Harvesting control in an integrodifference population model with concave growth term

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Abstract

We consider the harvest of a certain proportion of a population that is modeled by an integrodifference equation, which is discrete in time and continuous in the space variable. The dispersal of the population is modeled by an integral of the growth function evaluated at the current population density against a kernel function. A concave growth function is used. In our model, growth occurs first, then dispersal and lastly harvesting control before the next generation. With the goal of maximizing the discounted profit stream, the optimal control is characterized by an optimality system. Illustrative examples are computed numerically.

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1. Introduction

Integrodifference equations are models that are discrete in time and continuous in space and represent populations with separate growth and dispersal stages [8,13]. For certain populations, these equations do a better job of estimating the speed of invasion than reaction–diffusion equations [9]. A variety of dispersal mechanisms can be included by changing the kernel in the integral term.

We consider the harvest of a population modeled by the following integrodifference model:

\[ N_{t+1}(x) = (1 - \alpha_t(x)) \int_{\Omega} k(x, y) f(N_t(y), y) dy, \]

where \( t = 0, 1, \ldots, T - 1 \). The state variable \( N \) and the control \( \alpha \):

\[
N = N(\alpha) = (N_0(x), N_1(x), \ldots, N_T(x))
\]
\[
\alpha = (\alpha_0(x), \alpha_1(x), \ldots, \alpha_{T-1}(x)),
\]

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represent the population and proportion of the population to be harvested. We use $T$ time steps and $x$ as the spatial variable in a bounded domain $\Omega \subset \mathbb{R}^n$. The initial distribution $N_0(x)$ is given. We assume $0 \leq \alpha_t(x) \leq M < 1$ for all $t = 0, 1, \ldots, T - 1$, and $x \in \Omega$. On each time interval, growth occurs first, represented by $f(N_t(y), y)$. Then dispersal of the population occurs and is represented by a non-local operator, integration against the kernel $k(x, y)$. The harvesting occurs after the growth and dispersal stages.

Our goal is to maximize the profit (revenue less cost). The objective functional is defined as:

$$J(\alpha) = \sum_{t=0}^{T-1} \int_{\Omega} e^{-\delta t} \left[ A_t \alpha_t(x) \int_{\Omega} k(x, y) f(N_t(y), y) \, dy - \frac{B_t}{2} V(\alpha_t(x)) \right] \, dx. \quad (1.2)$$

The coefficient $e^{-\delta t}$ is a discount factor, $A_t$ gives the price factor, and $B_t$ is a weight factor that balances the two parts of the objective functional. Note that in this objective functional, the discount factor $e^{-\delta t}$ with $\delta > 0$ multiplies on the profit stream (revenue less cost), where in [6], we only discounted the revenue stream. The type of discounting here seems more realistic. We assume the cost of harvesting is a nonlinear function $V$, and assume that the $C^2$ function

$$V: [0, M] \rightarrow \mathbb{R}$$

is increasing and convex with

$$V''(\alpha) \geq b > 0 \quad \text{for all } \alpha \in [0, M].$$

In the numerical illustration, we choose a simple quadratic cost. Our goal is to find an $\alpha^*$ such that

$$J(\alpha^*) = \max_{\alpha} J(\alpha),$$

where the control set is:

$$U = \{\alpha \in \left(L^\infty(\Omega)\right)^T \mid 0 \leq \alpha_t(x) \leq M < 1, t = 0, 1, \ldots, T - 1\}.$$  

Optimal control of hybrid systems is a newly developing area and this paper is the third optimal control result in integro-difference equations. This work, using a concave growth function and convex cost function and a more realistic discount term, extends the work in the paper [6], which used a linear growth function. Such a nonlinear growth term causes the existence and sensitivity results to be more delicate than in [6], and requires more careful analysis. An optimal control result for a disease system with an integrodifference term in the spread of the pathogen is treated in [4], but this paper also has linear terms inside the integrand. Techniques used in this paper and [4,6] combine ideas from the discrete version of Pontryagin’s Maximum Principle [3,7,23] and optimal control of infinite dimensional systems [16,17,21]. See [9–11,22] for more background and stability results for integrodifference population models. See [2,5,12,14,15,18–20,24] for control of a variety of hybrid systems, which are quite different from the system treated here.

In Section 2, we state our assumptions and prove the existence of an optimal harvesting control. Section 3 gives the characterization of an optimal control in terms of an optimality system, which is the state equation coupled with an adjoint equation. Uniqueness of an optimal control is given in Section 4 under the assumption of the weight factors, $B_t$, being sufficiently large. The last section gives illustrative examples of numerical results.

2. Existence of an optimal control

2.1. Assumptions

1. The kernels are bounded and measurable such that

$$\left| \int_{\Omega} k(x, y) \, dx \right| \leq C \leq 1 \quad \text{for all } x \in \Omega \text{ and } 0 \leq k(x, y) \leq k_1 \text{ for } (x, y) \in \Omega \times \Omega.$$  

2. The function $f$ is twice differentiable in $N_t(y)$ and measurable in $y$.

$$f(N_t(y), y) \geq 0, \quad \forall N_t(y) \geq 0, \quad y \in \Omega$$

$$|f(N_t(y), y)| \leq C_R < \infty \quad \forall 0 \leq N_t(y) \leq R, \quad y \in \Omega.$$
Moreover, for almost all \( y \in \Omega \), \( f(\cdot, y) \) is nondecreasing and concave in the \( N \) variable (roughly speaking, \( \frac{\partial f(N(x), x)}{\partial N} \) is nonnegative and decreasing in the \( N \) variable).

Assuming \( M < 1 \), we define the control set as:

\[
U = \{ \alpha \in (L^\infty(\Omega))^T | 0 \leq \alpha_t(x) \leq M, t = 0, 1, \ldots, T - 1 \}.
\]

Given the above assumptions with \( N_0(x) \geq 0 \), and \( \alpha \in U \), the corresponding state \( N = N(\alpha) \) satisfies

\[
0 \leq N_t(x) \leq C_f(N_0),
\]

where \( C_f(N_0) \) is a constant that depends on the growth function value at \( N_0 \).

We note that this existence proof requires more careful functional analysis than the corresponding proof in [6] due to the nonlinear growth term.

**Theorem 2.1.** There exists an optimal control \( \alpha^* \) in \( U \) that maximizes the functional \( J(\alpha) \).

**Proof.** Due to \( L^\infty \) bounds on the state and control,

\[
\sup_{\alpha \in U} J(\alpha) < \infty.
\]

Let \( \{\alpha^n\} \) be a maximizing sequence of controls in \( U \), i.e.

\[
\lim_{n \to \infty} J(\alpha^n) = \sup_{\alpha \in U} J(\alpha).
\]

There exists \( \alpha^* \in U \) and \( N^* \in (L^\infty(\Omega))^T \) such that on a subsequence

\[
\alpha^n_t(x) \rightharpoonup \alpha^*_t(x) \quad \text{weakly in } L^2(\Omega), \ t = 0, \ldots, T - 1
\]

\[
N^n_t(x) \rightharpoonup N^*_t(x) \quad \text{weakly in } L^2(\Omega), \ t = 1, 2, \ldots, T. \tag{2.1}
\]

We do not know that \( N^*_t = N(\alpha^*)_t \), meaning we must show that \( N^*_t \) is the state corresponding to control \( \alpha^* \). There exists \( F \in (L^\infty(\Omega))^T \) such that for almost every \( x \in \Omega \)

\[
\int_{\Omega} k(x, y) f(N^n_t(y), y) \, dy \to \int_{\Omega} k(x, y) F_t(y) \, dy \quad \text{for all } t = 1, 2, \ldots, T - 1.
\]

The \( L^\infty \) bounds on the state, the assumption and Lebesgue’s dominant convergence theorem give

\[
\int_{\Omega} k(x, y) f(N^n_t(y), y) \, dy \to \int_{\Omega} k(x, y) F_t(y) \, dy \tag{2.2}
\]

strongly in \( L^2(\Omega), t = 1, 2, \ldots, T - 1. \)

We define \( F_0(y) = f(N_0(y), y) \). We will show by induction that

\[
\int_{\Omega} k(x, y) f((N(\alpha^*))_{t-1}(y), y) \, dy \geq \int_{\Omega} k(x, y) F_{t-1}(y) \, dy \quad \text{for } t = 1, 2, \ldots, T \tag{2.3}
\]

and

\[
(N(\alpha^*))_t(x) \geq N^*_t(x) \quad \text{for } t = 1, 2, \ldots, T. \tag{2.4}
\]

First from the definition of \( F_0(y) \), we have (2.3) is true for \( t = 1 \), which gives

\[
N^n_1(x) = (1 - \alpha^n_1(x)) \int_{\Omega} k(x, y) f(N_0(y), y) \, dy.
\]

Letting \( n \to \infty \),

\[
N^*_1(x) = (1 - \alpha^*_1(x)) \int_{\Omega} k(x, y) f(N_0(y), y) \, dy
\]

\[
= (N(\alpha^*))_1(x).
\]
Thus (2.3) and (2.4) hold for \( t = 1 \). Now suppose (2.3) and (2.4) hold for some \( 0 \leq t \leq T - 1 \). We shall prove that (2.3) and (2.4) hold for \( t + 1 \).

By (2.1) and Mazur’s theorem [25], there exists \( \beta^n_j, j = n, \ldots, m_n, \) satisfying

\[
\beta^n_j \geq 0, \quad n = 1, 2, \ldots, j = n, \ldots, m_n
\]

\[
\sum_{j=n}^{m_n} \beta^n_j = 1, \quad n = 1, 2, \ldots, \\
\sum_{j=n}^{m_n} \beta^n_j N^i_j(x) \to N^*_t \quad \text{strongly in } L^2(\Omega) \text{ for } t = 1, 2, \ldots, T.
\]

Using (2.2) and the \( \beta^n_j \) constants, we have

\[
\int_{\Omega} k(x, y)F_t(y) \, dy = \lim_{n \to \infty} \int_{\Omega} k(x, y) f(N^n_i(y), y) \, dy \\
= \lim_{n \to \infty} \sum_{j=n}^{m_n} \beta^n_j \int_{\Omega} k(x, y) f(N^i_j(y), y) \, dy \quad \text{in } L^2(\Omega).
\]

The concavity of \( f(\cdot, y) \), implies

\[
\sum_{j=n}^{m_n} \beta^n_j \int_{\Omega} k(x, y) f(N^i_j(y), y) \, dy \leq \int_{\Omega} k(x, y) f \left( \sum_{j=n}^{m_n} \beta^n_j N^i_j(y), y \right) \, dy.
\]

For a.e. \( x \in \Omega \), we have by (2.5)

\[
\int_{\Omega} k(x, y) f \left( \sum_{j=n}^{m_n} \beta^n_j N^i_j(y), y \right) \, dy \to \int_{\Omega} k(x, y) f(N^*_t(y), y) \, dy.
\]

This convergence together with (2.6) gives

\[
\int_{\Omega} k(x, y)F_t(y) \, dy \leq \int_{\Omega} k(x, y) f(N^*_t(y), y) \, dy.
\]

The monotonicity of \( f(\cdot, y) \) and the induction hypothesis implies

\[
\int_{\Omega} k(x, y) f(N^*_t(y), y) \, dy \leq \int_{\Omega} k(x, y) f((N(\alpha^*))_t(y), y) \, dy,
\]

which together with (2.8) gives (2.3) for \( t + 1 \).

The definition of \( N(\alpha^*) \) is directly

\[
(N(\alpha^*))_{t+1}(x) = (1 - \alpha^*_t(x)) \int_{\Omega} k(x, y) f((N(\alpha^*))_t(y), y) \, dy.
\]

By (2.1), (2.2), (2.5) and (2.8)

\[
N^*_t + 1(x) = (1 - \alpha^*_t(x)) \int_{\Omega} k(x, y) F_t(y) \, dy \\
\leq (1 - \alpha^*_t(x)) \int_{\Omega} k(x, y) f(N^*_t(y), y) \, dy.
\]

By (2.7)–(2.11), we conclude

\[
N^*_t + 1(x) \leq (N(\alpha^*))_{t+1}(x).
\]
Thus by induction, (2.3) and (2.4) hold. We also need to use the lower semicontinuity of the control sequence in the convex function $V$. Now we show that $\alpha^*$ achieves the maximum of $J$.

$$J(\alpha^*) = \sum_{t=0}^{T-1} \int_\Omega e^{-\delta t} \left[ A_r(\alpha^*_{i}) \int_\Omega k(x, y) f(N(\alpha^*)_i(y), y) \, dy - \frac{B_t}{2} V(\alpha^*_{i}(x)) \right] \, dx$$

$$\geq \sum_{t=0}^{T-1} \int_\Omega e^{-\delta t} \left[ A_r(\alpha^*_{i}) \int_\Omega k(x, y) F_i(y) \, dy - \frac{B_t}{2} V(\alpha^*_{i}(x)) \right] \, dx$$

$$\geq \lim_{n \to \infty} \sum_{t=0}^{T-1} \int_\Omega e^{-\delta t} \left[ A_r(\alpha^*_{i}) \int_\Omega k(x, y) F_i(y) \, dy - \frac{B_t}{2} V(\alpha^*_{i}(x)) \right] \, dx$$

$$= \lim_{n \to \infty} \sum_{t=0}^{T-1} \int_\Omega e^{-\delta t} \left[ A_r(\alpha^*_{i}) \int_\Omega k(x, y) f(N_i(y), y) \, dy - \frac{B_t}{2} V(\alpha^*_{i}(x)) \right] \, dx$$

$$= \lim_{n \to \infty} J(\alpha^*), \quad \text{using (2.2) and (2.3).} \quad \square$$

3. Characterization of an optimal control

To characterize an optimal control, we must differentiate the map

$$\alpha \to J(\alpha)$$

which requires first the differentiation of the solution map $\alpha \to N = N(\alpha)$. The directional derivative $\psi$ of this solution map is called the sensitivity of the state with respect to the control.

**Theorem 3.1.** The mapping $\alpha \in U \to N \in (L^\infty(\Omega))^T$ is differentiable in the following sense:

$$\frac{N^\epsilon_r(x) - N_r(x)}{\epsilon} \to \psi_r(x)$$

weakly in $L^2(\Omega)$ as $\epsilon \to 0$ for any $\alpha \in U$ and $l \in (L^\infty(\Omega))^T$ such that $(\alpha + \epsilon l) \in U$ for $\epsilon$ small, where $N^\epsilon = N(\alpha + \epsilon l)$. Also $\psi$, depending on $N$, $\alpha$ and $l$, satisfies:

$$\psi_{t+1}(x) = (1 - \alpha_t(x)) \int_\Omega k(x, y) \frac{\partial f(N_t(y), y)}{\partial N} \psi_t(y) \, dy - l_t(x) \int_\Omega k(x, y) f(N_t(y), y) \, dy \quad (3.1)$$

$$\psi_0(x) = 0$$

for $t = 0, 1, \ldots, T - 1$.

**Proof.** Consider the control-to-solution map:

$$\alpha + \epsilon l \to N^\epsilon = N(\alpha + \epsilon l).$$

Since

$$N^\epsilon_{t+1}(x) = (1 - \alpha_t(x) - \epsilon l_t(x)) \int_\Omega k(x, y) f(N^\epsilon_t(y), y) \, dy,$$

we have

$$\frac{N^\epsilon_{t+1}(x) - N_{t+1}(x)}{\epsilon} = (1 - \alpha_t(x)) \int_\Omega k(x, y) \left[ f(N^\epsilon_t(y), y) - f(N_t(y), y) \right] \, dy \quad - l_t(x) \int_\Omega k(x, y) f(N^\epsilon_t(y), y) \, dy. \quad (3.2)$$

Using $N^\epsilon_0 = N_0$ and

$$\frac{N^\epsilon_0(x) - N_1(x)}{\epsilon} = -l_0 \int_\Omega k(x, y) f(N_0(y), y) \, dy$$
we obtain,
\[
\frac{N_1^\epsilon(x) - N_1(x)}{\epsilon} \leq C_1 \quad \text{for all } x \in \Omega
\]
and that quotient is independent of \(\epsilon\),
\[
\frac{N_1^\epsilon(x) - N_1(x)}{\epsilon} = \psi_1.
\]
This gives uniform convergence of \(N_1^\epsilon\) to \(N_1\), which then from (3.2) gives pointwise convergence of the quotient
\[
\frac{N_2^\epsilon(x) - N_2(x)}{\epsilon}
\]
using the regularity properties of \(\epsilon\). Then by iteration,
\[
\frac{N_t^\epsilon(x) - N_t(x)}{\epsilon} \leq C_2 \quad \text{for all } x \in \Omega, \ t = 1, 2, \ldots T.
\]
For all \(t = 1, 2, \ldots T\), we have the existence of \(\psi_t\) and the convergences
\[
\frac{N_t^\epsilon(x) - N_t(x)}{\epsilon} \to \psi_t \quad \text{pointwise}
\]
and
\[
N_t^\epsilon(x) \to N_t(x) \quad \text{uniformly.}
\]
Then passing to the limit using the pointwise convergence of the quotients, we obtain that \(\psi\) satisfies (3.1). \(\square\)

**Theorem 3.2.** Given an optimal control \(\alpha^*\) and corresponding state solution \(N^* = N(\alpha^*)\), there exists a weak solution \(p \in (L^\infty(\Omega))^T\) satisfying the adjoint system:
\[
p_{t-1}(x) = \frac{\partial f(N_{t-1}(x), x)}{\partial N} \int_\Omega (1 - \alpha_{t-1}(y)) k(y, x) p_t(y) \, dy + \frac{\partial f(N_{t-1}(x), x)}{\partial N} \int_\Omega A_{t-1} e^{-\delta(t-1)} \alpha_{t-1}^*(y) k(y, x) \, dy
\]
\[
p_T(x) = 0
\]
where \(t = T, \ldots, 2, 1\). Furthermore, for \(t = 0, 1, 2, \ldots, T-1\)
\[
V'(\alpha_t^*(x)) = \frac{2}{B_t} (A_t - p_{t+1}(x) e^{\delta t}) \int_\Omega k(x, y) f(N_t^*(y), y) \, dy
\]
on the interior of the control set.

**Proof.** Let \(\alpha^*\) be an optimal control (which exists by Theorem 2.1) and \(N(\alpha^*)\) be its corresponding state. For variation \(l\) with \((\alpha^* + \epsilon l) \in U\) for \(\epsilon > 0\) sufficiently small, let \(N^\epsilon\) be the corresponding solution of the state equation (1.1). Since the adjoint system is linear, there exists a weak solution \(p\) satisfying (3.2). We compute the directional derivative of the functional \(J(\alpha)\) with respect to \(\alpha\) in the direction \(l\). Since \(J(\alpha^*)\) is the maximum value, we have
\[
0 \geq \lim_{\epsilon \to 0^+} \frac{J(\alpha^* + \epsilon l) - J(\alpha^*)}{\epsilon}
\]
\[
= \lim_{\epsilon \to 0^+} \sum_{t=0}^{T-1} e^{-\delta t} \frac{1}{\epsilon} \left[ \int_\Omega \left( A_t(\alpha^* + \epsilon l)_{t}(x) \int_\Omega k(x, y) f(N_t^*(y), y) \, dy - \frac{B_t}{2} V((\alpha^* + \epsilon l)_{t}(x)) \right) \, dx 
\right.
\]
\[
- \int_\Omega \left( A_t\alpha_t^*(x) \int_\Omega k(x, y) f(N_t^*(y), y) \, dy - \frac{B_t}{2} V(\alpha_t^*(x)) \right) \, dx \right]
\]
\[
= \lim_{\epsilon \to 0^+} \sum_{t=0}^{T-1} e^{-\delta t} \int_\Omega A_t\alpha_t^*(x) \int_\Omega k(x, y) \left[ \frac{f(N_t^*(y), y)}{\epsilon} - \frac{f(N_t^*(y), y)}{\epsilon} \right] \, dy \, dx
\]
\[ + \int_{\Omega} e^{-\delta t} \left[ A_t l_t(x) \int_{\Omega} k(x, y) f(N_t^s(y), y) \, dy - \frac{B_t}{2} V'(\alpha_t^s) l_t \right] \, dx \]

\[ = \sum_{t=0}^{T-1} \left\{ e^{-\delta t} \int_{\Omega} A_t \alpha_t^s(x) \int_{\Omega} k(x, y) \frac{\partial f(N_t^s(y), y)}{\partial N} \psi_t(y) \, dy \, dx \right\} \]

\[ + \int_{\Omega} A_t e^{-\delta t} l_t(x) \int_{\Omega} k(x, y) f(N_t^s(y), y) \, dy \, dx - \int_{\Omega} \frac{B_t}{2} e^{-\delta t} V'(\alpha_t^s) l_t \, dx \]

\[ = \sum_{t=0}^{T-1} \left\{ \int_{\Omega} \psi_t(y) \frac{\partial f(N_t^s(y), y)}{\partial N} \int_{\Omega} A_t e^{-\delta t} \alpha_t^s(x) k(x, y) \, dy \, dx \right\} \]

\[ + \int_{\Omega} A_t e^{-\delta t} l_t(x) \int_{\Omega} k(x, y) f(N_t^s(y), y) \, dy \, dx - \int_{\Omega} e^{-\delta t} \frac{B_t}{2} V'(\alpha_t^s) l_t \, dx \}

where the last equality is obtained by switching the order of integration of the first integral. We use the coefficient of the \( \psi_t \) term as the non-homogeneous term in the adjoint system.

Substituting from the adjoint system (3.2), we simplify the first term in the \( J \) derivative expression

\[ \sum_{t=0}^{T-1} \int_{\Omega} \psi_t(y) \frac{\partial f(N_t^s(y), y)}{\partial N} \int_{\Omega} A_t e^{-\delta t} \alpha_t^s(x) k(x, y) \, dy \, dx \]

\[ = \sum_{t=0}^{T-1} \left\{ \int_{\Omega} \psi_t(y) \left[ p_t(y) - \frac{\partial f(N_t^s(y), y)}{\partial N} \int_{\Omega} (1 - \alpha_t^s(x)) k(x, y) p_{t+1}(x) \, dy \right] \right\} \]

\[ = \sum_{t=0}^{T-1} \int_{\Omega} p_{t+1} \psi_{t+1}(x) \, dx - \sum_{t=0}^{T-1} \int_{\Omega} p_{t+1}(y)(1 - \alpha_t^s(y)) \int_{\Omega} k(y, x) \frac{\partial f(N_t^s(x), x)}{\partial N} \psi_t(x) \, dx \, dy. \quad (3.5) \]

In the last equality, we rewrote first sum, added \( p_T \psi_T \) term, and used \( p_T = 0, \psi_0 = 0 \).

Then using the \( \psi \) system, (3.5) becomes:

\[ \sum_{t=0}^{T-1} \int_{\Omega} p_{t+1}(x) \left[ \psi_{t+1}(x) - (1 - \alpha_t(x)) \int_{\Omega} k(x, s) \frac{\partial f(N_t^s(s), s)}{\partial N} \psi_t(s) \, dx \right] \, dx \]

\[ = \sum_{t=0}^{T-1} \int_{\Omega} p_{t+1}(x) \left( -l_t(x) \int_{\Omega} k(x, y) f(N_t(y), y) \, dy \right) \, dx. \]

We go back to differentiating \( J \), substituting for the first term from the above \( p, \psi \) calculation,

\[ 0 \geq \sum_{t=0}^{T-1} \left\{ \int_{\Omega} \psi_t(y) \frac{\partial f(N_t^s(y), y)}{\partial N} \int_{\Omega} A_t e^{-\delta t} \alpha_t^s(x) k(x, y) \, dy \, dx \right\} \]

\[ + \int_{\Omega} A_t e^{-\delta t} l_t(x) \int_{\Omega} k(x, y) f(N_t^s(y), y) \, dy \, dx - \int_{\Omega} \frac{B_t}{2} e^{-\delta t} V'(\alpha_t^s) l_t \, dx \]

\[ = \sum_{t=0}^{T-1} \int_{\Omega} l_t(x) \left[ -p_{t+1}(x) \int_{\Omega} k(x, y) f(N_t^s(y), y) \, dy - \frac{B_t}{2} e^{-\delta t} V'(\alpha_t^s(x)) \right. \]

\[ + A_t e^{-\delta t} \int_{\Omega} k(x, y) f(N_t^s(y), y) \, dy \left. \right] \, dx. \]

For \( t = 0, 1, \ldots, T - 1 \), we obtain:

\[ V'(\alpha_t^s(x)) = \frac{2}{B_t} (A_t - p_{t+1}(x)e^{\delta t}) \int_{\Omega} k(x, y) f(N_t^s(y), y) \, dy \]

on the interior of the control set. \( \square \)
Remark. If \( V(\alpha) = \alpha^2 \), then we can solve explicitly for \( \alpha^* \) in the optimal control characterization

\[
\alpha_t^* = \min \left( \max \left( \left( \frac{1}{B_t} (A_t - p_{t+1}(x)e^{\delta t}) \int_\Omega k(x, y) f(N_t^*(y), y) \, dy \right), 0 \right), M \right)
\]  

by choosing appropriate choices of variation \( l \) on the size of \( \alpha^* \). In our numerical calculations, we will be solving the optimality system, which is (1.1) and (3.3) with \( N_0(x) \) known, \( P_T(x) = 0 \), and the characterization condition (3.6).

4. Uniqueness result

Theorem 4.1. If \( B_t, t = 0, 1, \ldots, T - 1 \) are sufficiently large, then the optimal control is unique.

Proof. We show uniqueness by showing strict concavity of the map:

\[ \alpha \in U \rightarrow J(\alpha). \]

The concavity follows from showing that for all \( \alpha, l \in U, 0 < \epsilon < 1 \)

\[ g''(\epsilon) < 0 \]

where \( g(\epsilon) = J(\epsilon l + (1 - \epsilon)\alpha) = J(\alpha + \epsilon(l - \alpha)). \)

First, we calculate

\[
g'(\epsilon) = \lim_{\tau \to 0} \frac{J(\alpha + (\epsilon + \tau)(l - \alpha)) - J(\alpha + \epsilon(l - \alpha))}{\tau}
\]

\[
= \sum_{t=0}^{T-1} e^{-\delta t} \left[ \int_\Omega A_t(\alpha_t + \epsilon(l_t - \alpha_t)) \int_\Omega k(x, y) \frac{\partial f(N_t^*(y))}{\partial N_t^*(y)} \psi_t^*(y) \, dy \, dx \right. \\
+ \left. \int_\Omega A_t(l_t - \alpha_t) \int_\Omega k(x, y) f(N_t^*(y)) \, dy \, dx - \frac{B_t}{2} \int_\Omega V'(\alpha_t + \epsilon(l_t - \alpha_t))(l_t - \alpha_t) \, dx \right]
\]

where, for \( t = 0, 1, \ldots, T - 1 \)

\[ N^\epsilon = N(\alpha + \epsilon(l - \alpha)) \]

\[ N^{\epsilon, \tau} = N(\alpha + (\epsilon + \tau)(l - \alpha)) \]

\[ \frac{N^\epsilon_{t, \tau} - N^\epsilon_t}{\tau} \to \psi_t^\epsilon \quad \text{as} \quad \tau \to 0 \]

\[ \psi_{t+1}^\epsilon(x) = (1 - (\alpha_t + \epsilon(l_t - \alpha_t))) (x) \int_\Omega k(x, y) \frac{\partial f(N_t^*(y))}{\partial N_t^*(y)} \psi_t^* \, dy - (l_t - \alpha_t) \int_\Omega k(x, y) f(N_t^*(y)) \, dy \]

\[ \psi_0^\epsilon \equiv 0. \]

Note that \( \psi_t^\epsilon(x) \) can be estimated in terms of \( l - \alpha \). For example,

\[ |\psi_2^\epsilon(x)| \leq C_1 \left[ \int_\Omega |l_0 - \alpha_0| \, dy + |l_1 - \alpha_1| \right], \]

where \( C_1 \) depends on \( L_\infty \) bounds on the function value at initial state. One can continue to estimate iteratively. Next we obtain

\[
g''(\epsilon) = \sum_{t=0}^{T-1} e^{-\delta t} \left[ \int_\Omega A_t(\alpha_t + \epsilon(l_t - \alpha_t)) \int_\Omega k(x, y) \left[ \frac{\partial^2 f(N_t^*(y))}{\partial(N_t^*)^2} \psi_t^*(x) \right]^2 \right. \\
+ \left. \frac{\partial f(N_t^*(y))}{\partial N_t^*(y)} \sigma_t^*(y) \right] \, dy \, dx + \int_\Omega A_t(l_t - \alpha_t) \int_\Omega k(x, y) \frac{\partial f(N_t^*(y))}{\partial N_t^*(y)} \psi_t^*(y) \, dy \, dx \\
- \frac{B_t}{2} \int_\Omega V''(\alpha_t + \epsilon(l_t - \alpha_t))(l_t - \alpha_t)^2 \, dx \]

where, for \( t = 0, 1, \ldots, T - 1 \)
\[
\sigma_{t+1}^\alpha(x) = (1 - (\alpha_t + \epsilon(l_t - \alpha_t))) (x) \int_\Omega k(x, y) \left[ \frac{\partial^2 f(N_t^\alpha(y))}{\partial (N_t^\alpha)^2} \left( \psi_t^\epsilon(y) \right)^2 + \frac{\partial f(N_t^\alpha(y))}{\partial N_t^\alpha} \sigma_t^\epsilon(y) \right] dy
- 2(l_t - \alpha_t) \int_\Omega k(x, y) \frac{\partial f(N_t^\alpha(y))}{\partial N_t^\alpha} \psi_t^\epsilon(y) dy
\]
\( \sigma_0^\epsilon \equiv 0. \)

We can estimate \( \int_\Omega |\sigma_t^\epsilon(y)| dy \) in terms of \( \int_\Omega (l_t - \alpha_t)^2 dy \), \( k = 0, 1, \ldots, t - 1 \). Using the estimates,
\[
g''(\epsilon) \leq \sum_{t=0}^{T-1} \left( C_2 - b \frac{B_t}{2} \right) e^{-\delta t} \int_\Omega (l_t - \alpha_t)^2 dy,
\]
which gives the desired concavity for \( B_t \)'s sufficiently large. \( \square \)

5. Numerical results

Dispersal kernels have been measured for a wide variety of organisms. The resulting data display an incredible variety of patterns and strategies. Some dispersal kernels have been found to be leptokurtic (fat-tailed) rather than normal [10]. From the literature on integrodifference models, we explored the following four dispersal kernels: the finite range dispersal kernel [11]
\[
k(x, y) = \begin{cases} 0, & x \leq y - R, \\ \frac{\pi}{4R} \cos \left( \frac{\pi}{2R} |x - y| \right), & y - R \leq x \leq y + R, \\ 0, & x \geq y + R, \end{cases}
\]
the Laplace kernel [8],
\[
k(x, y) = \frac{1}{2} \beta \exp(-\beta |x - y|),
\]
normal dispersal [10],
\[
k(x, y) = \sqrt{\frac{\beta}{\pi}} \exp\left( -\beta (x - y)^2 \right),
\]
and a Double-Weibull-type dispersal [22],
\[
k(x, y) = \frac{3a}{2} |x - y|^2 \exp\left( -a(x - y)^3 \right).
\]

We use an iterative method to solve the optimality system. With the initial estimates for the adjoint, \( p, N_0 \) and \( \alpha \), the iterative process begins by solving the equations for \( N \) forward in time. Then using these new estimates for \( N \), new \( p \) values are calculated backwards through time. The control \( \alpha \) is updated using a convex combination of the old value and the value calculated from characterization (3.6). Note we are using
\[
V(\alpha) = \alpha^2
\]
in these calculations. The new estimates of \( N \) and \( p \) are compared with those from the previous iteration. The composite trapezoid rule is used to estimate the needed integrals [1]. If the difference is less than one percent for every point, the final control \( \alpha \) is saved as the optimal solution.

For each of our numerical solutions, we used 10 time steps. We also assumed that \( \Omega = [0, 1] \), which was divided into 5000 spatial grid points for the trapezoid rule. The figures are generated using the Beverton–Holt type growth function, which satisfied the needed concavity assumptions and is commonly used for modeling fishery populations. We use
\[
\frac{r N_y}{1 + \left( \frac{r-1}{\kappa} \right) N_y},
\]
where $r$ and $K$ are the growth rate and carrying capacity respectively. The following parameter values are used to generate the illustrated figures: $r = 1.8, K = 100, A_t = 1.0, B_t = 1600, N_0(x) = 5.0$, and for the kernels $R = 0.5, a = 25$.

We illustrate the numerical results here for only two kernels, the finite range and the Double-Weibull type dispersal kernels. The first four figures (Figs. 1–4) for $\delta = 0.05$, which is an interest rate of 5%, show that the harvest begins on the edge of the domain earlier than in the middle. With the Double-Weibull kernel with non-zero mode, the peak of the population is higher and there is more harvesting at the edges initially.
Figs. 5 and 6 show the contrasting effects of using a much larger discount factor, $\delta = 0.25$, which causes a lower population and more harvesting at the beginning (to make more profit by investing).

The proximity of the initial population to the carrying capacity also affects the predicted optimal harvest. For a carrying capacity of 100, compare the harvest and populations in Figs. 7 and 8 with an initial population of 75 with those in Figs. 1 and 2 with an initial population of 5. The growth function used modulates the growth rate as the population approaches the carrying capacity such that it is slow enough that over 10 time steps, there isn’t enough benefit to wait to harvest until a later time if the population is already near carrying capacity. If the initial population
is far below carrying capacity, it is optimal to harvest a bit later to wait for the growth to happen. The length of time to wait to harvest will depend upon how far the population is below carrying capacity.

The optimal control of this integrodifference model has been completely characterized through the analysis in this paper and the optimal harvesting strategy for particular examples were calculated numerically using the optimality system. With this model and control analysis, one can see the effects on the harvesting strategy of the various parameters.

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