THE REGULARITY OF SEMIHYPERBOLIC PATCHES NEAR SONIC LINES FOR THE 2-D EULER SYSTEM IN GAS DYNAMICS

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Abstract. We study the regularity of semihyperbolic patches of self-similar solutions near sonic lines to a Riemann problem for the two-dimensional (2-D) Euler system. As a result, it is verified that there exists a global solution in the semihyperbolic patch up to the sonic boundary and that the sonic boundary has $C^1$-regularity. The study of the semihyperbolic patches of solutions for the Euler system was initiated by Li and Zheng [Arch. Rational Mech. Anal., 201 (2011), pp. 1069–1096]. This type of solution appears in the transonic flow over an airfoil and Guderley reflection and is common in the numerical configurations of 2-D Riemann problems.

Key words. semihyperbolic patches, regularity, two-dimensional Riemann problem, characteristic decomposition, sonic line

AMS subject classifications. Primary, 35L65, 35J70, 35R35; Secondary, 35J65

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1. Introduction. Previously, Li and Zheng [8] established the global existence of smooth solutions in the semihyperbolic patches for the isentropic Euler system, which is frequently observed in several numerical configurations of the two-dimensional (2-D) Riemann problems of the Euler system. In this paper, we improve the result of [8]; namely, we obtain the global existence of solutions up to the sonic lines, where degeneracy of hyperbolicity occurs, in the semihyperbolic region for the isentropic Euler system, and, moreover, we also verify that the sonic lines are $C^1$ continuous.

This paper concerns sonic lines and semihyperbolic regions arising from the transonic problems of the 2-D isentropic Euler system. Recently, there have been many of works on multidimensional transonic problems using various approximate models of multidimensional gas flows. In particular, the research in [1, 7, 12, 13, 16, 17] was developed under specific physical models such as infinite long nozzle problems or shock reflection problems, etc. It is verified that the problems on the sonic lines and semihyperbolic regions are not only from the above physical situations but also 2-D four-wave Riemann problems.

In general, the initial value problem of the 2-D full Euler system is regarded as very difficult. One of the ways to handle this problem is to consider simple initial data. The Riemann problem is such a case. In this paper, we will consider the 2-D isentropic compressible Euler system

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0, \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0.
\end{align*}
\]

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Here $\rho$ is density, $(u,v)$ is velocity, and $p$ is pressure given by $p(\rho) = A\rho^\gamma$, where $A > 0$ can be scaled to be one and $\gamma > 1$ is the gas constant. In the self-similar plane $(\xi, \eta) = (x/t, y/t)$, system (1.1) can be changed into

(1.2) \[
\begin{cases}
U_i\xi + V_i\eta + 2\kappa_i(u_i + v_i) = 0, \\
U u_i + V u_i + i_i = 0, \\
U v_i + V v_i + i_i = 0,
\end{cases}
\]

where $i = \frac{c^2}{\gamma - 1}$ is the enthalpy, $c = \sqrt{\gamma p/\rho}$ is sound speed, $(U,V) = (u - \xi, v - \eta)$ is the pseudovelocity, and $\kappa = (\gamma - 1)/2$. The eigenvalues of (1.2) are

$$\Lambda_{\pm} := \frac{UV \pm c\sqrt{U^2 + V^2 - c^2}}{U^2 - c^2}, \quad \Lambda_0 := \frac{V}{U}. $$

The pseudowave characteristics for system (1.2) are defined by the integral curves of

(1.3) \[
\frac{d\eta}{d\xi} = \Lambda_{\pm}.
\]

We shall assume that the flow is irrotational with $u_\eta = v_\xi$. Then system (1.2) can be rewritten as

(1.4) \[
\begin{cases}
(c^2 - U^2)u_\xi - UV(u_\eta + v_\xi) + (c^2 - V^2)v_\eta = 0, \\
u_\eta - v_\xi = 0,
\end{cases}
\]

supplemented by the pseudo-Bernoulli’s law,

(1.5) \[
\frac{c^2}{\gamma - 1} + \frac{U^2 + V^2}{2} = -\varphi, \quad \varphi_\xi = U, \quad \varphi_\eta = V,
\]

where $\varphi$ is a pseudovelocity potential.

By the standard diagonalization, we can rewrite system (1.4) as

(1.6) \[
\partial^\pm u + \Lambda_{\pm} \partial^\pm v = 0,
\]

where $\partial^\pm := \partial_\xi + \Lambda_{\pm} \partial_\eta$. Let us define the inclination angles $(\alpha, \beta)$ by

$$\tan \alpha = \Lambda_+, \quad \tan \beta = \Lambda_-$$

for $\alpha \in [\pi/2, \pi]$ and $\beta \in [-\pi/2, 0]$. We also define

$$\varpi = \frac{\alpha - \beta}{2}, \quad \sigma = \frac{\alpha + \beta}{2}, \quad \nu = \frac{\gamma + 1}{2(\gamma - 1)},$$

$$\mu = m - \tan^2 \varpi, \quad m = \frac{3 - \gamma}{\gamma + 1}, \quad M := \frac{q}{c} = \frac{1}{\sin \varpi},$$

where $q = \frac{\sqrt{U^2 + V^2}}{c}$ and $M$ is called pseudo-Mach number. We introduce new directional derivatives

$$\overrightarrow{\partial}^+ := \cos \alpha \partial_\xi + \sin \alpha \partial_\eta, \quad \overrightarrow{\partial}^- := \cos \beta \partial_\xi + \sin \beta \partial_\eta, \quad \overrightarrow{\partial}^0 := -\cos \sigma \partial_\xi - \sin \sigma \partial_\eta.$$
Using this notation, system (1.4) can be changed into the diagonal form

\[
\begin{align*}
\partial^+ (\Psi(\varpi) - \beta) &= J(c, \varpi), \\
\partial^- (\Psi(\varpi) + \alpha) &= J(c, \varpi), \\
\partial^0 [c^2(1 + \kappa M^2)] &= 2ckM,
\end{align*}
\]

where

\[
\Psi(\varpi) = \sqrt{2}\nu \arctan \left( \cot \frac{\varpi}{\sqrt{2}\nu} \right), \quad J(c, \varpi) = \frac{\sin^2 \varpi(\cos 2\varpi - \kappa)}{c(\sin^2 \varpi + \kappa)}.
\]

Here, \((\Psi(\varpi) + \alpha, \Psi(\varpi) - \beta)\) are called Riemann variables. There is a one-to-one correspondence between \((\alpha, \beta)\) and \((\Psi(\varpi) + \alpha, \Psi(\varpi) - \beta)\). According to the definitions of \(\Lambda^+, \Lambda^-, \) we can find

\[
U = -c \frac{\cos \sigma}{\sin \varpi}, \quad V = -c \frac{\sin \sigma}{\sin \varpi}.
\]

Then

\[
u = U + \xi = \xi - c \cdot \frac{\cos \sigma}{\sin \varpi}, \quad v = V + \eta = \eta - c \cdot \frac{\sin \sigma}{\sin \varpi}.
\]

In particular, later \(\partial^\pm\) are used to represent first-order normalized derivatives of other variables, and it is shown that \(\partial^+ c + \partial^- c = 0\) at sonic points [8].

There have been many attempts to verify the global structures of solutions for 2-D four-wave Riemann problems of the Euler systems by rigorous theories, numerical simulations, and various transonic models [3, 6, 7, 14, 15, 16, 17]. As a result, it is verified that the structures of solutions are composed of common features such as constant regions, simple wave zones, elliptic/hyperbolic regions, sonic lines, transonic shocks, etc. Here, a simple wave means a region in which one family of characteristics consist of all straight lines.

The numerical result in [3] indicates that there is something different than the common phenomenon in the previous paragraph, and this happens in most four-wave Riemann problems. This paper confirms that characteristics are indeed extremely complicated near sonic curves and that a transonic shock forms near the sonic curves, see Figure 1. These properties invite us to seek a better understanding of the characteristics and the domain of dependence of hyperbolic initial boundary value problems.

The complicated behavior of characteristics presents a particular difficulty for the method of characteristics, which is the main tool in handling the hyperbolic part. Usually, after the hyperbolic part has been constructed by the method of characteristics, one wishes to have a simple connection zone, preferably a single smooth sonic curve or a single transonic shock at the end of the domain of dependence to connect the hyperbolic part to the elliptic part. But the numerical result in [3] illustrates that the boundary of the domain of dependence is not as simple as we expect. And we come to know that a region, called a semihyperbolic patch, around the sonic boundary in the hyperbolic part is not easy to handle, either.

Semihyperbolic patches are not fully hyperbolic but partially hyperbolic. In the patch, one family of characteristics starts from the transonic shock and ends on the sonic line. This phenomenon has never been observed in typical hyperbolic regions.
However, we encounter this phenomenon in the transonic flow frequently. In fact, this type of solution appears in the transonic flow over obstacles like an airfoil, or for air accelerating in a planar tunnel over a small inward bulge on one of the walls, and on the Guderley reflection, etc.; see Courant and Friedrichs [2]. We see that the sonic line and the transonic shock are demarcation lines between the (semi-)hyperbolic region and the elliptic region. Furthermore, the exact locations of these lines are not known a priori. That is, they are free boundaries to be determined.

The study on these patches was initiated by Song and Zheng using the pressure-gradient system [9]. Later, the global existence of smooth solution in the semihyperbolic patch for the Euler system was obtained by Li and Zheng [8]. As in Figure 2, Li and Zheng established the global existence of solutions in the semihyperbolic patches formed by the interaction of two forward rarefaction waves and two backward rarefaction waves. Particularly, they obtained the global existence of solutions in the region of $ABC$ formed by two characteristics emitting from $B$, which is a Goursat boundary value problem. They solved this problem by considering that the sonic line is fixed. In order to resolve this problem, they rewrote system (1.7) in the form

\begin{align}
\begin{aligned}
(-\beta + \Psi(\varpi))_\xi + \tan \alpha (-\beta + \Psi(\varpi))_\eta &= J / \cos \alpha, \\
\left[\frac{c^2}{2} \left(\frac{\kappa}{\kappa - 1} + \frac{1}{\sin^2 \varpi}\right)\right]_\xi - \cot \sigma \left[\frac{c^2}{2} \left(\frac{\kappa}{\kappa - 1} + \frac{1}{\sin^2 \varpi}\right)\right]_\eta &= 0, \\
(\alpha + \Psi(\varpi))_\xi + \tan \beta (\alpha + \Psi(\varpi))_\eta &= J / \cos \beta,
\end{aligned}
\end{align}

(1.8)

in which the characteristic $\Lambda_0$ has to be between the positive characteristic $AB$ and the negative characteristic $BC$, rather than going from one to the other. In [8], Li and
Fig. 2. Setup for a semihyperbolic patch. We are given a positive characteristic $AB$ in a planar rarefaction wave $R_{14}$ with tangential extension $BD$ into a constant state, where both points $A$ and $D$ are sonic points. For a given strictly convex negative characteristic $BC$, where the point $C$ is sonic, find a solution in a curvilinear triangle region $ABC$, where $AC$ is a sonic curve. We note that the curvilinear triangle region $CBD$ is a simple wave zone. Here $CD$ is an envelope. The main problem is to determine the regularity of the sonic curve $AC$.

Zheng first proved that there exists a local smooth solution to the Goursat problem near $B$ using the scheme proposed in [11]. For details, see [5, 18] for the English version of [11]. Then they established uniform estimates on the derivatives of sound speed and obtained global existence of the solution in the region $ABC$, where the sonic line $AC$ is assumed fixed. The main result in [8] is as follows.

Theorem 1.1 (see [8]). There exists a smooth solution in a fixed region $ABC$ of Euler system (1.8) satisfying the boundary conditions

$$
\beta|_{AB} = 0, \quad \pi/2 \leq \alpha|_{BC} \leq \pi + \beta_C,
$$

where $\beta_C \in (-\pi/2, 0)$ denotes the inclination angle of the convex negative characteristic $BC$ at the point $C$. Also,

$$
\pi/2 \leq \alpha|_{AB} \leq \pi, \quad \beta_C \leq \beta|_{BC} \leq 0,
$$

$$
c|_{AB} = \kappa v + c_4, \quad c|_{BC} = y(\xi),
$$

where $c_4$ is a sound speed of the fourth quadrant, and $y(\xi)$ is a function determined by the given curve $BC$ and the equation of $c$ on it. Furthermore, the boundary $AC$ is sonic.

In [8, 9], however, the exact behavior of the global solution near the sonic line is not provided. Thus, the regularity of the sonic lines is not known. Recently, Wang and Zheng resolved the difficulty arising from the degeneracy of hyperbolicity near the sonic line of the pressure-gradient system [10]. They showed that the solutions are uniformly smooth up to the sonic boundaries and that the boundaries are $C^1$ continuous. We refer the reader [14, 17] for other directions of research on sonic lines.

In this paper, we extend the work of Wang and Zheng [10] to the Euler equations. We can see that the solutions obtained in [8] are uniformly smooth up to the sonic boundaries. The two derivatives $\partial^+ c$ and $-\partial^- c$ approach common values on the sonic curve at a rate of $O(\sqrt{q^2 - c^2})$, and, as a result, we also show the $C^1$ continuity of sonic lines. The main contribution of this paper is that we provide some insightful information about sonic curves which makes it possible to extend the solution into the
subsonic domain and construct the global solutions of 2-D Riemann problems. Our main result can be stated as follows.

**Theorem 1.2.** The solution to the Goursat boundary value problem for the 2-D Euler system as stated in Theorem 1.1 is smooth in the whole region ABC, up to the sonic boundary. The two derivatives $\alpha^+ c$ and $-\alpha^+ c$ can approach common values on the sonic curve at a rate of $O(\sqrt{q^2 - c^2})$. Moreover, the sonic boundary AC is $C^1$ continuous.

Our paper is organized as follows. In section 2, we employ a new coordinate system $(t, \varphi) = (\sqrt{q^2 - c^2}, \varphi)$ to derive new systems for $\alpha^+ c$. In section 3, we will use the bootstrap method to get the uniform bounds of $t^\delta R_\varphi$ and $t^\delta S_\varphi$ for any $\delta \in (1, 2)$. With the help of these estimates, we can prove Theorem 1.2 in section 4.

2. Preliminaries. There have been many attempts made with various methods, including the hodograph transformation method, to attack the transonic flow problems. But the hodograph transformation method, considering the velocity $(u, v)$, is known to be difficult in inversion back to the spatial variables $(x, y)$. Recently, it is known to be easier for the spatial coordinates $(\xi, \eta)$ or the coordinates $(\phi, \varphi)$ to take on boundary conditions, where $\phi$ is the stream function and $\varphi$ is the potential function. We mention this method in [4]. In this paper, we use a slight different system, $(\sqrt{q^2 - c^2}, \varphi)$, motivated from our study of the pressure-gradient system [10], which seems to capture the degeneracy of the problem at the sonic curve.

Let us introduce new independent variables

\begin{equation}
\begin{aligned}
&\begin{aligned}
&\begin{cases}
&\begin{aligned}
&t = \sqrt{U^2 + V^2 - c^2} = \sqrt{q^2 - c^2}, \\
&\varphi = \varphi(\xi, \eta),
\end{aligned}
\end{cases}
\end{aligned}
\end{aligned}
\end{equation}

where $\varphi$ is the pseudopotential function satisfying

$$\varphi_\xi = U, \quad \varphi_\eta = V.$$  

The new coordinates $(\varphi, t)$ can flatten the sonic boundary corresponding to $\{t = 0\}$.

Now we want to analyze the regularity of the sonic lines for the 2-D Riemann problem for the Euler system. Let us define level curves in the $(\xi, \eta)$-plane defined by

$$f^\xi(\xi, \eta) = q^2 - c^2 = \epsilon,$$

which is approximating the sonic line defined implicitly by $q^2 - c^2 = 0$. Then

$$f^\xi = 2qq_\xi - 2cc_\xi, \quad f^\eta = 2qq_\eta - 2cc_\eta.$$  

From

$$\partial_\xi = \frac{\sin \beta \alpha^+ - \sin \alpha \alpha^+}{\sin(\alpha - \beta)}, \quad \partial_\eta = \frac{\cos \beta \alpha^+ - \cos \alpha \alpha^+}{\sin(\alpha - \beta)},$$

we have

\begin{equation}
\begin{aligned}
&\begin{aligned}
&\begin{cases}
&f^\xi = -\frac{2q \sin \beta(-t - 2vc_\xi c)}{\sin(\alpha - \beta)}, \\
&f^\eta = \frac{2q \cos \beta(-t - 2vc_\eta c)}{\sin(\alpha - \beta)},
\end{cases}
\end{aligned}
\end{aligned}
\end{equation}

see more details in section 4. Since $\alpha - \beta = \pi$ on the sonic line, we can verify in Lemma 4.3 that

\begin{equation}
\begin{aligned}
&\begin{aligned}
&\begin{cases}
&f^\xi = -\frac{2q^2}{c} \left\{ \frac{\sin \alpha(t + vc_\xi (\xi + \eta c))}{t} + \frac{\epsilon}{q^2} \cos \alpha(t + 2vc_\xi c) \right\} + O(t), \\
&f^\eta = \frac{2q^2}{c} \left\{ \frac{\cos \alpha(t + vc_\eta (\xi + \eta c))}{t} - \frac{\epsilon}{q^2} \sin \alpha(t + 2vc_\eta c) \right\} + O(t).
\end{cases}
\end{aligned}
\end{aligned}
\end{equation}
So we see that the regularity of sonic curves strongly depends on how to control the quantity
\[
\frac{\overline{\partial}^+ c + \overline{\partial}^- c}{t},
\]
which is an indeterminate form, where \( \overline{\partial}^+ c + \overline{\partial}^- c = 0 \) on the sonic curve. Hence, we first come to focus on the quantities \( \overline{\partial}^+ c \).

From the second-order system of \( c \) from [8],
\[
\begin{align*}
\begin{cases}
\alpha \partial_t^2 + \nu \partial_t^2 + \nu \mu \cos(2\varpi) \partial_t^+ c &= \nu \mu \cos(2\varpi) \partial_t^- c, \\
\mu \cos(2\varpi) &= \frac{1}{\cos^2 \varpi} + 2(m + 1) \cos^2 \varpi - m - 3,
\end{cases}
\end{align*}
\]
system (2.2) can be changed into
\[
\begin{align*}
\begin{cases}
\overline{\partial}^+ \overline{\partial}^+ c &= \overline{\partial}^- c \left[ -\sin(2\varpi) + \frac{\nu}{\cos^2 \varpi} \overline{\partial}^- c + \nu \mu \cos(2\varpi) \right], \\
\overline{\partial}^- \overline{\partial}^+ c &= \overline{\partial}^+ c \left[ -\sin(2\varpi) + \frac{\nu}{\cos^2 \varpi} \overline{\partial}^+ c + \nu \mu \cos(2\varpi) \right].
\end{cases}
\end{align*}
\]
we can derive
\[
\begin{align*}
\begin{cases}
\overline{\partial}^+ (\overline{\partial}^- c) &= \overline{\partial}^- c \left[ -\sin(2\varpi) + \frac{\nu}{\cos^2 \varpi} \overline{\partial}^- c + \nu \mu \cos(2\varpi) \right], \\
\overline{\partial}^- (\overline{\partial}^+ c) &= \overline{\partial}^+ c \left[ -\sin(2\varpi) + \frac{\nu}{\cos^2 \varpi} \overline{\partial}^+ c + \nu \mu \cos(2\varpi) \right].
\end{cases}
\end{align*}
\]
Let
\[
R := \frac{\overline{\partial}^+ c}{c}, \quad S := \frac{\overline{\partial}^- c}{c}.
\]
Since
\[
\nu \mu \cos(2\varpi) = \frac{1}{\cos^2 \varpi} + 2(m + 1) \cos^2 \varpi - m - 3,
\]
system (2.2) can be changed into
\[
\begin{align*}
\begin{cases}
\overline{\partial}^+ R &= R \left[ -\sin(2\varpi) + \nu \frac{R + S}{\cos^2 \varpi} + \nu \left( 2(m + 1) \cos^2 \varpi - m - 3 \right) S \right], \\
\overline{\partial}^- S &= S \left[ -\sin(2\varpi) + \nu \frac{R + S}{\cos^2 \varpi} + \nu \left( 2(m + 1) \cos^2 \varpi - m - 3 \right) R \right].
\end{cases}
\end{align*}
\]
From the relations
\[
\partial_\xi = \frac{q q_\xi - c c_\xi}{t} \partial_t + U \partial_\phi, \quad \partial_\eta = \frac{q q_\eta - c c_\eta}{t} \partial_t + V \partial_\phi,
\]
we have
\[
\overline{\partial}^+ = \cos \alpha \partial_\xi + \sin \alpha \partial_\eta = \frac{q \overline{\partial}^+ + \nu \overline{\partial}^- c}{t} \partial_t + (U \cos \alpha + V \sin \alpha) \partial_\phi.
\]
Taking \( \overline{\partial}^+ \) on both sides of the Bernoulli’s law
\[
\frac{c^2}{\gamma - 1} + \frac{q^2}{2} = -\varphi,
\]
we have
\[
\frac{2c}{\gamma - 1} \overline{\partial}^+ c + q \overline{\partial}^+ q = -(U \cos \alpha + V \sin \alpha).
\]
We note that
\[ U \cos \alpha + V \sin \alpha = q(\cos \alpha \cos \theta + \sin \alpha \sin \theta) = q \cos(\alpha - \theta) = q \left[ \frac{1}{1 + \tan^2(\alpha - \theta)} \right]^{1/2}, \]
where \( \tan \theta = V/U \). Moreover,
\[ \tan(\alpha - \theta) = \frac{\tan \alpha - \tan \theta}{1 + \tan \alpha \cdot \tan \theta} = \frac{UV + c\nu(U^2 + V^2 - c^2)}{U}\nu + V^2 - c^2 \cdot \frac{V}{U} = c \left( \frac{1}{t} \right). \]
Thus,
\[ U \cos \alpha + V \sin \alpha = q \left[ \frac{1}{1 + \left( \frac{t}{t} \right)^2} \right]^{1/2} = t, \]
which implies \( q\theta^+ q - c\theta^+ c = -t - 2\nu c\theta^+ c \). Hence, from (2.4) we obtain
\[ \overrightarrow{\partial} = - \left( 1 + \frac{2\nu c^2}{t} \frac{\theta^+}{c} \right) \partial_t + t\partial_\varphi. \]
Similarly, we also have
\[ \overleftarrow{\partial} = - \left( 1 + \frac{2\nu c^2}{t} \frac{\theta^+}{c} \right) \partial_t + t\partial_\varphi. \]
Owing to \( \csc \varpi = M = q/c \), the factors of \( \sin(2\varpi), \cos^2 \varpi \) in the right-hand side of system (2.2) become
\[ \sin(2\varpi) = 2 \sin \varpi \cos \varpi = \frac{2t}{q^2}, \quad \cos^2 \varpi = \frac{t^2}{q^2}. \]
Hence, system (2.3) transforms into (2.6)
\[ \begin{cases} 
(1 + \frac{2\nu c^2}{t})R_t - tR_\varphi = R \left[ \frac{2m}{q^2} - \frac{\nu c^2}{t^2}(R + S) - \nu \left( 2(m + 1)\frac{t^2}{q^2} - m - 3 \right) S \right], \\
(1 + \frac{2\nu c^2}{t})S_t - tS_\varphi = S \left[ \frac{2m}{q^2} - \frac{\nu c^2}{t^2}(R + S) - \nu \left( 2(m + 1)\frac{t^2}{q^2} - m - 3 \right) R \right]. 
\end{cases} \]
The normalization of the coefficients of \( R_t, S_t \) yields that
\[ \begin{cases} 
R_t - \frac{t^2}{t^2 + 2\nu c^2} R_\varphi = R \left[ \frac{\nu c^2}{t^2 + 2\nu c^2} - \frac{\nu c^2}{t^2 + 2\nu c^2} \frac{R + S}{t} \right. \\
- \nu \left( 2(m + 1)\frac{t^2}{q^2} - m - 3 \right) \frac{t^2}{t^2 + 2\nu c^2} \right], \\
S_t - \frac{t^2}{t^2 + 2\nu c^2} S_\varphi = S \left[ \frac{2\nu c^2}{t^2 + 2\nu c^2} - \frac{\nu c^2}{t^2 + 2\nu c^2} \frac{R + S}{t} \right. \\
- \nu \left( 2(m + 1)\frac{t^2}{q^2} - m - 3 \right) \frac{t^2}{t^2 + 2\nu c^2} \right]. 
\end{cases} \]
In [8] it has been shown that both $R$ and $-S$ are uniformly bounded and positive in the domain $ABC$; thus (2.7) is equivalent to

\[
\begin{align*}
\left(\frac{1}{R}\right)_t - \frac{t^2S^{-1}}{tS^{-1} + 2\nu c^2} \left(\frac{1}{R}\right)_\varphi &= \frac{-2t^2}{q^2(q(tS^{-1} + 2\nu c^2))RS} + \frac{t\nu^2}{tS^{-1} + 2\nu c^2} \left(\frac{1}{R} + \frac{1}{S}\right) \frac{1}{t} + \nu \left(2(m + 1)\frac{t^2}{q^2} - m - 3\right) \frac{1}{tS^{-1} + 2\nu c^2} \frac{1}{R}, \\
\left(\frac{1}{S}\right)_t - \frac{t^2R^{-1}}{tR^{-1} + 2\nu c^2} \left(\frac{1}{S}\right)_\varphi &= \frac{-2t^2}{q^2(q(tR^{-1} + 2\nu c^2))RS} + \frac{t\nu^2}{tR^{-1} + 2\nu c^2} \left(\frac{1}{R} + \frac{1}{S}\right) \frac{1}{t} + \nu \left(2(m + 1)\frac{t^2}{q^2} - m - 3\right) \frac{1}{tR^{-1} + 2\nu c^2} \frac{1}{S}.
\end{align*}
\]  
(2.8)

Letting

\[W := \frac{1}{R} + \frac{1}{S},\]
we obtain the equation of $W$ from (2.8) satisfying

\[
W_t = \frac{\nu q^2}{t} \frac{(4\nu c^2 + tW)W}{(tR^{-1} + 2\nu c^2)(tS^{-1} + 2\nu c^2)} - \frac{2t^2}{q^2RS (tR^{-1} + 2\nu c^2)(tS^{-1} + 2\nu c^2)} \frac{tW + 4\nu c^2}{t} \frac{t^2S^{-1}}{tS^{-1} + 2\nu c^2} \left(\frac{1}{R}\right) \varphi + \frac{t^2R^{-1}}{tR^{-1} + 2\nu c^2} \left(\frac{1}{S}\right) \varphi \frac{2t(m + 1)\frac{t^2}{q^2} - m - 3}{t^2(R^{-2} + S^{-2}) + 2\nu c^2 W} \frac{t^2(R^{-2} + S^{-2}) + 2\nu c^2 W}{(tR^{-1} + 2\nu c^2)(tS^{-1} + 2\nu c^2)}.
\]  
(2.9)

According to (2.7), let us define new directional derivatives

\[
\begin{align*}
\partial_+ := \partial_t - \frac{t^2}{tS^{-1} + 2\nu c^2} \partial_\varphi, \\
\partial_- := \partial_t - \frac{t^2}{tR^{-1} + 2\nu c^2} \partial_\varphi.
\end{align*}
\]  
(2.10)

What should be mentioned is that here we use subscripts rather than superscripts to denote the directional derivatives. Then the corresponding characteristics are defined by

\[
\begin{align*}
\frac{d\varphi^+}{dt} &= \lambda_+ := -\frac{t^2}{tS^{-1} + 2\nu c^2} R, \\
\frac{d\varphi^-}{dt} &= \lambda_- := -\frac{t^2}{tR^{-1} + 2\nu c^2} S.
\end{align*}
\]  
(2.11)

Since it has been known that $R = -S$ on the sonic line $AC$ in [8], we can see that $W = 0$ on the sonic line. Now let us derive the uniform boundedness of $W/t$ around the sonic line. We first cite the commutator relation from [6], which can be obtained by direct computations.

**Lemma 2.1** ([6, commutator relation of $\partial_{\pm}$]). For any quantity $I$,

\[
\begin{align*}
\partial_- \partial_+ I - \partial_+ \partial_- I = \frac{\partial_- \lambda_+ - \partial_+ \lambda_-}{\lambda_+ - \lambda_-} (\partial_+ I - \partial_- I),
\end{align*}
\]  
(2.12)

where $\partial_{\pm} := \partial_t + \lambda_{\pm} \partial_\varphi$.

Let $G = \partial_+ R - \partial_- R$, $H = \partial_+ S - \partial_- S$. Then

\[
\begin{align*}
\partial_- G &= \frac{\partial_- \lambda_+ - \partial_+ \lambda_-}{\lambda_+ - \lambda_-} G + (\partial_+ \partial_- R - \partial_- \partial_+ R), \\
\partial_+ H &= \frac{\partial_- \lambda_+ - \partial_+ \lambda_-}{\lambda_+ - \lambda_-} H + (\partial_+ \partial_+ S - \partial_- \partial_- S).
\end{align*}
\]  
(2.13)
It can be calculated directly that
\[
\frac{\partial_+ \lambda_+ - \partial_- \lambda_-}{\lambda_+ - \lambda_-} = -\frac{1}{2\nu c^2(R - S)} \left( \frac{t \nu c^2 S}{t + 2\nu c^2 R} - \frac{t \nu c^2 R}{t + 2\nu c^2 S} \right) - \frac{2}{t(R - S)} \left( \frac{t \nu c^2 S}{t + 2\nu c^2 R} - \frac{t \nu c^2 R}{t + 2\nu c^2 S} \right) + \frac{1}{c(R - S)} \left( \frac{t \nu c^2 S}{t + 2\nu c^2 R} \left( 2c_R + cR_t \right) - \frac{t \nu c^2 R}{t + 2\nu c^2 S} \left( 2cR + cS_t \right) \right) - \frac{t^2}{c(R - S)} \left( \frac{t \nu c^2 R + cR_R}{t + 2\nu c^2 R} - \frac{t \nu c^2 S + cS_R}{t + 2\nu c^2 S} \right)
\]
\[
\Rightarrow = \frac{2}{t} + h(\varphi, t),
\]
where
\[
h(\varphi, t) = -\frac{1}{(t + 2\nu c^2 R)(t + 2\nu c^2 S)} \left[ 2t + \nu(2c^2 + q^2)(R + S) - 4t\nu c^2 RS \right] - \frac{2t^3}{q^2} - 2t\nu c^2 RS \left( 2(m + 1) \frac{t^2}{q^2} - m - 3 \right).
\]
It is obvious that \( h(t, \varphi) \) tends to zero as \( t \to 0^+ \). On the other hand,
\[
\partial_+ \partial_- R - \partial_- \partial_+ R = \left( \frac{t^2}{t + 2\nu c^2 S} - \frac{t^2}{t + 2\nu c^2 R} \right) (\partial_- R)_\varphi
\]
\[
= \frac{2t^2 \nu c^2 (R - S)}{(t + 2\nu c^2 R)(t + 2\nu c^2 S)} \left[ \frac{2t^2 R}{q^2(t + 2\nu c^2 S)} - \frac{\nu q^2 R}{t + 2\nu c^2 S} \right] \frac{R + S}{t} - \nu \left( \frac{2(m + 1) \frac{t^2}{q^2} - m - 3}{t} \right) \frac{tR}{t + 2\nu c^2 S} \varphi
\]
\[
\Rightarrow = f_1(\varphi, t)tR_\varphi + f_2(\varphi, t) tS_\varphi + f_3(\varphi, t) q_\varphi + f_4(\varphi, t)c_\varphi,
\]
where
\[
f_1(\varphi, t) = -E_1(\varphi, t) \left[ \nu q^2(2R + S) - \frac{2t^3}{q^2} + \left( 2(m + 1) \frac{t^2}{q^2} - m - 3 \right) t^2 S \right],
\]
\[
f_2(\varphi, t) = -E_1(\varphi, t) \left[ \nu q^2 R + \frac{4t^3 \nu c^2 R}{q^2(t + 2\nu c^2 S)} - \frac{2\nu^2 q^2 c^2 R}{t + 2\nu c^2 S} (R + S) \right]
\]
\[
+ \left( 2(m + 1) \frac{t^2}{q^2} - m - 3 \right) \frac{t^3 R}{t + 2\nu c^2 S},
\]
\[
f_3(\varphi, t) = -t^2 E_1(\varphi, t) \left[ \frac{2tqR(R + S)}{t} + \frac{4t^3 R}{q^2} - 4\nu(m + 1) \frac{t^3 RS}{q^3} \right],
\]
\[
f_4(\varphi, t) = t^2 E_1(\varphi, t) \left[ \frac{4\nu R S}{t + 2\nu c^2 S} \right] \left[ \frac{\nu q^2(R + S)}{t} - \frac{2t^2}{q^2} - \nu \left( 2(m + 1) \frac{t^2}{q^2} - m - 3 \right) tS \right],
\]
and
\[
E_1(\varphi, t) = \frac{2\nu c^2 (R - S)}{(t + 2\nu c^2 R)(t + 2\nu c^2 S)^2}.
\]
Owing to
\[
\mathbf{\mathcal{J}}^+ c = - \left( 1 + \frac{2\nu c^2 R}{t} \right) \partial_+ c + t\partial_+ c, \quad \mathbf{\mathcal{J}}^+ c = - \left( 1 + \frac{2\nu c^2 S}{t} \right) \partial_- c + t\partial_- c,
\]
we can reduce to

\[(2.17)\]

\[c_\varphi = -\frac{1}{2\nu c}.\]

Furthermore, from Bernoulli’s law \((2.5)\), we have

\[(2.18)\]

\[q_\varphi = -(\gamma - 1) + 2\nu c_\varphi = -\frac{1}{2\nu q}.\]

Here are terms of indeterminate form from \(f_3\) and \(f_4\), respectively,

\[-2\nu qR \left(\frac{R + S}{t}\right) q_\varphi, \quad \frac{4\nu^2 c^2 q RS}{t + 2\nu c^2 S} \left(\frac{R + S}{t}\right) c_\varphi.\]

But their addition yields a term that is determinate as follows:

\[(2.19)\]

\[-2\nu qR \left(\frac{R + S}{t}\right) q_\varphi + \frac{4\nu^2 c^2 q RS}{t + 2\nu c^2 S} \left(\frac{R + S}{t}\right) c_\varphi = \frac{(R + S)R}{t(t + 2\nu c^2 S)} \left[ -2\nu q q_\varphi + \frac{4\nu^2 q^2 c S}{t + 2\nu c^2 S} c_\varphi \right] = \frac{R(R + S)(1 + 2\nu St)}{(t + 2\nu c^2 S)}.\]

Hence, by \((2.17)\)–\((2.19)\) we find that

\[(2.20)\]

\[\partial_+ \partial_- R - \partial_- \partial_+ R = f_1(\varphi, t) t R_\varphi + f_2(\varphi, t) t S_\varphi + f_5(\varphi, t) t^2,\]

where \(f_i(\varphi, t)\) are all bounded near the sonic boundaries for \(i = 1, 2, 5\) and \(f_5(\varphi, t) \to 0\) as \(t \to 0^+\).

Similarly, we can obtain

\[(2.21)\]

\[\partial_+ \partial_+ S - \partial_- \partial_+ S = \left( \frac{t^2}{t + 2\nu c^2 S} - \frac{t^2}{t + 2\nu c^2 R} \right) (\partial_+ S)_{\varphi} = \frac{2t^2 \nu c^2 (R - S)}{(t + 2\nu c^2 R)(t + 2\nu c^2 S)} \left[ \frac{2t^2 S}{t + 2\nu c^2 R} - \frac{\nu q^2 S}{t + 2\nu c^2 R} \right] - \nu \left( 2(m + 1) \frac{t^2}{q^2} - m - 3 \right) \left( \frac{t R S}{t + 2\nu c^2 R} \right)_{\varphi} =: g_1(\varphi, t) t R_\varphi + g_2(\varphi, t) t S_\varphi + g_3(\varphi, t) q_\varphi + g_4(\varphi, t) c_\varphi =: g_1(\varphi, t) t R_\varphi + g_2(\varphi, t) t S_\varphi + g_5(\varphi, t) t^2,\]

where

\[g_1(\varphi, t) = -E_2(\varphi, t) \left[ \frac{\nu q^2 S}{t} + \frac{4t^3 \nu c^2 S}{q^2 (t + 2\nu c^2 R)} - \frac{2\nu q^2 c^2 S(R + S)}{t + 2\nu c^2 R} \right] + \left( 2(m + 1) \frac{t^2}{q^2} - m - 3 \right) \left( \frac{t^3 S}{t + 2\nu c^2 R} \right),\]

\[g_2(\varphi, t) = -E_2(\varphi, t) \left[ \frac{\nu q^2 (R + 2S)}{q^2} - \frac{2t^3}{q^2} + \left( 2(m + 1) \frac{t^2}{q^2} - m - 3 \right) t^2 R \right],\]

\[g_3(\varphi, t) = -t^2 E_2(\varphi, t) \left[ \frac{2\nu q(R + S)}{t} + \frac{4t^2 S}{q^3} - 4\nu (m + 1) \frac{t^3 R S}{q^3} \right],\]

\[g_4(\varphi, t) = \frac{4t^2 \nu c R S E_2(\varphi, t)}{t + 2\nu c^2 R} \left[ \frac{\nu q^2 (R + S)}{t} - \frac{2t^2}{q^2} - \nu \left( 2(m + 1) \frac{t^2}{q^2} - m - 3 \right) t R \right].\]
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Fig. 3. The regions of \( \mathcal{D} \) and \( \Omega(\varphi, 0) \).

and

\[ E_2(\varphi, t) = \frac{2\nu c^2 (R - S)}{(t + 2\nu c^2 R)^2(t + 2\nu c^2 S)}. \]

We easily see that \( g_i(\varphi, t) \) are also bounded for \( i = 1, 2, 5 \) when \( t \) is small, and \( g_5(\varphi, t) \to 0 \) as \( t \to 0^+ \).

3. Uniform bounds of \( t^5 R_{\varphi} \) and \( t^5 S_{\varphi} \). In this section, we will use formulas (2.9)–(2.21) to establish the regularity of semihyperbolic patches near sonic boundaries. Since we are concerned with the regularity of semihyperbolic patches near the sonic line, for any fixed point \( (\varphi, 0) \) on the sonic line \( AC \), we shall take a new point \( B'(\varphi, t_{B'}) \), where \( t_{B'} \) is positive and small so that \( B' \) remains in the domain \( ABC \). Then through the point \( B' \) we can draw the positive and the negative characteristic curves \( \varphi^+(B') \) and \( \varphi^-(B') \) until they intersect the sonic line \( AC \) at two points \( A' \) and \( C' \), respectively; see Figure 3. Let us denote the region \( A'B'C' \) by \( \mathcal{D} \).

Since \( R, -S \) are uniformly bounded and positive in the domain \( ABC \) and \( R + S = 0 \) on the sonic line, we see that \( h(\varphi, t), f_i(\varphi, t), g_i(\varphi, t) \) for \( i = 1, 2, 5 \) are also uniformly bounded in the small subdomain \( \mathcal{D} \). Let

\begin{align*}
K_1 &= \max_{(\varphi, t) \in \mathcal{D}} \left\{ |h(\varphi, t)|, |f_5(\varphi, t)|, |g_5(\varphi, t)| \right\}, \\
K_2 &= \max_{(\varphi, t) \in \mathcal{D}} \left\{ |f_i(\varphi, t)|, |g_i(\varphi, t)|, \quad i = 1, 2 \right\}, \\
K_3 &= \min_{(\varphi, t) \in \mathcal{D}} \left| \frac{2\nu c^2 (R - S)}{(t + 2\nu c^2 R)(t + 2\nu c^2 S)} \right|.
\end{align*}

We can see that \( K_1 \) tends to zero and \( K_2, K_3 \) have positive bounds as \( t_{B'} \to 0^+ \). If \( t_{B'} = 0 \), then the domain \( \mathcal{D} \) shrinks into a point on the sonic line, denoted by \( (\varphi_0, 0) \). At this sonic point, the two constants \( K_2 \) and \( K_3 \) satisfy

\[ 2K_2 = K_3 = \frac{1}{\nu c^2 R(\varphi_0, 0)} > 0. \]
Fig. 4. The region of $\Omega_\varepsilon$.

by the definitions of $K_2$ and $K_3$. Thus, $K_3 > \frac{2K_2}{\delta}$ for any fixed $\delta \in (1, 2)$ near the sonic line. Furthermore, we can let $K_3 > \frac{2K_2}{\delta} e^{2K_1 t_{B'}}$ and $K_1 < K_2$ by taking suitably small $t_{B'}$.

Let $\Omega(\varphi, 0)$ denote a bounded domain surrounded by a positive characteristic and a negative characteristic starting from $(\varphi, 0)$ and the characteristics $B'C'$ and $B'A'$. For any $(\varphi, t) \in \Omega(\varphi, 0)$, let $a$ and $b$ be intersections of the negative characteristic and the positive characteristic through $(\varphi, t)$ with the boundaries $B'C'$ and $B'A'$, respectively. Let

$$M_1 = \max \left\{ \max_{\Omega(\varphi, 0)} \frac{|H(t_a)|}{K_2 t_a^{2-\delta} e^{K_1 t_a}} + 1, \max_{\Omega(\varphi, 0)} \frac{|G(t_b)|}{K_2 t_b^{2-\delta} e^{K_1 t_b}} + 1 \right\},$$

where $t_a$ and $t_b$ are $t$-coordinates of points $a$ and $b$, respectively. Since $K_2$ has a positive lower bound and $t_a$ is strictly positive, $M_1$ is well-defined and uniformly bounded in the domain $\Omega(\varphi, 0)$, but dependent on $\delta$.

Next, for a fixed $\varepsilon \in (0, t_{B'})$, let

$$\Omega_\varepsilon := \left\{ (\varphi, t) | \varepsilon \leq t \leq t_{B'}, \varphi_-(B') \leq \varphi \leq \varphi_+(B') \right\} \cap \Omega(\varphi, 0);$$

see Figure 4. Define $M' = \max \{ |t^\delta R_\varphi|, |t^\delta S_\varphi| \}$; then we have

$$(3.4) \quad |R_\varphi| \leq \frac{M'}{t^\delta}, \quad |S_\varphi| \leq \frac{M'}{t^\delta}$$

in $\Omega_\varepsilon$ with $M'$ depending on $\varepsilon$. If for any $\varepsilon \in (0, t_{B'})$, we have $M' \leq M_1$, then (3.4) holds in the whole domain $\Omega(\varphi, 0)$. If there exists $\varepsilon_0 \in (0, t_{B'})$ making $M' > M_1$, then we let

$$M_2 = \max_{\Omega_{\varepsilon_0}} \{ |t^\delta R_\varphi|, |t^\delta S_\varphi| \}.$$

**Lemma 3.1.** For any $(\varphi, t) \in \Omega(\varphi, 0)$ and any $\delta \in (1, 2)$, there hold

$$(3.5) \quad |t^\delta R_\varphi| \leq M_2, \quad |t^\delta S_\varphi| \leq M_2.$$
According to (2.20) and (3.1)–(3.2), we have

Thus, we obtain

On the other hand,

Proof. For \((\varphi_{x_0}, \epsilon_0) \in \Omega(\varphi, 0)\), we denote the two characteristics passing through \((\varphi_{x_0}, \epsilon_0)\) by \(\varphi^a_{+}\) and \(\varphi^-_{+}\), where \(a, b\) are the intersection points with \(B'C'\) and \(B'A'\), respectively. Here \(a = a_{x_0}\) and \(b = b_{x_0}\) are dependent on \(\epsilon_0\), and for convenience we omit the subscript \(\epsilon_0\).

From system (2.13) we have

Integrating (3.6) along the negative characteristic passing through \((\varphi_{x_0}, \epsilon_0)\), we get

According to (2.20) and (3.1)–(3.2), we have

where we use the definition of \(M_1\) in the last inequality. Thus,

On the other hand,

Thus, we obtain

\[
|R_{\varphi}| < \frac{1}{t^3} \frac{2K_2M_2e^{2K_1t_0}}{\delta} \frac{2K_2M_2e^{2K_1t_0}}{2\nu c^2(R-S)} \leq \frac{M_2}{t^3}.
\]
So we have a strict inequality such that

\[ |R_\varphi| < M_2/\varepsilon_0^\delta \]

on the line segment \( t = \varepsilon_0 \).

Similarly, integration of the second equation of system (2.13) along the positive characteristic results in

\[ |H(\varphi, t)| < t^{2-\delta} \frac{2K_2M_2\varepsilon^{2K_1t_0}}{\delta}. \]

We also note

\[ H(\varphi, t) = t^2 \frac{2\nu c^2(R - S)}{(t + 2\nu c^2 R)(t + 2\nu c^2 S)} S_\varphi(\varphi, t). \]

Thus, the estimate

\[ |S_\varphi|_{t=\varepsilon_0} < \frac{M_2}{\varepsilon_0^\delta} \]

holds. According to (3.7) and (3.8), \( |t^{\delta} R_\varphi| \) and \( |t^{\delta} S_\varphi| \) do not have maximum on the line segment \( t = \varepsilon_0 \); that is, the maximum happens in the interior. Hence, we conclude that (3.5) holds in a larger domain \( \Omega_{\varepsilon'} \), where \( \varepsilon' < \varepsilon_0 \). Similarly, we can extend the domain larger step by step until the domain is extended to the whole domain \( \Omega(\varphi, 0) \). The proof is completed.

Next, for any \((\varphi, 0) \in A'C'\), we can take a small interval \((\varphi - \lambda, \varphi + \lambda) \subset A'C'\). Suppose the negative and the positive characteristics passing through points \( P(\varphi - \lambda, 0) \) and \( Q(\varphi + \lambda, 0) \) intersect the boundaries \( B'C' \) and \( B'A' \) at \( P' \) and \( Q' \), respectively; see Figure 5. Then we have the same estimate in the region of \( B'P'PQQ' \) in the following lemma. Its proof is very similar to that of the previous lemma so we omit it.

**Lemma 3.2.** For any \( \delta \in (1, 2) \), there exists a constant \( M \) depending only on the interval \((\varphi - \lambda, \varphi + \lambda) \) and \( \delta \) such that

\[ |t^{\delta} R_\varphi| \leq M, \quad |t^{\delta} S_\varphi| \leq M \]

hold in the domain \( B'P'PQQ' \).
4. Properties of the solutions. With the aid of uniform bounds of $t^3 R_\varphi$ and $t^3 S_\varphi$, we want to show properties of the solutions.

Lemma 4.1. $|W(t)|/t$ is uniformly bounded in the domain $ABC$.

Proof. In fact, for any point $(\varphi, 0) \in AC$, we take the domain $D$ as before. From (2.9) we have

\begin{equation}
W_t = \frac{\nu q^2(4\nu c^2 + tW)}{(tR^{-1} + 2\nu c^2)(tS^{-1} + 2\nu c^2)} \frac{W}{t} - \frac{t^3 R_\varphi}{(tS^{-1} + 2\nu c^2)R} + \frac{t^3 S_\varphi}{(tR^{-1} + 2\nu c^2)S} \frac{t^{2-\delta} \nu}{RS} + \frac{t}{(tR^{-1} + 2\nu c^2)(tS^{-1} + 2\nu c^2)} \cdot \left[ \nu \left( 2(m + 1) \frac{t^2}{q^2} - m - 3 \right) \left( t(R^{-2} + S^{-2}) + 2\nu c^2 W \right) - \frac{2t(W + 4\nu c^2)}{q^2 RS} \right]
\end{equation}

\begin{equation}
= l_1(\varphi, t) \frac{W}{t} + l_2(\varphi, t)t^{2-\delta} + l_3(\varphi, t)t
\end{equation}

It is obvious that $\lim_{t \to 0^+} l_1(\varphi, t) = 1$, $\lim_{t \to 0^+} l_3(\varphi, t) = 0$, and $l_2(\varphi, t)$ is bounded in the domain $D$. Moreover,

\frac{l_1(\varphi, t) - 1}{t} = \frac{t(4\nu c^2 - R^{-1} S^{-1}) + \nu W(q^2 - 2c^2)}{(tR^{-1} + 2\nu c^2)(tS^{-1} + 2\nu c^2)}

also tends to zero as $t \to 0^+$. Let

\begin{equation}
K_4 = \max_D \left\{ \left| l_1(\varphi, t) - 1 \right|, \ |l_3(\varphi, t)| \right\}, \ \ K_5 = \max_D |l_2(\varphi, t)|.
\end{equation}

Now we rewrite (4.1) in the form of

\begin{equation}
\partial_t \left( W(\varphi, t) \exp \left( \int_t^{t_B^*} \frac{l_1(\varphi, \tau)}{\tau^2} d\tau \right) \right) \left( l_2(\varphi, t)t^{2-\delta} + l_3(\varphi, t)t \right) \exp \left( \int_t^{t_B^*} \frac{l_1(\varphi, \tau) - 1}{\tau} d\tau \right),
\end{equation}

that is,

\begin{equation}
\partial_t \left( W \frac{t}{t} \exp \left( \int_t^{t_B^*} \frac{l_1(\varphi, \tau) - 1}{\tau} d\tau \right) \right) = \left( l_2(\varphi, t)t^{2-\delta} + l_3(\varphi, t) \right) \exp \left( \int_t^{t_B^*} \frac{l_1(\varphi, \tau) - 1}{\tau} d\tau \right).
\end{equation}

Integration of the above equation from $t$ to $t_B^*$ yields that

\begin{equation}
\frac{W}{t} \left( t_B^* \right) = \frac{W}{t} \left( t \right) \exp \left( \int_t^{t_B^*} \frac{l_1(\varphi, \tau) - 1}{\tau} d\tau \right)
\end{equation}

\begin{equation}
= \int_t^{t_B^*} \left( l_2(\varphi, \tau)t^{1-\delta} + l_3(\varphi, \tau) \right) \exp \left( \int_\tau^{t_B^*} \frac{l_1(\varphi, s) - 1}{s} ds \right) d\tau.
\end{equation}

Thus,

\begin{equation}
\begin{aligned}
e^{-K_4 t_B^*} \left| \frac{W}{t} \left( t \right) \right| & \leq \exp \left( \int_t^{t_B^*} \frac{l_1(\varphi, \tau) - 1}{\tau} d\tau \right) \left| \frac{W}{t} \left( t \right) \right| \\
& \leq \frac{W}{t} \left( t_B^* \right) + \int_t^{t_B^*} \left| l_2(\varphi, \tau)t^{1-\delta} + l_3(\varphi, \tau) \right| \exp \left( \int_\tau^{t_B^*} \left| \frac{l_1(\varphi, s) - 1}{s} \right| ds \right) d\tau
\end{aligned}
\end{equation}

\begin{equation}
\leq \frac{W}{t} \left( t_B^* \right) + e^{K_4 t_B^*} \left( \frac{K_5 t_B^{2-\delta}}{2} + K_4 t_B^* \right),
\end{equation}
that is,

\[
\left| \frac{W}{t}(t) \right| < e^{K_4 t_{B'}} \left\{ \left| \frac{W}{t}(t_{B'}) \right| e^{K_4 t_{B'}} + e^{K_4 t_{B'}} \left( \frac{K_4 t_{B'}^2 - \delta}{2 - \delta} + K_4 t_{B'} \right) \right\} : = \tilde{M}.
\]

Because of the arbitrariness of \( t \) and the continuity of \( W(\varphi, t) \), we obtain that \( |W(t)| \leq \tilde{M} t \) holds for any \( t \in [0, t_{B'}] \). □

Thus by Lemma 4.1 and (2.7), we can see that the two derivatives \( R_\varphi \) and \( S_\varphi \) are uniformly bounded in the whole domain. Hence, \( \overline{\partial}_c^+ \) and \( -\overline{\partial}_c^- \) can approach common values on the sonic curve at a rate of \( O(\sqrt{q^2 - c^2}) \).

**Lemma 4.2.** \( R, S \), and \( |W(t)|/t \) are uniformly continuous in the whole domain \( ABC \), including the sonic line \( AC \).

**Proof.** For any two points \( P_i(\varphi_i, 0) (i = 1, 2) \) on the sonic line, we draw the negative and the positive characteristics passing through \( P_1 \) and \( P_2 \), respectively, denoting the intersection point by \( Q(\varphi_0, t_0) \). It is easy to see that the point \( Q \) will approach the sonic line, i.e., \( t_0 \to 0^+ \) as \( |\varphi_2 - \varphi_1| \to 0 \).

From system (2.7), we know that as \( t \to 0^+ \),

\[
\begin{align*}
\left| \partial_- R \right| &= \frac{\nu c^2 R}{\tau + 2 c^2 S} \left| \frac{R + S}{t} \right| + O(t) \leq K, \\
\left| \partial_+ S \right| &= \frac{\nu c^2 S}{\tau + 2 c^2 R} \left| \frac{R + S}{t} \right| + O(t) \leq K
\end{align*}
\]

for some uniform bound \( K \), depending on the bound of \( \frac{R + S}{t} \).

Integrating (4.5) along the characteristics \( \varphi_- \) and \( \varphi_+ \), respectively, we get

\[
\begin{align*}
\left| R(\varphi_1, 0) - R(\varphi_0, t_0) \right| &\leq K t_0, \\
\left| S(\varphi_2, 0) - S(\varphi_0, t_0) \right| &\leq K t_0.
\end{align*}
\]

Recall that \( R = -S \) on the sonic line and \( R, S \) are both continuous in the interior domain \( ABC \), i.e.,

\[
\begin{align*}
R(\varphi_i, 0) = -S(\varphi_i, 0), & \quad i = 1, 2, \\
\left| R(\varphi_0, t_0) + S(\varphi_0, t_0) \right| &\to 0 \quad \text{as} \quad t_0 \to 0^+.
\end{align*}
\]

Thus,

\[
\left| R(\varphi_1, 0) - R(\varphi_2, 0) \right|, \quad \left| S(\varphi_1, 0) - S(\varphi_2, 0) \right| \leq \omega(t_0),
\]

where \( \omega(t_0) = 2Kt_0 + \left| R(\varphi_0, t_0) + S(\varphi_0, t_0) \right| \to 0 \) as \( |\varphi_1 - \varphi_2| \to 0 \), which imply that \( R \) and \( S \) are both continuous on the sonic line \( AC \), as well as uniformly continuous in the whole domain \( ABC \).

Next we will consider the modulus of continuity of \( \frac{W(\varphi,t)}{t} \). For arbitrary \( \varepsilon > 0 \), taking \( t_{B'} \), which will be determined later, integrating (4.2) from 0 to \( t_{B'} \), we obtain

\[
\begin{align*}
\frac{W}{t}(\varphi,t_{B'}) - \frac{W}{t}(\varphi,t = 0) &\exp \left( \int_0^{t_{B'}} \frac{l_1(\varphi, \tau) - 1}{\tau} d\tau \right) \\
&= \int_0^{t_{B'}} \left( l_2(\varphi, \tau) \tau^{1-\delta} + l_3(\varphi, \tau) \right) \exp \left( \int_{\tau}^{t_{B'}} \frac{l_1(\varphi, s) - 1}{s} ds \right) d\tau.
\end{align*}
\]
Then
\[ \left| W_t^B(\varphi, tB') - W_t^B(\varphi, t = 0) \right| \leq \left| W_t^B(\varphi, t = 0) \right| \left( \exp \int_0^{tB'} \left| l_1(\varphi, \tau) - 1 \right| d\tau - 1 \right) + \int_0^{tB'} \left| l_2(\varphi, \tau) \right|^{1-\delta} + l_3(\varphi, \tau) \right| \exp \left( \int_0^{tB'} \left| l_1(\varphi, s) - 1 \right| ds \right) d\tau \leq 2\tilde{M}K_4tB' + e^{K_4tB'} \left( \frac{K_5tB'}{2-\delta} + K_4tB' \right) \leq Kt^{2-\delta}.
\]

Let us choose \( \varphi_1, \varphi_2 \in (\varphi_C, \varphi_A) \), where \( \varphi_C, \varphi_A \) are \( \varphi \)-coordinate of \( C, A \), respectively. Then we can take small \( tB' \) such that \( Kt^{2-\alpha} < \frac{\epsilon}{4} \). For this fixed \( tB' \), taking \( d > 0 \) such that \( |\varphi_1 - \varphi_2| \leq d \), we have
\[ \left| \frac{W}{t}(\varphi_1, tB') - \frac{W}{t}(\varphi_2, tB') \right| \leq \frac{\epsilon}{4}.
\]

Then for any \( |\varphi_1 - \varphi_2| \leq d \), we obtain
\[ \left| \frac{W}{t}(\varphi_1, 0) - \frac{W}{t}(\varphi_2, 0) \right| \leq \left| \frac{W}{t}(\varphi_1, 0) - \frac{W}{t}(\varphi_1, tB') \right| + \left| \frac{W}{t}(\varphi_1, tB') - \frac{W}{t}(\varphi_2, tB') \right| + \left| \frac{W}{t}(\varphi_2, 0) - \frac{W}{t}(\varphi_2, tB') \right| \leq \epsilon.
\]

Thus, \( \frac{W}{t}(\varphi, t) \) is uniformly continuous in the whole domain. \( \Box \)

**Lemma 4.3.** The sonic line \( AC \) is \( C^1 \) the continuous.

**Proof.** Let us consider level curves in the \((\xi, \eta)\)-plane defined by
\[ f^\xi(\xi, \eta) = q^2 - c^2 = \epsilon,
\]
which is approximating the sonic line \( q^2 - c^2 = 0 \). Then
\[ f^\xi_\xi = 2qq_\xi - 2cc_\xi, \quad f^\eta_\eta = 2qq_\eta - 2cc_\eta.
\]

From
\[ \begin{cases}
  \partial_\xi = -\frac{\sin \beta \overrightarrow{c} - \sin \alpha \overrightarrow{c}}{\sin(\alpha - \beta)}, \\
  \partial_\eta = \frac{\cos \beta \overrightarrow{c} - \cos \alpha \overrightarrow{c}}{\sin(\alpha - \beta)},
\end{cases}
\]

we have
\[ \begin{align*}
  f^\xi_\xi &= -2\sin \beta q \overrightarrow{c} q - \sin \alpha \overrightarrow{c} q - c(\sin \beta \overrightarrow{c} c - \sin \alpha \overrightarrow{c} c) \\
  &= -2\sin \beta q \overrightarrow{c} q - \sin \alpha \overrightarrow{c} q - c, \\
  f^\eta_\eta &= 2\cos \beta q \overrightarrow{c} q - \cos \alpha \overrightarrow{c} q - c(\cos \beta \overrightarrow{c} c - \cos \alpha \overrightarrow{c} c) \\
  &= 2\cos \beta q \overrightarrow{c} q - \cos \alpha \overrightarrow{c} q - c.
\end{align*}
\]

Since
\[ q \overrightarrow{c} q - \overrightarrow{c} c = -t - 2vc \overrightarrow{c} c, \quad q \overrightarrow{c} q - \overrightarrow{c} c = -t - 2vc \overrightarrow{c} c,
\]
we have

\[
\begin{align*}
    f_{\xi} &= -2\frac{\sin\beta(-t-2\nu\vartheta^+)}{\sin(\alpha-\beta)} - \frac{\sin\alpha(-t-2\nu\vartheta^+)}{\sin(\alpha-\beta)}, \\
    f_{\eta} &= 2\frac{\cos\beta(-t-2\nu\vartheta^+)}{\sin(\alpha-\beta)} - \frac{\cos\alpha(-t-2\nu\vartheta^+)}{\sin(\alpha-\beta)}.
\end{align*}
\]

(4.7)

Since \(\alpha - \beta = \pi\) on the sonic line, and

\[
\sin(\alpha - \beta) = \sin 2\varpi = 2\sin \varpi \cos \varpi = \frac{2c}{q} \sqrt{1 - \left(\frac{c}{q}\right)^2} = \frac{2ct}{q^2},
\]

we have \(\alpha - \beta = \pi - \arcsin \frac{2ct}{q^2}\) in the domain \(D\). Thus,

\[
\sin \alpha + \sin \beta = \sin \alpha + \sin \left(\alpha - \pi + \arcsin \frac{2ct}{q^2}\right) = \sin \alpha \left(1 - \cos \left(\arcsin \frac{2ct}{q^2}\right)\right) - \frac{2ct}{q^2} \cos \alpha = -\frac{2ct}{q^2} \cos \alpha + O(t^2).
\]

In the same way, we can prove

\[
\cos \alpha + \cos \beta = \cos \alpha + \cos \left(\alpha - \pi + \arcsin \frac{2ct}{q^2}\right) = \cos \alpha \left(1 - \cos \left(\arcsin \frac{2ct}{q^2}\right)\right) + \frac{2ct}{q^2} \sin \alpha = \frac{2ct}{q^2} \sin \alpha + O(t^2).
\]

Then system (4.7) can be rewritten as

\[
\begin{align*}
    f_{\xi} &= -\frac{2q^2}{c} \left\{ \sin \alpha \left(\frac{t + \nu \vartheta^2}{\nu} + S\right) + \frac{\nu}{q^2} \cos \alpha \left(\frac{t + \nu \vartheta^2}{\nu} + S\right) \right\} + O(t), \\
    f_{\eta} &= \frac{2q^2}{c} \left\{ \cos \alpha \left(\frac{t + \nu \vartheta^2}{\nu} + S\right) - \frac{\nu}{q^2} \sin \alpha \left(\frac{t + \nu \vartheta^2}{\nu} + S\right) \right\} + O(t).
\end{align*}
\]

From the uniform boundedness of \(W(t)/t\), we see that

\[
\frac{t + \nu \vartheta^2 (R + S)}{\nu} = 1 + \nu \vartheta^2 \frac{R + S}{\nu}
\]

is uniformly bounded around the sonic line. Thus \(f_{\xi}, f_{\eta}\) are both uniformly bounded in the domain \(D\). Furthermore,

\[
|f_{\xi}|^2 + |f_{\eta}|^2 = \frac{4q^4}{c^2} \left\{ \left(\frac{t + \nu \vartheta^2 (R + S)}{\nu}\right)^2 + \frac{c^2}{q^4} (t + 2\nu \vartheta^2)^2 \right\} + O(t) \neq 0.
\]

According to Lemma 4.2, we know that \(f_{\xi}, f_{\eta}\) are also uniformly continuous with respect to \(t\). Hence, we conclude that the sonic line defined by \(f(\xi, \eta) = 0\) is \(C^1\) continuous.

According to the previous lemmas, we can finish the proof of Theorem 1.2.

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REFERENCES


