configurations of these limit cycles are shown in Figs. 1(a) and 1(b);

(ii) there exists a \( \Delta^* \) with \( A_2^* \neq 0 \) and \( A_i^* = B_i^* = 0 \) \( (i = 3, 4, 5) \) such that system (1.3) has at least 24 or 18 limit cycles. The configurations of these limit cycles are shown in Figs. 1(c) and 1(d).

**Remark 1.1.** (i) System (1.3) with \( A_i^* = B_i^* = 0 \) \( (i = 3, 4, 5) \) is \( Z_6 \)-equivariant.

(ii) The configuration of the 23 limit cycles in Theorem 1.1 is new and different from that obtained in [2], where the unperturbed system is a \( Z_3 \)-equivariant quintic Hamiltonian system. Our unperturbed system is different from that in [26], see Figs. 2(a) and 2(b). The configurations of limit cycles in Figs. 1(b) and 1(d) are also new.

This paper is organized as follows. In Section 2, we will give some preliminaries. We will study the expansion of the Abelian integral \( I(h) \) near a homoclinic loop and give some criterions for determining the zeros of \( I(h) \). In Section 3, we will obtain the specific expressions of \( I(h) \) for system (1.3) near homoclinic loops. In Section 4, we shall give the proof of Theorem 1.1.

**2. Preliminaries**

Consider the following near-Hamiltonian systems

\[
\begin{align*}
\dot{x} &= H_y(x, y) + \epsilon P(x, y, \delta), \\
\dot{y} &= -H_x(x, y) + \epsilon Q(x, y, \delta),
\end{align*}
\]

\[
(2.1)
\]
Fig. 1. The phase plane for $Z_3$-equivariant system (1.3) with (a) 23, (b) 20 limit cycles, and for $Z_6$-equivariant system (1.3) with (c) 24, (d) 18 limit cycles.

Fig. 2. (a) The phase portraits of system (3.1). (b) The phase portraits of the unperturbed system in [26].
where $0 < |\epsilon| \ll 1$ and $H, P, Q$ are analytic in $(x, y, \delta) \in D \subset \mathbb{R}^2 \times \mathbb{R}^m$ with $D$ being bounded. We assume system $(2.1)_{\epsilon=0}$ has a continuous family of periodic orbits $\bigcup_{h \in \Sigma} \Gamma_h$, where $\Gamma_h = \{(x, y) : H(x, y) = h\}$. For the orientation of a periodic orbit $\Gamma_h$, we have

**Lemma 2.1.** (See [5].) If the closed curve $\Gamma_h$ is oriented clockwise, then it expands with $h$ increasing. If the closed curve $\Gamma_h$ is oriented counter clockwise, then it shrinks with $h$ increasing.

Let us take a segment $\sigma$, which is transversal to each of the ovals $\bigcup_{h \in \Sigma} \Gamma_h$, and parameterize $\sigma$ by the values of $H(x, y)$. Denote by $\Gamma_\sigma(h, \delta)$ a piece of orbit of system (2.1) between the starting point $h$ on $\sigma$ and the next intersection point $P_\epsilon(h, \delta)$ with $\sigma$. As usual, we call

$$d_\epsilon(h, \delta) = P_\epsilon(h, \delta) - h$$  \hspace{1cm} (2.2)

a displacement function of system (2.1). Obviously, if there are $\epsilon^* \neq 0$ and $\delta^* \in R^m$ such that $d_\epsilon^*(h, \delta^*) \neq 0$ and $d_\epsilon^*(h_0, \delta^*) = 0$, then $\Gamma_\sigma(h_0, \delta^*)$ is a limit cycle of system (2.1) with $\epsilon = \epsilon^*$ and $\delta = \delta^*$.

**Lemma 2.2 (Poincaré–Pontryagin).** (See [19].) The displacement function $d_\epsilon(h, \delta)$ has the following asymptotic expansion:

$$d_\epsilon(h, \delta) = \epsilon(I(h, \delta) + \epsilon \varphi_\epsilon(h, \delta)), \hspace{0.5cm} \epsilon \to 0,$$  \hspace{1cm} (2.3)

where

$$I(h, \delta) = \int_{\Gamma_h} Q \, dx - P \, dy,$$  \hspace{1cm} (2.4)

and $\varphi_\epsilon(h, \delta)$ is analytic and uniformly bounded for $(h, \epsilon)$ in a compact neighborhood of $(h, 0)$. $I(h, \delta)$ is called the Abelian integral of system (2.1).

The Poincaré–Pontryagin theorem has the following immediate corollaries.

**Corollary 2.1.** (See [10].) (i) If an $h_0$ is the simple zero of $I(h)$, then for sufficiently small $\epsilon$, $d_\epsilon(h)$ also has one simple zero close to $h_0$. Therefore, system (2.1) has one limit cycle close to $\Gamma(h_0)$.

(ii) If $I(h)$ is well defined in $(h_1, h_2)$ and $I(h_1^+I(h_2^-)) < 0$, then for sufficiently small $\epsilon$, $d_\epsilon(h)$ has at least one zero. It follows that system (2.1) has at least one limit cycle.

**Lemma 2.3.** (See [5, 7, 16].) Suppose that system $(2.1)_{\epsilon=0}$ has a homoclinic loop $\Gamma_0$ corresponding to $H(x, y) = 0$ through a hyperbolic saddle $p$. Assume the periodic orbits $\bigcup_{h \in \Sigma} \Gamma_h$ expand to $\Gamma_0$ as $h$ decreases to 0. Then, for $0 < h \ll 1$, there exist analytical functions $a_1(h, \delta)$ and $b_1(h, \delta)$ with $b_1(0, \delta) = 0$ such that $I(h, \delta) = a_1(h, \delta) + b_1(h, \delta) \ln h$, that is,

$$I(h, \delta) = a_1(0, \delta) + b_1'(0, \delta) h \ln h + a_1'(0, \delta) h + h.o.t.$$

where

$$a_1(0, \delta) = \int_{\Gamma_0} Q \, dx - P \, dy, \hspace{0.5cm} b_1'(0, \delta) = -\frac{1}{\lambda}(P_x + Q_y)(p),$$

$$a_1'(0, \delta) = \int_{\Gamma_0} (P_x + Q_y) \, dt \hspace{0.5cm} \text{if} \hspace{0.5cm} b_1'(0, \delta) = 0,$$

and $\lambda > 0$ is the eigenvalue of the saddle $p$. 
Lemma 2.4. (See [5,7,16].) Assume system (2.1) has a continuous family of periodic orbits $\bigcup_{h_1<h<h_2} \Gamma_h$ and $\Gamma_{h_2}$ is a homoclinic loop through a hyperbolic saddle $p$. Denote $\lambda > 0$ the eigenvalue of the saddle $p$.

(i) Suppose that $\bigcup_{h_1<h<h_2} \Gamma_h$ expand to $\Gamma_{h_2}$ as $h$ decreases to $h_2$. Then, for $0 < h - h_2 \ll 1$,

$$I(h) = c_0(\delta) + c_1(\delta)(h - h_2) \ln(h - h_2) + c_2(\delta)(h - h_2) + h.o.t.,$$

where

$$c_0(\delta) = \int_{I_{h_2}} Q \, dx - P \, dy, \quad c_1(\delta) = -\frac{1}{\lambda}(P_x + Q_y)_p,$$

$$c_2(\delta) = \int_{I_{h_2}} (P_x + Q_y) \, dt \quad \text{if } c_1(\delta) = 0.$$  (2.6)

(ii) Suppose that $\bigcup_{h_1<h<h_2} \Gamma_h$ expand to $\Gamma_{h_2}$ as $h$ increases to $h_2$. Then, for $0 < |h - h_2| \ll 1$ with $h < h_2$,

$$I(h) = c_0(\delta) + c_1(\delta)(h_2 - h) \ln(h_2 - h) + c_2(\delta)(h_2 - h) + h.o.t.$$

where $c_0(\delta)$ is given in (2.6), and

$$c_1(\delta) = \frac{1}{\lambda}(P_x + Q_y)(p), \quad c_2(\delta) = -\int_{I_{h_2}} (P_x + Q_y) \, dt \quad \text{if } c_1(\delta) = 0.$$  (2.9)

3. The expressions of the related Abelian integrals

For $\epsilon = 0$, system (1.3) is given as follows:

$$\dot{x} = H_y(x,y) = yH_1(x,y), \quad \dot{y} = -H_x(x,y) = -xH_2(x,y),$$  (3.1)

where

$$H_1(x,y) = [-5 + 14(x^2 + y^2) + 2(3x^4 - 19x^2y^2 - 2y^4)]/5,$$

$$H_2(x,y) = [-5 + 14(x^2 + y^2) + 9x^4 - 12x^2y^2 + 19y^4]/5.$$  (3.2)

The phase portraits of system (3.1) are shown in Fig. 2.

System (3.1) is a $\mathbb{Z}_6$-equivariant quintic system with 4 saddles on the axes of coordinates:

$$S_1(1,0), \quad S_2(-1,0), \quad S_3\left(0, \frac{\sqrt{7} - \sqrt{29}}{2}\right), \quad S_4\left(0, -\frac{\sqrt{7} - \sqrt{29}}{2}\right)$$ (3.4)

and 4 centers on the axes of coordinates:

$$C_1\left(\frac{\sqrt{5}}{3},0\right), \quad C_2\left(-\frac{\sqrt{5}}{3},0\right), \quad C_3\left(0, \frac{\sqrt{7} + \sqrt{29}}{2}\right), \quad C_4\left(0, -\frac{\sqrt{7} + \sqrt{29}}{2}\right).$$ (3.5)

Let $\Gamma_1$ denote the homoclinic loop passing the saddle $S_1(1,0)$. Since

$$H(C_1) \approx -0.1131687243, \quad H(S_1) = -0.1,$$
the periodic orbits inside the $\Gamma_1$ expand to $\Gamma_1$ as $h$ increases to $-0.1$, and $\Gamma_1$ is orientated clockwise. Let $\Gamma_2$ denote the homoclinic loop passing the saddle $(-1,0)$. We know that $\Gamma_2$ and $\Gamma_1$ are symmetric about $y$-axis, see Fig. 3(a).

Let $\Gamma_3$ denote the homoclinic loop passing the saddle $S_3(0,y_3)$ with $y_3 = \frac{\sqrt{7} - \sqrt{29}}{2}$ given in (3.4). Since

$$H(C_3) \approx 1.2050, \quad H(S_3) \approx -0.096542,$$

the periodic orbits inside the $\Gamma_3$ expand to $\Gamma_3$ as $h$ decreases to $-0.096542$, and $\Gamma_3$ is orientated counterclockwise. Let $\Gamma_4$ denote the homoclinic loop passing the saddle $S_4(0,-y_3)$. We know that $\Gamma_3$ and $\Gamma_4$ are symmetric about $x$-axis, see Fig. 3(b).

We introduce the following notations. For $i = 1, \cdots, 4$, let $\Gamma_i$ be expressed as $(x_i(t), y_i(t))$ for $t \in (-\infty, +\infty)$. Denote $D_i$ the region enclosed by $\Gamma_i$ and set

$$\gamma_i(t) = (x_i(t), y_i(t)), \quad (3.6)$$

$$\omega(x, y) := \frac{\partial P_3}{\partial x} + \frac{\partial Q_5}{\partial y} = \sum_{i=0, \neq 3}^5 \omega_i(x, y)A_i + \nu_4(x, y)B_4 + \nu_5(x, y)B_5, \quad (3.7)$$

$$\omega_0(x, y) = 2, \quad \omega_1(x, y) = 4(x^2 + y^2),$$

$$\omega_2(x, y) = 6(x^2 + y^2)^2, \quad \omega_4(x, y) = 2x(x^2 - 3y^2), \quad \omega_5(x, y) = 4\omega_4(x, y),$$

$$\nu_4(x, y) = 2y(3x^2 - y^2), \quad \nu_5(x, y) = -4\nu_4(x, y). \quad (3.8)$$

**Theorem 3.1.** The homoclinic loop $\Gamma_1$ can be expressed as $y = y_1^+(x) \ (> 0)$ and $y = y_1^-(x)$ with $y_1^-(x) = -y_1^+(x)$. It intersects $x$-axis at the points $(\frac{\sqrt{3}}{3}, 0)$ and $(1, 0)$. We express $\Gamma_2$ as $y = y_2^+(x) \ (> 0)$ and $y = y_2^-(x) \ (< 0)$. Then $y_2^+(x) = -y_2^+(x)$ and $y_2^-(x) = y_2^+(x)$, see Fig. 4. Suppose the Abelian integral of system (1.3) near $\Gamma_i$ is expressed as

$$I^{(i)}(h) = c_{i0} + c_{i1}(h_1 - h)\ln(h_1 - h) + c_{i2}(h_1 - h) + h.o.t., \quad i = 1, 2, \quad (3.9)$$

where $h_1 = H(S_1)$. Then
Fig. 4. About the calculation of $c_{12}$ and $c_{22}$: $x(t^*) = \frac{\sqrt{3}}{3}, x(t_2) = x_2^*, x(t_1) = x_1^*, C_1 = (\frac{\sqrt{3}}{3}, 0)$.

\[ \begin{align*}
(i) & \quad \begin{cases} 
    c_{10} = (\alpha_0, \alpha_1, \alpha_2, \alpha_4, 4\alpha_4) \cdot (A_0, A_1, A_2, A_4, A_5)^T, \\
    c_{20} = (\alpha_0, \alpha_1, \alpha_2, -\alpha_4, -4\alpha_4) \cdot (A_0, A_2, A_2, A_4, A_5)^T, \\
    \alpha_i = \int_{\tilde{D}_1} \omega_i(x, y) \, dx \, dy;
  \end{cases} \\
(ii) & \quad \begin{cases} 
    c_{11} = \frac{2}{\lambda_1} (A_0 + 2A_1 + 3A_2 + A_4 + 4A_5), \\
    c_{21} = \frac{2}{\lambda_1} (A_0 + 2A_1 + 3A_2 - A_4 - 4A_5),
  \end{cases}
\end{align*} \]

where $\lambda_1 = \sqrt{\frac{24}{5}}$ is the eigenvalue of system (3.1) at the saddle $S_1(1, 0)$.

(iii) For $x_1^*$ and $x_2^*$ satisfying $\frac{\sqrt{3}}{3} < x_2^* < \frac{\sqrt{3}}{3} < x_1^* < 1$, and $y_2^* = -y_+(x_2^*)$ (see Fig. 4), $c_{12}$ and $c_{22}$ can be expressed as

\[ c_{12} = -(J_{1(1)} + J_{2(1)} + J_{3(1)}), \quad c_{22} = -(J_{1(2)} + J_{2(2)} + J_{3(2)}), \]

where

\[ \begin{align*}
J_{1(1)}^{(1)} &= 4 \int_{x_1^*}^{1} \left. \frac{2yA_1 + 3y(2x^2 + y^2)A_2 - 3xyA_4 - 12xyA_5}{H_1(x, y)} \right|_{y = y_+(x)} dx \\
&\quad + 4 \int_{x_1^*}^{1} \left. \frac{x - 1}{y} \cdot \frac{\beta_1 A_1 + \beta_2 A_2 + \beta_4 B_4 + 4\beta_4 B_5}{H_1(x, y)} \right|_{y = y_+(x)} dx, \\
J_{1(2)}^{(1)} &= 4 \int_{x_1^*}^{1} \left. \frac{2yA_1 + 3y(2x^2 + y^2)A_2 + 3xyA_4 + 12xyA_5}{H_1(x, y)} \right|_{y = y_+(x)} dx \\
&\quad + 4 \int_{x_1^*}^{1} \left. \frac{x - 1}{y} \cdot \frac{\beta_1 A_1 + \beta_2 A_2 - \beta_4 B_4 - 4\beta_4 B_5}{H_1(x, y)} \right|_{y = y_+(x)} dx;
\end{align*} \]
Then, by (2.9), we get
\[
c_{12} = - \frac{\partial}{\partial t_1} \omega(\gamma_1(t)) \ dt = -(J_1^{(1)} + J_2^{(1)} + J_3^{(1)}).
\]
By system (3.1), we get
\[
J_1^{(1)} = \int_1^{x(t_1)} \frac{\omega}{H_y} (x, y_-(x)) \ dx + \int_{x(t_4)}^{1} \frac{\omega}{H_y} (x, y_+(x)) \ dx.
\]
\[
\omega(x, y) = \int_{x(t_1)}^{1} \frac{\omega(x, y_+(x)) + \omega(x, y_-(x))}{H_y(x, y_+(x))} \, dx.
\]

It is easy to know that \(\omega(x, y)\) under (3.15) can be expressed as

\[
\omega(x, y) = f(x, y) + 2(x - 1)(\beta_1 A_1 + \beta_2 A_2 + \beta_3 B_4 + 4\beta_4 B_5)
\]

where \(\beta_i (i = 1, \cdots, 4)\) are given by (3.13) and

\[
f(x, y) = 2y[2yA_1 + 3y(2x^2 + y^2)A_2 - 3xyA_4 - 12xyA_5 + (3x^2 - y^2)B_4 - 4(3x^2 - y^2)B_5].
\]

Noting that \(y_-(x) = -y_+(x)\), we get

\[
J_1^{(1)} = \int_{x(t_1)}^{1} \left[ \frac{4y[2yA_1 + 3y(2x^2 + y^2)A_2 - 3xyA_4 - 12xyA_5]}{yH_1(x, y)} \right]_{y=y_+(x)} \, dx
\]

\[
+ \int_{x(t_1)}^{1} \left[ \frac{4(x - 1)(\beta_1 A_1 + \beta_2 A_2 + \beta_3 B_4 + 4\beta_4 B_5)}{yH_1(x, y)} \right]_{y=y_+(x)} \, dx
\]

which gives the first formula of (3.10). Similarly, we can get

\[
J_2^{(1)} = \int_{x(t_2)}^{x(t_1)} \frac{\omega(x, y_+(x)) + \omega(x, y_-(x))}{H_y(x, y_+(x))} \, dx.
\]

We know that \(\omega(x, y)\) under (3.15) can also be expressed as

\[
\omega(x, y) = \sum_{i=1, i\neq 3}^{5} \omega_i(x, y)A_i + \sum_{i=4}^{5} \nu_i(x, y)B_i - 4A_1 - 6A_2 - 2A_4 - 8A_5
\]

\[
= \sum_{i=1, i\neq 3}^{5} \tilde{\omega}_i(x, y)A_i + \nu_4(x, y)B_4 + \nu_5(x, y)B_5,
\]

where \(\tilde{\omega}_i(x, y)\) are given by (3.14) and \(\tilde{\omega}_5(x, y) = 4\tilde{\omega}_4(x, y)\). Substituting (3.18) into (3.17), we can get the first formula of (3.11). Noting that \(x(-y) = x(y)\), we have

\[
J_3^{(1)} = \left( \int_{t_2}^{t_3} \omega(\gamma(t)) \, dt \right) + \left( \int_{0}^{-y(t_2)} \frac{-y(t_2)}{0} \omega(x(y), y) \, dy \right).
\]

Let \(u = -y\), then

\[
\int_{y(t_2)}^{0} \frac{\omega(x(y), y)}{-x(y)H_2(x(y), y)} \, dy = \int_{0}^{-y(t_2)} \frac{\omega(x(-u), -u)}{-x(-u)H_2(x(-u), -u)} \, du.
\]

Since \(H_2(x, -u) = H_2(x, u)\), thus
By \( \nu(x, -y) = -\nu(x, y) \) and \( \tilde{\omega}_i(x, -y) = \tilde{\omega}_i(x, y) \), we can get the first formula of (3.12).

Next, let us consider the expressions of \( c_{2i} \) \( (i = 0, 1, 2) \). By Lemma 2.4(ii), we get

\[
c_{20} = \int_{D_2} Q_5 \, dx - P_5 \, dy = \sum_{i=1, \neq 3}^5 A_i \iint_{D_2} \omega_i(x, y') \, dx \, dy + \sum_{i=4}^5 B_i \iint_{D_2} \nu_i(x, y') \, dx \, dy.
\]

Noting that

\[
\iint_{D_2} \omega_i(x, y') \, dx \, dy = \iint_{D_1} \omega_i(x, y) \, dx \, dy, \quad i = 1, 3, 6;
\]
\[
\iint_{D_2} \omega_4(x, y') \, dx \, dy = -\iint_{D_1} \omega_4(x, y) \, dx \, dy;
\]

we get

\[
c_{20} = (\alpha_0, \alpha_1, \alpha_2, -\alpha_4, -4\alpha_4) \cdot (A_0, A_1, A_2, A_4, A_5)^T,
\]

where \( \alpha_i \) \( (i = 1, \ldots, 4) \) are given in Theorem 3.1(i). By Lemma 2.4(ii) and (3.7), we have

\[
c_{21} = \frac{1}{\lambda_1} \omega(-1, 0) = \frac{2}{\lambda_1} (A_0 + 2A_1 + 3A_2 - A_4 - 4A_5).
\]

It follows from \( c_{21} = 0 \) that

\[
A_0 = -2A_1 - 3A_2 + A_4 + 4A_5.
\]

(3.19)

Let us give the expression of \( c_{22} \). For \(-\infty < \tau_1 < \tau_2 < \tau^* < \tau_3 < \tau_4 < +\infty \) such that

\[
x(\tau_1) = x(\tau_4) = -x(t_1), \quad x(\tau_2) = x(\tau_3) = -x(t_2), \quad x(\tau^*) = -x(t^*),
\]

see Fig. 4, let

\[
J^{(2)}_1 = \left( \int_{-\infty}^{\tau_1} + \int_{\tau_4}^{+\infty} \right) \omega(\gamma_2(t)) \, dt,
\]
\[
J^{(2)}_2 = \left( \int_{\tau_1}^{\tau_2} + \int_{\tau_3}^{\tau_4} \right) \omega(\gamma_2(t)) \, dt,
\]
\[
J^{(2)}_3 = \int_{\tau_2}^{\tau_3} \omega(\gamma_2(t)) \, dt.
\]

Then, by Lemma 2.4(ii), we get

\[
c_{22} = -\int_{\tau_2} \omega(\gamma_2(t)) \, dt = -(J^{(2)}_1 + J^{(2)}_2 + J^{(2)}_3). \quad \text{By (3.1), we get}
\]
\[ J^{(2)}_1 = \int_{-1}^{x(\tau_1)} \frac{\omega(x, y'_+(x))}{H_y(x, y'_+(x))} \, dx + \int_{x(\tau_1)}^{-1} \frac{\omega(x, y'_-(x))}{H_y(x, y'_-(x))} \, dx \]

\[ = \int_{-1}^{x(\tau_1)} \frac{\omega(x, y'_+(x)) + \omega(x, y'_-(x))}{H_y(x, y'_+(x))} \, dx. \]

It is easy to know that \( \omega(x, y) \) under (3.19) can be expressed as

\[ \omega = f(x, y) + 2(x + 1)(\beta_1 A_1 + \beta_2 A_2 + \beta_4 B_4 + 4\beta_4 B_5), \quad (3.20) \]

where \( f(x, y) \) is given by (3.16) and

\[ \beta_1(x) = 2(x - 1), \quad \beta_2(x) = 3(x - 1)(x^2 + 1), \quad \beta_4(x) = x^2 - x + 1. \]

Hence,

\[ J^{(2)}_1 = \int_{-1}^{x(\tau_1)} \frac{4y[2yA_1 + 3y(2x^2 + y^2)A_2 - 3xyA_4 - 12xyA_5]}{yH_1(x, y)} \bigg|_{y = y'_+(x)} \, dx \]

\[ + \int_{-1}^{x(\tau_1)} \frac{4(x + 1)(\beta_1 A_1 + \beta_2 A_2 + \beta_4 B_4 + 4\beta_4 B_5)}{yH_1(x, y)} \bigg|_{y = y'_+(x)} \, dx \]

\[ = 4 \int_{-1}^{x(\tau_1)} \frac{2yA_1 + 3y(2x^2 + y^2)A_2 - 3xyA_4 - 12xyA_5}{H_1(x, y)} \bigg|_{y = y'_+(x)} \, dx \]

\[ + 4 \int_{-1}^{x(\tau_1)} \frac{x + 1}{y'_+(x)} \cdot \frac{\beta_1 A_1 + \beta_2 A_2 + \beta_4 B_4 + 4\beta_4 B_5}{H_1(x, y)} \bigg|_{y = y'_+(x)} \, dx. \]

Since \( y'_+(-x) = y_+(x) \) and

\[ \tilde{\beta}_1(-x) = -\beta_1(x), \quad \tilde{\beta}_2(-x) = -\beta_2(x), \quad \tilde{\beta}_4(-x) = \beta_4(x), \]

we get

\[ J^{(2)}_1 = 4 \int_{x(\tau_1)}^1 \frac{2yA_1 + 3y(2x^2 + y^2)A_2 + 3xyA_4 + 12xyA_5}{H_1(x, y)} \bigg|_{y = y_+(x)} \, dx \]

\[ + 4 \int_{x(\tau_1)}^1 \frac{x - 1}{y_+(x)} \cdot \frac{\beta_1 A_1 + \beta_2 A_2 - \beta_4 B_4 - 4\beta_4 B_5}{H_1(x, y)} \bigg|_{y = y_+(x)} \, dx, \]

which yields the second formula of (3.10). Similarly, we have

\[ J^{(2)}_2 = \int_{x(\tau_1)}^{x(\tau_2)} \frac{\omega(x, y'_+(x)) + \omega(x, y'_-(x))}{H_y(x, y'_+(x))} \, dx. \]
We can check that \( \omega(x, y) \) under (3.19) can also be expressed as

\[
\omega(x, y) = \sum_{i=1, \neq 3}^{5} \tilde{\omega}_i(x, y)A_i + \nu_2(x, y)B_4 + \nu_5(x, y)B_5,
\]

\[
\tilde{\omega}_i(x, y) = \tilde{\omega}_i(x, y), \quad i = 1, 2;
\]

\[
\tilde{\omega}_4(x, y) = 2x(x^2 - 3y^2) + 2, \quad \tilde{\omega}_5(x, y) = 4\tilde{\omega}_4.
\]

Since \( \tilde{\omega}_i(-x, y) = \tilde{\omega}_i(x, y) \) for \( i = 1, 2 \) and \( \tilde{\omega}_4(-x, y) = -\tilde{\omega}_4(x, y) \), thus,

\[
J^{(2)}_2 = 2 \sum_{i=1, 2} A_i \int_{x(t_2)}^{x(t_1)} \frac{\tilde{\omega}_4}{H_y}(x, y(x)) \, dx - 2(A_4 + 4A_5) \int_{x(t_2)}^{x(t_1)} \frac{\tilde{\omega}_4}{H_y}(x, y(x)) \, dx,
\]

which yields the second formula of (3.11). For \( J^{(2)}_3 \), we have

\[
J^{(2)}_3 = \left( \int_{\tau_2}^{\tau_1} + \int_{\tau_2}^{\tau_2} \right) \omega(\gamma_2(t)) \, dt = \left( \int_{0}^{y(\tau_3)} + \int_{0}^{y(\tau_2)} \right) \omega \frac{x'(y)}{H_x} \, dy.
\]

Let \( u = -y \), then

\[
\int_{0}^{y(\tau_3)} \frac{\omega(x'(y), y)}{-x(y)H_2(x'(y), y)} \, dy = - \int_{0}^{y(\tau_3)} \frac{\omega(x'(-u), -u)}{-x'(-u)H_2(x'(-u), -u)} \, du.
\]

Since \( x'(-y) = x'(y) \) and \( H_2(x, -u) = H_2(x, u) \), we have

\[
J^{(2)}_3 = \int_{-y(\tau_3)}^{0} \frac{\omega(x'(y), y) + \omega(x'(y), -y)}{-H_x(x'(y), y)} \, dy.
\]

Noting that \( x'(y) = -x'(-y) \), we get

\[
J^{(2)}_3 = 2 \sum_{i=1, 2} A_i \int_{-y(t_2)}^{0} \frac{\tilde{\omega}_4}{H_x}(x(y), y) \, dy - 2(A_4 + 4A_5) \int_{0}^{-y(t_2)} \frac{\tilde{\omega}_4}{H_x}(x(y), y) \, dy,
\]

which yields the second formula of (3.12). This completes the proof. \( \square \)

**Theorem 3.2.** Suppose the Abelian integral of system (1.3) near the homoclinic loop \( \Gamma_i \) is expressed as

\[
I^{(i)}(h) = c_{i0} + c_{i1}(h - h_3) \ln(h - h_3) + c_{i2}(h - h_3) + h.o.t., \quad i = 3, 4,
\]

where \( h_3 = H(S_3) \). Then
\[
\begin{align*}
\begin{cases}
c_{30} = (\eta_0, \eta_1, \eta_2, \eta_4, -4\eta_4) \cdot (A_0, A_1, A_2, B_4, B_5)^T, \\
c_{40} = (\eta_0, \eta_1, \eta_2, -\eta_4, 4\eta_4) \cdot (A_0, A_1, A_2, B_4, B_5)^T, \\
\eta_i = -\int_{D_3} \omega_i(x, y) \, dx \, dy, \quad i = 0, 1, 2; \\
\eta_4 = -\int_{D_3} \nu_4(x, y) \, dx \, dy,
\end{cases}
\end{align*}
\]

where \(\omega_i(x, y)\) are given in (3.8).

**Proof.** We express the homoclinic loop \(\Gamma_3\) as \(x = x_{+}(y) > 0\) and \(x = x_{-}(y) = -x_{+}(y)\), and express \(\Gamma_4\) as \(x = x^{\prime}_{+}(x) > 0\) and \(x = x^{\prime}_{-}(y) < 0\). Then \(x^{\prime}_{-}(y) = -x^{\prime}_{+}(y)\) and \(x^{\prime}_{-}(-y) = x^{\prime}_{+}(y)\). By Lemma 2.4(i) and (3.7), we have

\[
c_{30} = \oint_{\Gamma_3} Q_5 \, dx - P_5 \, dy
\]

\[
= -\sum_{i=0,1,2} A_i \oint_{D_3} \omega_i(x, y) \, dx \, dy - (B_4 - 4B_5) \oint_{D_3} \nu_4(x, y) \, dx \, dy,
\]

which gives the first formula in (i). By Lemma 2.4(i), we get \(c_{31} = -\frac{1}{\lambda_3} \omega(S_3)\). Noting the orbits of system (3.1) are symmetric about both of the \(x\)-axis and \(y\)-axis, we can obtain the formula of \(c_{4i}\) by the similar arguments to that of \(c_{2i}\) \((i = 0, 1)\). This ends the proof. \(\square\)

For system (3.1), there exists a heteroclinic-polycycle \(S^{(5)}(6)\) through 6 saddles (see Fig. 3(b)). It is orientated counter-clockwise and corresponds to \(H(x, y) = H(S_3)\). It follows from [7] and [16] that for \(0 < h - H(S_3) \ll 1\), \(I(h)\) can be expressed as

\[
I^{(5)}(h) = \int_{S^{(5)}} Q_5(x, y) \, dx - P_5(x, y) \, dy + \text{h.o.t.}
\]

Denote \(c_{60} = \int_{S^{(6)}} Q_5 \, dx - P_5 \, dy\). Since system (1.3) is \(Z_3\)-equivariant, we get

\[
c_{60} = -3 \left( \int_{\Theta_2 S_3} + \int_{S_3 \Theta_1} \right) [Q_5(x, y) \, dx - P_5(x, y) \, dy],
\]

(3.22)

where \(\Theta_2(\xi_3, \eta_3)\) is obtained by rotating \(S_3\) counter-clockwise with \(\frac{\pi}{3}\), and \(\Theta_1(\xi_3, \eta_3)\) is obtained by rotating \(S_3\) clockwise with \(\frac{\pi}{3}\). By symmetry, we get \(x_3 = -x_{3r}\) and \(y_{3r} = y_{3}\). Hence,

\[
-\frac{1}{3} c_{60} = \left( \int_{0}^{x_{3r}} + \int_{x_{3r}}^{0} \right) [Q_5(x, y(x)) - P_5(x, y(x)) y'(x)] \, dx.
\]

Let \(u = -x\). Since along \(\Theta_2 S_3\) and \(S_3 \Theta_1\), we have \(y(-x) = y(x)\) and \(y'(-x) = -y'(x)\), then
\[
\int_{x_3}^{x_3} \left[ Q_5(x, y(x)) - P_5(x, y(x))y'(x) \right] dx \\
= \int_{-x_3}^{0} \left[ Q_5(-u, y(-u)) - P_5(-u, y(-u))y'(-u) \right] d(-u) \\
= \int_{x_3}^{0} \left[ Q_5(-u, y(u)) + P_5(-u, y(u))y'(u) \right] du.
\]

So,

\[
-\frac{1}{3} c_{60} = \int_{x_3}^{0} \left[ Q_5(x, y(x)) + Q_5(-x, y(x)) \right] dx \\
- \int_{x_3}^{0} \left[ P_5(x, y(x)) - P_5(-x, y(x)) \right] y'(x) dx \\
= 2 \int_{x_3}^{0} \left\{ y(x^2 + y^2)^2 A_2 + y(x^2 + y^2) A_1 + y A_0 \right\} \bigg|_{y=y(x)} dx \\
- 2 \int_{x_3}^{0} \left\{ x(x^2 + y^2)^2 A_2 + x(x^2 + y^2) A_1 + x A_0 \right\} \bigg|_{y=y(x)} y'(x) dx.
\] (3.23)

**Remark 3.1.** In the calculations of \(c_{i2} (i = 1, 2)\), we find that \(H_y(x, y)\) has the factor \(y\), which leads to the integrals cannot be calculated directly by Matlab even though we know that \(c_{i2}\) exists by Lemma 2.4. We deal with this problem by the way of (3.16) and (3.20).

4. **Proof of the main results**

Set \(x_1^* = 0.95, x_2^* = \sqrt{\frac{3}{4}} + 0.1\) and \(\frac{x-1}{y(x)} \approx -\lambda^{-1}\) for \(x \in (0.95, 1)\). Then by Matlab 7.5, we get

(i) The values in Theorems 3.1(i) and 3.2(i):

\[
\alpha_0 \approx 0.22126, \quad \alpha_1 \approx 0.24487, \quad \alpha_2 \approx 0.21722, \quad \eta_0 \approx 1.981659229; \\
\eta_1 \approx 7.62447, \quad \eta_2 \approx 27.60503, \quad \eta_3 \approx -5.21282, \quad \eta_4 \approx 4 \times 5.2128.
\]

(ii) The values for (3.10)–(3.12):
\[
\begin{aligned}
\int_{x_1}^{1} \frac{2y}{H_1(x, y)} \bigg|_{y=y_+(x)} \, dx &\approx 6.806860858 \times 10^{-4}, \\
\int_{x_1}^{1} \frac{3y(2x^2 + y^2)}{H_1(x, y)} \bigg|_{y=y_+(x)} \, dx &\approx 1.907295061 \times 10^{-3}, \\
\int_{x_1}^{1} \frac{\beta_3}{H_1(x, y)} \bigg|_{y=y_+(x)} \, dx &\approx 9.192955257 \times 10^{-1}, \\
\int_{x_1}^{1} \frac{\beta_6}{H_1(x, y)} \bigg|_{y=y_+(x)} \, dx &\approx 2.108043101 \times 10^{-1}, \\
\int_{x_1}^{1} \frac{\beta_4}{H_1(x, y)} \bigg|_{y=y_+(x)} \, dx &\approx 5.337516448 \times 10^{-2}, \\
\end{aligned}
\]

\[
\begin{aligned}
\int_{x_2}^{x_1} \frac{\tilde{\omega}_3}{H_y} (x, y_+(x)) \, dx &\approx -2.363016631, \\
\int_{x_2}^{x_1} \frac{\tilde{\omega}_6}{H_y} (x, y_+(x)) \, dx &\approx -5.808452011, \\
\int_{0}^{-y_2^*} \frac{\tilde{\omega}_3}{H_y} (x(y), y) \, dy &\approx -4.828232524, \\
\int_{0}^{-y_2^*} \frac{\tilde{\omega}_6}{H_y} (x(y), y) \, dy &\approx -9.852007882.
\end{aligned}
\]

(iii) The values for (3.23):

\[
\begin{aligned}
\int_{x_{31}}^{0} y(x^2 + y^2) \bigg|_{y=y(x)} \, dx &\approx 3.092091723 \times 10^{-2}, \\
\int_{x_{31}}^{0} (x^4 - y^4) \bigg|_{y=y(x)} \, dx &\approx -2.580407313 \times 10^{-2}, \\
\int_{x_{31}}^{0} y(x^2 + y^2) \bigg|_{y=y(x)} \, dx &\approx 9.015509722 \times 10^{-2}, \\
\int_{x_{31}}^{0} y(x) \, dx &\approx 2.646601805 \times 10^{-1},
\end{aligned}
\]
\[
\begin{align*}
\int_{x_M}^{} x(x^2 + y^2)^2 \bigg|_{y=y(x)} y'(x) \, dx &\approx -1.303820399 \times 10^{-2}, \\
\int_{x_M}^{} 2xy(x^2 + y^2) \bigg|_{y=y(x)} y'(x) \, dx &\approx -3.064747205 \times 10^{-2}, \\
\int_{x_M}^{} x(x^2 + y^2) \bigg|_{y=y(x)} y'(x) \, dx &\approx -3.064747205 \times 10^{-2}, \\
\int_{x_M}^{} xy'(x) \, dx &\approx -1.088923565 \times 10^{-1}.
\end{align*}
\]

**Remark 4.1.** It is crucial for us to obtain the expressions of the homoclinic loops $\Gamma_1$ and $\Gamma_3$. To do this, we solve the equations $H(x, y) = H(S_1)$ and $H(x, y) = H(S_3)$ by the formula of root for the equation $s^3 + as^2 + bs + c = 0$, where $s \in \mathbb{R}$ is variable. There are three branches for each of $H(x, y) = H(S_1)$ and $H(x, y) = H(S_3)$. Then, we plot the image for each of the branches to make sure that the expressions are right.

Let $A_4 = -4A_5$, then we have $\omega(O) = 2A_0$, and

\[
\begin{align*}
c_{10} &= c_{20} = \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \\
c_{11} &= c_{21} = 2(A_0 + 2A_1 + 3A_2)/\lambda_1, \\
c_{12} &\approx 16.05817189A_1 + 31.69816485A_2 + 0.09744927200B_4 + 0.3897970879B_5, \\
c_{22} &\approx 16.05817189A_1 + 31.69816485A_2 - 0.09744927200B_4 - 0.3897970879B_5, \\
\omega(C_1) = \omega(C_2) &= \frac{2}{27}[27A_0 + 30A_1 + 25A_2] \approx 2.40 + 2.2222A_1 + 1.8519A_2.
\end{align*}
\]

\[
\begin{align*}
c_{30} &= \eta_0 A_0 + \eta_1 A_1 + \eta_2 A_2 + \eta_4 B_4 - 4\eta_4 B_5, \\
c_{31} &= -\frac{1}{\lambda_3}[2A_0 + (7 - \sqrt{29})A_1 + 3(7 - \sqrt{29})^2 A_2 - \frac{1}{4}(7 - \sqrt{29})^2 B_4 + (7 - \sqrt{29})^\frac{3}{2} B_5] \\
&\approx -\frac{1}{\lambda_3}[2A_0 + 1.614835193A_1 + 0.9778847629A_2 - 0.5130176625B_4 + 2.05207650B_5], \\
\omega(C_3) &= 2A_0 + (7 + \sqrt{29})A_1 + 3(7 + \sqrt{29})^2 A_2 - \frac{1}{4}(7 + \sqrt{29})^3 B_4 + (7 + \sqrt{29})^\frac{3}{2} B_5 \\
&\approx 2A_0 + 12.385A_1 + 57.521A_2 - 10.896B_4 + 43.584B_5;
\end{align*}
\]

\[
\begin{align*}
c_{40} &= \eta_0 A_0 + \eta_1 A_1 + \eta_2 A_2 - \eta_4 B_4 + 4\eta_4 B_5, \\
\omega(C_4) &\approx 2A_1 + 12.385A_1 + 57.521A_2 + 10.896B_4 - 43.584B_5, \\
c_{60} &\approx -0.9346069440A_0 - 0.3570457510A_1 - 0.1072962794A_2 \\
&+ 0.3387092711B_4 + 1.354837084B_5,
\end{align*}
\]

where $\lambda_1 > 0$ and $\lambda_3 > 0$ are respectively the eigenvalues of system (3.1) at the saddles $S_1$ and $S_3$.

**4.1. The 23 limit cycles for Z_3-equivariant quintic systems**

Let
\[
\begin{align*}
\mu_1 &= \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \\
\mu_2 &= \eta_0 A_0 + \eta_1 A_1 + \eta_2 A_2 + \eta_4 B_4 - 4 \eta_4 B_5, \\
\mu_3 &= - \left[2A_0 + (7 - \sqrt{29})A_1 + \frac{3}{8}(7 - \sqrt{29})^2 A_2 - \frac{1}{4}(7 - \sqrt{29})^3 B_4 + (7 - \sqrt{29})^3 B_5 \right].
\end{align*}
\]

(4.4)

Then we get

\[
\begin{align*}
A_0 &\approx 2.447341576 A_2 - 3.448106551 \mu_1 - 0.0961219038 \mu_2 - 0.9767044561 \mu_3, \\
A_1 &\approx -3.098455496 A_2 + 7.199444833 \mu_1 + 0.08685397331 \mu_2 + 0.885320703 \mu_3, \\
B_4 &\approx 1.694040964 A_2 + 4B_5 + 9.219378069 \mu_1 - 0.1013395027 \mu_2 + 0.9195296149 \mu_3, \\
\omega(C_1) &\approx -0.138921654 A_2 + 9.102553198 \mu_1 + 0.000765022 \mu_2 + 0.007773466 \mu_3, \\
\omega(C_3) &\approx 5.58255808 A_2 - 18.19016710 \mu_1 + 1.987717275 \mu_2 - 1.042889053 \mu_3, \\
\omega(C_4) &\approx 42.50127484 A_2 + 182.7303625 \mu_1 - 0.220803343 \mu_2 + 18.99668151 \mu_3, \\
\omega(O) &\approx 2(2.447341576 A_2 - 3.448106551 \mu_1 - 0.0961219038 \mu_2 - 0.9767044561 \mu_3),
\end{align*}
\]

and

\[
\begin{align*}
I^{(1)}(h) &= I^{(2)}(h) = \mu_1 + c_{11}^*(h_1 - h) \ln(h_1 - h) + \text{h.o.t.}, \\
I^{(3)}(h) &= \mu_2 + \frac{h_3}{\lambda_3} (h - h_3) \ln(h - h_3) + \text{h.o.t.}, \\
I^{(4)}(h) &= c_{40}^* + \text{h.o.t.}, \\
I^{(6)}(h) &= c_{60}^* + \text{h.o.t.},
\end{align*}
\]

(4.6)

with

\[
\begin{align*}
c_{11}^* &\approx \frac{2}{\lambda_1} (-0.749569416 A_2 + 10.95078312 \mu_1 + 0.0775860428 \mu_2 + 0.7883596849 \mu_3), \\
c_{40}^* &\approx 17.66147727 A_2 + 96.11800402 \mu_1 - 0.0565301320 \mu_2 + 9.586693437 \mu_3, \\
c_{60}^* &\approx -0.7145209599 A_2 + 2.709674167 B_5 \\
&+ 3.774781965 \mu_1 + 0.02450072754 \mu_2 + 0.9091836462 \mu_3.
\end{align*}
\]

(4.7)

Denote \( \mu = (\mu_1, \mu_2, \mu_3) \). Since

\[
\begin{align*}
\lim_{\mu \to 0} c_{11}^* &\approx \frac{2}{\lambda_1} (-0.749569416 A_2), \\
\lim_{\mu \to 0} c_{40}^* &\approx 17.66147727 A_2, \\
\lim_{\mu \to 0} c_{60}^* &\approx -0.7145209599 A_2 + 2.709674167 B_5, \\
\lim_{\mu \to 0} \omega(C_1) &= \lim_{\mu \to 0} \omega(C_2) \approx -0.138921654 A_2, \\
\lim_{\mu \to 0} \omega(C_3) &\approx 5.58255808 A_2, \\
\lim_{\mu \to 0} \omega(C_4) &\approx 42.50127484 A_2, \\
\lim_{\mu \to 0} \omega(O) &\approx 2 \times 2.447341576 A_2,
\end{align*}
\]

for some \( A_k^* \) and \( B_k^* \) such that \( A_k^* > 0 \) and \(-0.7145209599 A_2^2 + 2.709674167 B_5^* > 0 \), there exists a neighborhood \( U \) of \( \mu = 0 \) such that for \( \mu \in U \) we have

\[
c_{11}^* < 0, \quad \omega(C_1) < 0; \quad \omega(C_3) > 0; \quad c_{40}^* > 0, \quad \omega(C_4) > 0; \quad c_{60}^* > 0, \quad \omega(O) > 0.
\]
Set some $\mu^* \in U$ such that $\mu_i^* < 0$ ($i = 1, 2, 3$) with $|\mu_2| \ll |\mu_3|$. For $\mu = \mu^*$ and $\Delta = \Delta^*$ with $A_0^*, A_1^*$ and $B_1^*$ determined by (4.5), we get simultaneously the following facts.

1. For $j = 1, 2, 3$, $I^{(j)}(h)$ has 1 zero.
2. For $j = 1, 2, 3$, the stability of limit cycles (generated by homoclinic bifurcation) and the stability of centers $C_j$ yields 1 limit cycle in the region enclosed by $\Gamma_j$ in view of Poincaré–Bendixson theorem.
3. The broken position of the separatrixes of $S_4$ and the stability of the center $C_4$ yields one limit cycle.
4. The broken position of the separatrixes of the polycycle $S^{(6)}$ and the stability of the center $O$ yields one limit cycle.

From (1)–(4), we obtain $9 + 9 + 3 + 1 = 22$ limit cycles altogether. Finally, let us consider the limit cycles bifurcating from the unbounded period annulus. Let $V_i$ be the singular points obtained by rotating the saddle $S_1$ around the origin counter-clockwise with $\frac{2\pi}{6}i$ ($i = 1, 2, 3, 4, 5$) and $V_3 = S_2$. Then, we obtain a heteroclinic polycycle through the saddles $S_1, V_1, \ldots, V_5$, denoted as $V^{(6)}$, see Fig. 3(a). The unperturbed system has an unbounded period annulus with the boundary $V^{(6)}$ for $h \in (\infty, H(S_1))$ see Fig. 2(a). For $A_0^*, A_1^*, A_2^*, B_1^*$ and $B_5^*$ defined above and $h \in (\infty, H(S_1))$, we have

$$I(h) = \int_{\Gamma_h} Q_5(x, y) \, dx - \mathcal{P}_5(x, y) \, dy$$

$$= -\int_D \left( \frac{\partial \mathcal{P}_5}{\partial x} + \frac{\partial Q_5}{\partial y} \right) \, dx \, dy$$

$$= -\int_D \left\{ 2A_0^* + 4(x^2 + y^2)A_1^* + 6(x^2 + y^2)^2 A_2^* + 2y(3x^2 - y^2)B_1^* - 8y(3x^2 - y^2)B_5^* \right\} \, dx \, dy,$$

where $D$ is the region enclosed by $\Gamma_h$. It follows from (4.8) that $I(h)$ has the opposite sign with $A_2^*$ for $|h|$ sufficiently large. Hence, there exists an $h^*$ with $h^* < H(S_1)$ and $|h^*| \gg 1$ such that $I(h^*) < 0$. In the meanwhile, by (1.5), (1.6) and (3.7),

$$I(H(S_1)) = 3 \left( \int_{V_1 V_2} + \int_{V_2 V_3} \right) Q_5(x, y) \, dx - \mathcal{P}_5(x, y) \, dy$$

$$= -\int_{V_1 V_2 \cup V_2 V_1} \omega(x, y) \, dx \, dy - 3 \int_{V_2 V_3 \cup V_3 V_2} \omega(x, y) \, dx \, dy$$

$$+ 3 \left( \int_{V_2 V_1} + \int_{V_3 V_2} \right) Q_5(x, y) \, dx - \mathcal{P}_5(x, y) \, dy.$$

It is easy to see that both $\int_{V_1 V_2 \cup V_2 V_1} \omega(x, y) \, dx \, dy$ and $\int_{V_2 V_3 \cup V_3 V_2} \omega(x, y) \, dx \, dy$ are linear functions of $A_0^*$, $A_1^*, A_2^*, B_1^*, B_5^*$ and $\mu_i^*$ ($i = 1, 2, 3$), and

$$\left( \int_{V_2 V_1} + \int_{V_3 V_2} \right) Q_5(x, y) \, dx - \mathcal{P}_5(x, y) \, dy$$

$$= \int_{-1/2}^{1/2} Q_5 \left( x, \frac{\sqrt{3}}{2} \right) \, dx + \int_{-1}^{-1/2} \left( Q_5(x, \sqrt{3}(x + 1)) - \sqrt{3}\mathcal{P}_5(x, \sqrt{3}(x + 1)) \right) \, dx$$

$$= \sqrt{3}A_0^* + \frac{5}{6} \sqrt{3}A_1^* + \frac{7}{10} \sqrt{3}A_2^* + \frac{77}{64} B_6.$$
Hence, there exists a $B^*_6 > 0$ such that for $B_5 = B^*_6$, we have $I(H(S_1)) > 0$. Then, it follows from Corollary 2.1(ii) that there exists at least one "big" limit cycle around all 25 singular points for system (1.3). To sum up, we have obtained 23 limit cycles for system (1.3). The distribution of the 23 limit cycles is shown in Fig. 1(a).

4.2. The 20 limit cycles for $Z_3$-equivariant quintic systems

Set

$$
\begin{align*}
\mu_1 &= \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \\
\mu_2 &= 2(A_0 + 2A_1 + 3A_2), \\
\mu_3 &= \eta_0 A_0 + \eta_1 A_1 + \eta_2 A_2 + \eta_4 B_4 - 4\eta_4 B_5.
\end{align*}
$$

Then we get

$$
\begin{align*}
A_0 &\approx 1.518694660A_2 + 10.11889703\mu_1 - 1.238907158\mu_2; \\
A_1 &\approx -2.259347330A_2 - 5.059448517\mu_1 + 1.119453579\mu_2; \\
B_4 &\approx 2.568326255A_2 + 4.000000000B_5 - 3.553433097\mu_1 \\
&\quad + 1.166383357\mu_2 - 0.191834571\mu_3,
\end{align*}
$$

and

$$
\begin{align*}
f^{(1)}(h) &= \mu_1 + \mu_2/\lambda_1(h_1 - h) \ln(h_1 - h) + c_{12}^*(h_1 - h) + \text{h.o.t.}, \\
f^{(2)}(h) &= \mu_1 + \mu_2/\lambda_1(h_1 - h) \ln(h_1 - h) + c_{22}^*(h_1 - h) + \text{h.o.t.}, \\
f^{(3)}(h) &= \mu_3 + c_{31}^*(h - h_3) \ln(h - h_3) + \text{h.o.t.}, \\
f^{(4)}(h) &= c_{40}^* + \text{h.o.t.}, \\
f^{(6)}(h) &= c_{60}^* + \text{h.o.t.},
\end{align*}
$$

where

$$
\begin{align*}
c_{12}^* &\approx -4.33254141A_2 + 0.7795941759B_5 - 81.59177342\mu_1 + 18.09004120\mu_2 - 0.01869413936\mu_3, \\
c_{22}^* &\approx -4.83310445A_2 - 0.7795941759B_5 - 80.89921450\mu_1 + 17.86271478\mu_2 + 0.01869413936\mu_3, \\
c_{31}^* &\approx -\frac{1}{\lambda_3}[-0.95079622A_2 + 13.89059250\mu_1 - 1.268456544\mu_2 + 0.09841452355\mu_3], \\
c_{40}^* &\approx 26.7764924A_2 - 37.04684784\mu_1 + 12.16030402\mu_2 - 0.999999998\mu_3, \\
c_{60}^* &\approx 0.1499274240A_2 + 2.709674168B_5 - 8.854317565\mu_1 + 1.153259946\mu_2 - 0.06497614795\mu_3
\end{align*}
$$

and

$$
\begin{align*}
\omega(C_1) &= \omega(C_2) \approx -0.131530670A_2 + 8.99457515\mu_1 + 0.009860304\mu_2, \\
\omega(C_3) &\approx 4.59098313A_2 - 3.70382038\mu_1 - 1.32285945\mu_2 + 2.090352704\mu_3, \\
\omega(C_4) &\approx 60.56324797A_2 - 81.14479894\mu_1 + 24.09646495\mu_2 - 2.090352704\mu_3, \\
\omega(O) &\approx 2(1.518694660A_2 + 10.11889703\mu_1 - 1.238907158\mu_2).
\end{align*}
$$

Denote $\mu = (\mu_1, \mu_2, \mu_3)$. Then, we get
Choose $A_2^*, B_5^*$ such that $A_2^* > 0$ and

$$-4.83310445A_2^* - 0.7795941759B_5^* > 0, \quad 0.0249879037A_2^* + 0.4516123614B_5^* < 0.$$ 

It is easy to know that

$$-4.83310445A_2^* - 0.7795941759B_5^* > 0 \iff A_2^* < -18.07323923B_5^*,$$

$$0.0249879037A_2^* + 0.4516123614B_5^* < 0 \iff A_2^* < -0.1613029853B_5^*.$$

Then, there is a neighborhood $U$ of $\mu = 0$ such that for $\mu \in U$ we have $c_{60}^* < 0, \omega(O) < 0$; and

$$c_{22}^* > 0, \quad \omega(C_2) < 0; \quad c_{31}^* > 0, \quad \omega(C_3) > 0; \quad c_{40}^* > 0, \quad \omega(C_4) > 0.$$ 

Set some $\mu^* \in U$ such that $\mu_i^* < 0 (i = 1, 2, 3)$ with $\mu_i^* > 0$ and $|\mu_1| \ll |\mu_2|$. For $\mu = \mu^*$ and $\Delta = \Delta^*$ with $A_0^*, A_1^*$ and $B_5^*$ determined by (4.9) we get simultaneously the following facts.

1. $I^{(1)}(h)$ has 1 zero, $I^{(2)}(h)$ has 2 zeros and $I^{(3)}(h)$ has 1 zero.

2. The stability of limit cycles (generated by homoclinic bifurcation) and the stability of the center $C_3$ yields 1 limit cycle in the region enclosed by $I_3$.

3. The broken position of the separatrixes of $S_4$ and the stability of the center $C_4$ yields one limit cycle.

4. The broken position of the separatrixes of the polycycle $S^{(6)}$ and the stability of the center $O$ yields one limit cycle.

5. By the similar arguments as in Section 4.1, we can obtain a “big” limit cycle around all 25 singular points.

To sum up, we have obtained 20 limit cycles for system (1.3). The distribution of the 20 limit cycles is shown in Fig. 1(b).

### 4.3. The 24 limit cycles for $Z_6$-equivariant quintic systems

For $P_5, Q_5$ given in (1.5) and (1.6), let $A_i^* = B_i^* = 0 (i = 3, 4, 5)$. Then system (1.3) with $0 < \epsilon \ll 1$ is $Z_6$-equivariant. Let

$$\mu_1 = \alpha_0A_0 + \alpha_1A_1 + \alpha_2A_2, \quad \mu_2 = \eta_0A_0 + \eta_1A_1 + \eta_2A_2. \quad (4.10)$$

Then we get
\[
\begin{align*}
A_0 & \approx 4.246715224A_2 + 6.344517305\mu_1 - 0.2037625303\mu_2, \\
A_1 & \approx -4.72436223A_2 - 1.648988847\mu_1 + 0.1841160513\mu_2, \\
\omega(C_1) & \approx -0.15324264A_2 + 9.024614950\mu_1 + 0.0016217201\mu_2, \\
\omega(C_3) & \approx 7.50386299A_2 - 7.73396403\mu_1 + 1.872782579\mu_2,
\end{align*}
\]

and
\[
\begin{align*}
I^{(1)}(h) &= \mu_1 + \frac{c_{11}^*}{\lambda_1} (h_1 - h) \ln(h_1 - h) + \text{h.o.t.}, \\
I^{(3)}(h) &= \mu_2 + c_{31}^* (h - h_3) \ln(h - h_3) + \text{h.o.t.}, \\
c_{11}^* &\approx -2.010102733A_2 + 2.781097447\mu_1 + 0.1501394912\mu_2, \\
c_{31}^* &\approx -\frac{1}{\lambda_3} \left(1.842290815A_2 + 10.02618939\mu_1 - 0.1102079814\mu_2\right).
\end{align*}
\]

Denote \( \mu = (\mu_1, \mu_2) \). Since

\[
\begin{align*}
\lim_{\mu \to 0} c_{11}^* &\approx -2.010102733A_2, \\
\lim_{\mu \to 0} c_{31}^* &\approx -\frac{1.842290815}{\lambda_3} A_2, \\
\lim_{\mu \to 0} \omega(C_1) &\approx -0.15324264A_2, \\
\lim_{\mu \to 0} \omega(C_3) &\approx 7.50386299A_2,
\end{align*}
\]

for \( A_2^* > 0 \), there exists a neighborhood \( U \) of \( \mu = 0 \) such that for \( \mu \in U \) we have

\[
c_{11}^* < 0, \omega(C_1) < 0; \quad c_{31}^* < 0, \omega(C_3) > 0.
\]

Set some \( \mu^* \in U \) such that \( \mu_i^* < 0 \) \( (i = 1, 2) \). For \( \mu = \mu^* \) and \( \Delta = \Delta^* \) with \( A^*_i = A_i^* = B_i^* = B_5^* = 0 \) and \( A_{0,1}^* \) determined by \((4.11)\), we get simultaneously the following facts.

(1) For \( j = 1, 3 \), \( I^{(j)}(h) \) has 1 zero, which yields \( 2 \times 6 = 12 \) limit cycles (generated by homoclinic bifurcation) for system \((1.3)\) by Corollary 2.1 and rotating the vector fields around the origin by \( 2\pi/6 \) since system \((1.3)\) is \( Z_6 \)-equivariant.

(2) For \( j = 1, 3 \), the stability of limit cycles (generated by homoclinic bifurcation) and the stability of the centers \( C_j \) yields 1 limit cycle in the region enclosed by \( \Gamma_j \) in view of Poincaré–Bendixson theorem. So, we obtain another \( 2 \times 6 = 12 \) limit cycles for system \((1.3)\). The distribution of the 24 limit cycles is shown in Fig. 1(c).

4.4. The 19 limit cycles for \( Z_6 \)-equivariant quintic systems

Let

\[
\mu_1 = \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \quad \mu_2 = \frac{2}{\lambda_1} (A_0 + 2A_1 + 3A_2).
\]

Then we get
\[
\begin{align*}
A_0 & \approx 1.518694660A_2 + 10.11889704\mu_1 - 1.357154793\mu_2, \\
A_1 & \approx -2.259347330A_2 - 5.059448518\mu_1 + 1.226299954\mu_2,
\end{align*}
\]

and
\[ \begin{align*}
I^{(1)}(h) &= \mu_1 + \mu_2(h_1 - h) \ln(h_1 - h) + c_{12}^*(h_1 - h) + \text{h.o.t.}, \\
I^{(3)}(h) &= c_{30} + \text{h.o.t.}, \\
c_{12}^* &\approx -4.58282293A_2 - 81.24549397\mu_1 + 19.69213545\mu_2, \\
c_{30} &\approx 13.38823462A_2 - 18.52342392\mu_1 + 6.66047813\mu_2, \\
\omega(C_3) &\approx 32.57711555A_2 - 42.42430966\mu_1 + 12.47361745\mu_2.
\end{align*} \]

Denote \( \mu = (\mu_1, \mu_2) \). Since

\[
\lim_{\mu \to 0} c_{12}^* \approx -4.58282293A_2,
\lim_{\mu \to 0} c_{30}^* \approx 13.38823462A_2,
\lim_{\mu \to 0} \omega(C_3) \approx 32.57711555A_2,
\]

for \( A_2^* > 0 \), there exists a neighborhood \( U \) of \( \mu = 0 \) such that for \( \mu \in U \) we have

\[
c_{12}^* < 0, \quad c_{30}^* > 0, \quad \omega(C_3) > 0.
\]

Choose some \( \mu^* \in U \) such that \( \mu^*_2 < 0 \) and \( \mu^*_1 < 0 \) with \( |\mu_1| \ll |\mu_2| \). Then \( I^{(1)}(h) \) with \( \mu^*, A_2^*, A_4^* = A_5^* = B_4^* = B_5^* = 0 \) and \( A_{3,1}^* \) determined by (4.14) has 2 zeros. Simultaneously, the broken position of the separatrices of the saddle \( S_3 \) and the stability of center \( C_3 \) leads another 1 limit cycle in the region enclosed by \( I_3 \) according to Poincaré–Bendixson theorem. So, system (1.3) has \((2+1) \times 6 = 18\) limit cycles by Corollary 2.1 and rotating the vector fields around the origin by \( 2\pi/6 \), see Fig. 1(d). This completes the proof of Theorem 1.1. \( \square \)

**Remark 4.2.** In Section 4.1, we obtain a “big” limit cycle around all 25 singular points of system (1.3) and a limit cycle around only the origin. The configuration of the 23 limit cycles is new and different from the result obtained in [2].

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