Symmetric solutions for a class of singular biharmonic elliptic systems involving critical exponents

Zhiying Deng\(^a\), Yisheng Huang\(^b\)

\(^a\) School of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, PR China
\(^b\) Department of Mathematics, Soochow University, Suzhou 215006, Jiangsu, PR China

**Keywords:**
- G-symmetric solution
- Critical exponent
- Symmetric criticality principle
- Biharmonic elliptic systems

**Abstract**

This paper deals with a class of singular biharmonic elliptic systems involving critical exponents in a bounded symmetric domain. By using the variational method and the symmetric criticality principle of Palais, we obtain several existence and multiplicity results of G-symmetric solutions for the systems.

© 2015 Elsevier Inc. All rights reserved.

1. Introduction

In this paper, we investigate the existence and multiplicity of nontrivial solutions for the following singular biharmonic elliptic system

\[
\begin{align*}
\Delta^2 u &= K(x) \left( |u|^{2^*-2} u + \eta \frac{\alpha}{2} |u|^{\alpha-2} u |v|^{\beta} \right) + \lambda \frac{q_1}{(q_1 + q_2)} |u|^{q_1} |v|^{q_2}, \quad \text{in } \Omega, \\
\Delta^2 v &= K(x) \left( |v|^{2^*-2} v + \eta \frac{\beta}{2} |u|^{\alpha} |v|^{\beta-2} v \right) + \lambda \frac{q_2}{(q_1 + q_2)} |u|^{q_1} |v|^{q_2}, \quad \text{in } \Omega, \\
u &= \frac{\partial u}{\partial n} = 0, \quad v = \frac{\partial v}{\partial n} = 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N (N > 4) \) is a smooth bounded domain, \( 0 \in \Omega \) and \( \Omega \) is G-symmetric with respect to a closed subgroup \( G \) of \( O(N) \) (see Section 2 for details), \( \eta \geq 0, \lambda \geq 0, 0 \leq \zeta < 4, q_1, q_2 > 1 \) and \( 2 < q_1 + q_2 < 2^* (\zeta) \), with \( 2^* (\zeta) = \frac{2N}{N-4} \), \( \alpha, \beta > 1 \) satisfy \( \alpha + \beta = 2^* (0), 2^* (0) = 2^* \frac{2N}{N-4} \) is the critical Sobolev exponent, \( \frac{\partial}{\partial n} \) is the outer normal derivative, and \( K \in C(\Omega) \cap L^\infty (\Omega) \) fulfills some symmetry conditions which will be specified later.

The critical growth in singular elliptic problems of second order has been extensively studied in recent years, starting with the seminal paper \[1\]. The results relating to these problems can be found in \[2–4\], and the references therein. In a recent paper, Deng and Jin \[5\] considered the existence of nontrivial solutions of the following critical singular problem

\[
-\Delta u = \mu \frac{u}{|x|^r} + K(x)|x|^{-\zeta} u^{2^* (\zeta) - 1}, \quad \text{and } u > 0 \quad \text{in } \mathbb{R}^N,
\]
where $\mu \in \left[0, \frac{(N-2)^2}{4}\right)$, $s \in [0, 2)$, are constants, $N > 2$, $2^*(s) = \frac{2(N-s)}{N-2}$ is the critical Hardy–Sobolev exponent and $2^*(0) = 2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, and $K$ fulfills certain symmetry conditions with respect to a subgroup $G$ of $O(\mathbb{R})$. By using analytic techniques together with standard variational arguments, the authors proved the existence and multiplicity of $G$-symmetric solutions to (1.2) under certain hypotheses on $K$. Subsequently, Waliullah [6] improved the results in [5] by establishing a variant of concentration–compactness principle due to Lions. In particular, Waliullah studied the following elliptic problem

$$(-\Delta)^m u = K(x)|u|^{2^{*m}-2}u \quad \text{in } \mathbb{R}^N,$$

where $m > 1$, $N > 2m$, $Z^p(m) = \frac{2N}{N-2m}$ and $K$ is $G$-symmetric. Using the minimizing sequence and the concentration–compactness principle, the author obtained the existence of nontrivial $G$-symmetric solutions to Eq. (1.3). Very recently, Deng and Huang [7–9] extended the results in [5,6] to the scalar weighted elliptic problems in a bounded $G$-symmetric domain. Besides, we also mention that when $\mu = s = 0$ and the right-hand side term $|x|^{-s}u^{2^*(s)-1}$ is replaced by $|x|^{q-1}$, $0 < q < \frac{2N}{N-2}$ or $q = \frac{2N}{N-2}$ in (1.2), some elegant results of $G$-symmetric solutions of (1.2) were established in [10–12]. Finally, when $G = O(\mathbb{R})$, we remark that Su and Wang [13] proved the existence of nontrivial radial solutions for a class of quasilinear singular equations such as (1.2) with radial potentials by establishing several new embedding theorems.

On the other hand, there have been many papers concerned with the existence and multiplicity of nontrivial solutions for second-order semilinear elliptic systems. In [14], Wu considered the following elliptic system with subcritical nonlinearity and sign-changing weights

$$\begin{align*}
-\Delta u & = \frac{2\alpha}{\alpha + \beta} K(x)|u|^{\alpha-2}u|v|^\beta + \lambda f(x)|u|^{\alpha-2}u, \quad \text{in } \Omega, \\
-\Delta v & = \frac{2\beta}{\alpha + \beta} K(x)|u|^\alpha|v|^{\beta-2}v + \sigma h(x)|v|^{\alpha-2}v, \quad \text{in } \Omega, \\
u & = v = 0, \quad \text{on } \partial\Omega,
\end{align*}$$

where $1 < q < 2$, $\alpha, \beta > 1$ satisfy $2 < \alpha + \beta < 2^*$ and the weight function $K, f, h$ fulfills certain suitable conditions. Applying the analytic techniques of Nehari manifold, the author proved that the system (1.4) had at least two nonnegative solutions if the parameters $\lambda$ and $\sigma$ satisfied an appropriate condition. Subsequently, Nyamoradi [15] generalized some results of [14] to the singular elliptic systems involving critical Hardy–Sobolev exponents. Very recently, Huang and Kang [16] studied the existence and asymptotic properties of positive solutions of the following critical singular elliptic systems

$$\begin{align*}
-\Delta u - \mu_1 \frac{u}{|x-a_1|^2} & = |u|^{2^*-2}u + \frac{\eta\alpha}{\alpha + \beta} |u|^{\alpha-2}u|v|^\beta + \lambda_1 |u|^{\alpha_1-2}u, \quad \text{in } \Omega, \\
-\Delta v - \mu_2 \frac{v}{|x-a_2|^2} & = |v|^{2^*-2}v + \frac{\eta\beta}{\alpha + \beta} |u|^\alpha|v|^{\beta-2}v + \lambda_2 |v|^{\alpha_2-2}v, \quad \text{in } \Omega, \\
v & = v = 0, \quad \text{on } \partial\Omega,
\end{align*}$$

where $\eta > 0$, $a_i \in \Omega$, $\lambda_1 > 0$, $\mu_i < \left(\frac{N-2}{2}\right)^2$, $2 \leq q_i < 2^*(i = 1, 2)$, and $\alpha, \beta > 1$ satisfy $\alpha + \beta = 2^*$. Note that $|u|^{\alpha-2}u|v|^\beta$ and $|u|^\alpha|v|^{\beta-2}v$ in (1.5) are called strongly coupled terms, and $|u|^{2^*-2}u$, $|v|^{2^*-2}v$ are weakly coupled terms. By employing the Moser iteration method and variational methods, the authors obtained the existence of positive solutions and some properties of the nontrivial solutions to (1.5). Other results about existence and multiplicity of nontrivial solutions, also for second-order elliptic systems, can be found in [17–19]. For the systems of fourth-order elliptic equations with concave–convex nonlinearities, various studies concerning the solutions structures have also been studied by several authors, see [20] and references therein.

However, concerning the existence and multiplicity of $G$-symmetric solutions for elliptic systems, we can only find some existence results for singular second-order elliptic systems in [21,22] and when $G = O(\mathbb{R})$, some radial and nonradial results for nonsingular second-order elliptic systems in [23]. Stimulated by [5,10,16], in the present paper, we shall study the existence and multiplicity of $G$-symmetric solutions for the fourth-order elliptic system (1.1). Due to the singular perturbations $|x|^{-s}|u|^{\alpha-2}u|x|^{\beta|u|^{\alpha-2}u}$, $|x|^{-s}|v|^{\beta|u|^{\alpha-2}u}$, $|x|^{-s}|u|^{\alpha|v|^{\beta-2}v}$ and weakly coupled terms $|u|^{2^*-2}u$ and $|v|^{2^*-2}v$ with the critical Sobolev exponents, compared with (1.2) and (1.3), the singular biharmonic elliptic system (1.1) becomes more complicated to deal with and therefore we have to face more difficulties. As far as we know, it seems that there are no results for (1.1) even in the scalar case $\eta = \lambda = 0$ and $u = v$. Many attractive and challenging topics on singular fourth-order elliptic systems remain unsolved. Hence, it makes sense for us to investigate system (1.1) thoroughly. Let $K_0 > 0$ be a constant. Note that here we will try to treat both the cases of $\lambda = 0$, $K(\mathbb{x}) \neq K_0$, and $\lambda > 0$, $K(\mathbb{x}) \equiv K_0$.

This paper is organized as follows. In Section 2, we will establish the suitable Sobolev space which is useful to discuss the elliptic system (1.1), and we will state the main results of this paper. In Section 3, we detail the proofs of several existence and multiplicity results for the cases $\lambda = 0$ and $K(\mathbb{x}) \neq K_0$ in (1.1). In Section 4, we present the proofs of existence results for the cases $\lambda > 0$ and $K(\mathbb{x}) \equiv K_0$ in (1.1). Our methods in this paper are mainly based upon the symmetric criticality principle of Palais (see [24]) and variational arguments.

2. Preliminaries and main results

Let $O(\mathbb{R})$ be the group of orthogonal linear transformations of $\mathbb{R}^N$ with natural action and let $G \subset O(\mathbb{R})$ be a closed subgroup. For any point $x \in \mathbb{R}^N$, the set $G_x = \{x_1 \in \mathbb{R}^N; x_1 = gx, g \in G\}$ is called an orbit of $x$. The number of points contained in the orbit $G_x$
will be denoted as $|G_x|$. If this number is infinite, then we write $|G_x| = +\infty$. Denote $|G| = \inf_{0 \neq x \in \mathbb{R}^N} |G_x|$ (|G| may be, in particular, $+\infty$). We call $\Omega$ a G-symmetric subset of $\mathbb{R}^N$, if $x \in \Omega$, then $gx \in \Omega$ for all $g \in G$. A measurable function $f$ is called G-symmetric if for all $x \in \mathbb{R}^N \setminus \{0\}$, $f(gx) = f(x)$ holds. Consider the natural action of $O(N)$ on $\mathbb{R}^N \setminus \{0\}$, for example. We easily see that in this case, orbits are the sphere $\partial B_R(0) (R > 0)$, $|G| = +\infty$ and G-symmetric functions are the radial functions. Other further examples of G-symmetric functions can be found in [5].

Let $H^2_0(\Omega)$ denote the closure of $C^\infty_0(\Omega)$ functions with respect to the norm $\|u\| = (\int_{\Omega} |\Delta u|^2 dx)^{1/2}$, associated with inner product given by $(u, \varphi) = \int_{\Omega} \Delta u \Delta \varphi dx$. We recall that the Hardy–Rellich inequality (see [25,26]) states that for any $u \in H^2_0(\Omega)$, there exists a constant $C = C(N, q, s) > 0$ such that

\[
\left( \int_{\Omega} |x|^{-s} |u|^q dx \right)^{2/q} \leq C \int_{\Omega} |\Delta u|^2 dx, \tag{2.1}
\]

where $0 \leq s \leq 4$ and $2 \leq q \leq 2^*^*(s) = 2(N-s)/(N-4)$. Moreover, we define the Hilbert space $(H^2_0(\Omega))^2$ endowed with the norm

$\| (u, v) \| = \left( \| u \|^2 + \| v \|^2 \right)^{1/2}$, $\forall (u, v) \in (H^2_0(\Omega))^2$.

For a bounded and G-symmetric domain $0 \in \Omega \subset \mathbb{R}^N$, the natural functional space to study problem (1.1) is the Hilbert space $(H^2_{0,G}(\Omega))^2$, which is the subspace of $(H^2_0(\Omega))^2$ consisting of all G-symmetric functions. In the present paper, we are concerned with the following problems:

$$\begin{cases}
\Delta^2 u = K(x) \left( |u|^{2^*-2} u + \frac{\eta \alpha}{2^*} |u|^{\alpha-2} u |v|^\beta \right) + \lambda \frac{q_1 |u|^{q_1-2} u |v|^\beta}{(q_1 + q_2)|x|^s}, & \text{in } \Omega, \\
\Delta^2 v = K(x) \left( |v|^{2^*-2} v + \frac{\eta \beta}{2^*} |u|^{\alpha} |v|^{\beta-2} v \right) + \lambda \frac{q_2 |u|^{q_2-2} v}{(q_1 + q_2)|x|^s}, & \text{in } \Omega, \\
(u, v) \in \left( H^2_{0,G}(\Omega) \right)^2, & \text{and } u \neq 0, \ v \neq 0,
\end{cases} \tag{\mathcal{K}^K_{\alpha}}$$

To mention our main results, we need to introduce two notations $S$ and $y_\varepsilon(x)$, which are, respectively, defined by

$$S \triangleq \inf_{u \in H^2_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\Delta u|^2 dx}{\int_{\Omega} |u|^{2^*} dx}, \tag{2.2}$$

and

$$y_\varepsilon(x) \triangleq C \varepsilon^{-\frac{N+2}{2}} (\varepsilon + |x|)^{-\frac{N+4}{2}}, \tag{2.3}$$

where $\varepsilon > 0$, and $C = C(N) > 0$, depends only on $N$. From [27], we see that $S$ is independent of $\Omega$ and $y_\varepsilon$ satisfies the equations

$$\int_{\mathbb{R}^N} |\Delta y_\varepsilon|^2 dx = 1 \tag{2.4}$$

and

$$\int_{\mathbb{R}^N} y_\varepsilon^{2^*-1} \phi dx = S^{-\frac{2}{2^*}} \int_{\mathbb{R}^N} \Delta y_\varepsilon \Delta \phi dx \tag{2.5}$$

for all $\phi \in H^2(\mathbb{R}^N)$, where $H^2(\mathbb{R}^N)$ is the closure of $C^\infty_0(\mathbb{R}^N)$ functions with respect to the norm $(\int_{\mathbb{R}^N} |\Delta \cdot|^2 dx)^{1/2}$. In particular, we have (let $\phi = y_\varepsilon$)

$$\int_{\mathbb{R}^N} y_\varepsilon^{2^*-1} dx = S^{-\frac{2}{2^*}} = S^{\frac{4}{N+4}}. \tag{2.6}$$

We suppose that the function $K(x)$ verifies the following hypotheses.

(K.1) $K(x)$ is G-symmetric.

(K.2) $K \in C(\overline{\Omega}) \cap L^\infty(\Omega)$, and $K_+ \neq 0$, where $K_+ = \max \{0, K\}$.

Since $0 \in \Omega$, we can choose $q > 0$ small enough such that $B_{2q}(0) \subset \Omega$ and define a function $\phi \in C^\infty_0(\Omega)$ such that $\phi(x) = 1$ on $B_q(0)$, $\phi(x) = 0$ on $\Omega \setminus B_{2q}(0)$. Setting $V_\varepsilon = \phi y_\varepsilon / \| \phi y_\varepsilon \|$, we obtain (see (3.15) for details)

$$\| V_\varepsilon \| = 1, \quad \text{and} \quad \int_{\Omega} |V_\varepsilon|^{2^*} dx = \int_{\Omega} |V_\varepsilon|^{\alpha+\beta} dx = S^{\frac{4}{N+4}} + O(\varepsilon^{\frac{N+4}{2}}).$$

The main results of this paper are the following.

**Theorem 2.1.** Suppose that (K.1) and (K.2) hold. If

$$\int_{\Omega} K(x) |V_\varepsilon|^{2^*} dx \geq S^{\frac{4}{N+4}} \max \left\{ |G| \frac{4}{N+4} \| K_+ \|_\infty, K_+(0) \right\} > 0 \tag{2.7}$$

for some $\varepsilon > 0$, then problem $(\mathcal{K}^K_{\alpha})$ has at least one nontrivial solution in $(H^2_{0,G}(\Omega))^2$. 

Corollary 2.1. Suppose that (K.1) and (K.2) hold. Then problem \((\mathcal{P}_0^K)\) has at least one nontrivial solution in \((H^2_{0,C}(\Omega))^2\) if
\[ K(0) > 0, \quad K(\gamma) = |G| \frac{\gamma^2}{|K|} \|K\|_{\infty} \] (2.8)
and \(K(x) \geq K(0) + \gamma_0|K|^2\) for some \(\gamma_0 > 0, \gamma \in (0, N-4)\) and \(|x|\) small.

Theorem 2.2. Suppose that \(K_*(0) = 0\) and \(|G| = +\infty\). Then problem \((\mathcal{P}_0^K)\) has infinitely many \(G\)-symmetric solutions.

Corollary 2.2. If \(K\) is a radially symmetric function such that \(K_*(0) = 0\), then problem \((\mathcal{P}_0^K)\) has infinitely many solutions which are radially symmetric.

Theorem 2.3. Let \(K(x) \equiv K_0 > 0\) be a constant and \(\lambda > 0, \eta > 0, \alpha > 1, 1 < \beta < 2\) with \(\alpha + \beta = 2^*\). If \(q_1, q_2 > 1\) satisfy
\[ \max \left\{ 2, \frac{N-\zeta}{N-4}, \frac{2(4-\zeta)}{N-4} \right\} < q_1 + q_2 < 2^*(\zeta), \] (2.9)
where \(0 \leq \zeta < 4\) and \(2^*(\zeta) = 2(N-\zeta)/(N-4)\), then problem \((\mathcal{P}_0^K)\) possesses at least one nontrivial solution in \((H^2_{0,C}(\Omega))^2\).

Remark 2.1. The main results of this paper extend and complement some results of the aforementioned papers [5,6,21]. Even in the scalar case \(\eta = \lambda = 0\) and \(u = v \in H^2_{0,C}(\Omega)\), the above results to problem \((\mathcal{P}_0^K)\) are new in the \(G\)-symmetric bounded domain.

Throughout this paper, we denote by \((H^2_{0,C}(\Omega))^2\) the subspace of \((H^2(\Omega))^2\) consisting of all \(G\)-symmetric functions. The dual space of \((H^2_{0,C}(\Omega))^2\) is the space \((H^{-2}_{0,C}(\Omega))^2\), respectively. \(O(\epsilon^4)\) denotes the quantity satisfying \(|O(\epsilon^4)|\epsilon^4 \leq C\), \(o(\epsilon^4)\) means \(|O(\epsilon^4)|\epsilon^4 \to 0\) as \(\epsilon \to 0\) and \(o(1)\) is a generic infinitesimal value. The ball of center \(x\) and radius \(\tau\) is denoted by \(B_{\tau}(x)\).

We always denote positive constant as \(C\). In a given Banach space \(X\), we denote by “\(\rightarrow\)” and “\(\rightharpoonup\)” strong and weak convergence, respectively. A functional \(F \in C^1(X, \mathbb{R})\) is said to satisfy the Palais–Smale condition at level \(c\) (the (PS) condition for short) if each sequence \((u_n)\) in \(X\) satisfying \(F(u_n) \to c\). \(F(u_n) \to 0\) in \(X^*\) possesses a strongly convergent subsequence in \(X\). Hereafter, \(L^q(\Omega, |x|^{-\delta})\) denotes the weighted \(L^q(\Omega)\) space with the norm \((\int_{\Omega} |x|^{-\delta} |u|^q dx)^{1/q}\).

3. Existence and multiplicity results for problem \((\mathcal{P}_0^K)\)

The energy functional corresponding to \((\mathcal{P}_0^K)\) is defined on \((H^2_{0,C}(\Omega))^2\) by
\[ J(u, v) = \frac{1}{2} \int_{\Omega} (|\Delta u|^2 + |\Delta v|^2) dx - \frac{K(x)}{2^*} \left( |u|^{2^*} + |v|^{2^*} + \eta |u|^\alpha |v|^\beta \right) dx. \] (3.1)
Then \(J \in C^1((H^2_{0,C}(\Omega))^2, \mathbb{R})\) and it is well known that the nontrivial critical points of \(J(u, v)\) on \((H^2_{0,C}(\Omega))^2\) correspond to the nontrivial weak solutions of problem \((\mathcal{P}_0^K)\). More precisely, by the following principle of symmetric criticality due to Palais (see Lemma 3.1), we say that \((u, v) \in (H^2_{0,C}(\Omega))^2\) is a weak solution of problem \((\mathcal{P}_0^K)\), if for any \((\varphi_1, \varphi_2) \in (H^2(\Omega))^2\), there holds
\[ \int_{\Omega} (\Delta u \Delta \varphi_1 + \Delta v \Delta \varphi_2) dx - \int_{\Omega} K(x) \left( |u|^{2^*} - 2u \varphi_1 + |v|^{2^*} - 2v \varphi_2 \right) dx \]
\[ - \frac{\eta}{2^*} \int_{\Omega} K(x) \left( |u|^{\alpha} - 2u |\varphi_1|^{\alpha/2} + |v|^{\beta} - 2v |\varphi_2|^{\beta/2} \right) dx = 0. \] (3.2)

Lemma 3.1. Let \(K(x)\) be a \(G\)-symmetric function; \(J(u, v) = 0\) in \((H^2_{0,C}(\Omega))^2\) implies \(J(u, v) = 0\) in \((H^2(\Omega))^2\).

Proof. Similar to the proof of the symmetric criticality principle in [10, Lemma 1] (see also [23, Proposition 2.8]).

For any \(\alpha, \beta > 1\) and \(\alpha + \beta = 2^*\), we define
\[ S_{\eta, \alpha, \beta} \triangleq \inf_{(u, v) \in (H^2(\Omega))^2} \frac{\int_{\Omega} (|\Delta u|^2 + |\Delta v|^2) dx}{\left[ \int_{\Omega} \left( |u|^{2^*} + |v|^{2^*} + \eta |u|^{\alpha} |v|^{\beta} \right) dx \right]^{2^*}}, \] (3.3)
\[ h(\tau) \triangleq \frac{1 + \tau^2}{(1 + \eta \tau^\alpha + \tau^{2^*})^{2^*}}, \quad \tau \geq 0, \] (3.4)
\[ h(\tau_{\min}) \triangleq \min_{\tau \geq 0} h(\tau) > 0, \] (3.5)
where \(\tau_{\min} \geq 0\) is a minimal point of \(h(\tau)\).

Lemma 3.2. Let \(y_\varepsilon(x)\) be the minimizers of \(S\) defined in (2.2) and \(\eta \geq 0, \alpha, \beta > 1\) with \(\alpha + \beta = 2^*\). Then we have the following statements.

(i) \(\tau_{\min} \geq 0, S_{\eta, \alpha, \beta} = h(\tau_{\min}) S, \) and \(S_{0, \alpha, \beta} = S.\)
(ii) \(S_{\eta, \alpha, \beta}\) has the minimizers \((y_\varepsilon(x), \tau_{\min}, y_\varepsilon(x))\), and \(S_{0, \alpha, \beta}\) has the minimizers \((y_\varepsilon(x), 0)\) for all \(\varepsilon > 0.\)
Theorem 2.1. Similar to the proof of the concentration-compactness principle in [28] (see also [26, Lemma 2.2]).

Proof. The proof is similar to that of Theorem 1.1 in Huang and Kang [17] and is omitted here. □

Lemma 3.3. Let \( \{(u_n, v_n)\} \) be a weakly convergent sequence to \((u, v)\) in \((H^2_{x, T}(\Omega))^2\) such that \(|\Delta u_n|^2 \rightarrow \eta_n|\), \(|\Delta v_n|^2 \rightarrow \eta_n|\), \(|u_n|^2 \rightarrow \sigma^{(1)}\), \(|v_n|^2 \rightarrow \sigma^{(2)}\), \(|u_n|^2|v_n|^{\beta} \rightarrow \nu\) in the sense of measures. Then there exists some at most countable set \( J \), \( \{\eta_n^{(1)} \geq 0\}_{J \in J}, \{\eta_n^{(2)} \geq 0\}_{J \in J}, \{v \geq 0\}_{J \in J}, \{x_j\}_{J \in J} \subset \Omega \) such that

\[
\begin{align*}
(a) & \quad \eta_n^{(1)} \geq |\Delta u|^2 + \sum_{J \in J} \eta_n^{(1)} \delta_{x_j}, \\
(b) & \quad \sigma_n^{(1)} = |u|^2 + \sum_{J \in J} \sigma_n^{(1)} \delta_{x_j}, \\
(c) & \quad v = |u|^2 + \sum_{J \in J} v \delta_{x_j}, \\
(d) & \quad S_{\eta, \sigma, \nu}(\sigma_n^{(1)} + \eta_n^{(2)} + \eta_n) \leq \eta_n^{(1)} + \eta_n^{(2)}, \\
(e) & \quad S(\sigma_n^{(2)} + \nu) \leq \eta_n^{(2)},
\end{align*}
\]

where \( \delta_{x_j} (j \in J) \) is the Dirac mass of 1 concentrated at \( x_j \in \Omega \).

Proof. Similar to the proof of the concentration-compactness principle in [28] (see also [26, Lemma 2.2]). □

To prove the existence results of problem (\( P_{0, T} \)) we need the following local \((PS)_c\) condition which is crucial for the proof of Theorem 2.1.

Lemma 3.4. Suppose that (K.1) and (K.2) hold. Then the \((PS)_c\) condition in \((H^2_{x, T}(\Omega))^2\) holds for \((u, v)\) if

\[
c < c_0^* \triangleq \frac{2}{N} S_{\eta, \sigma, \nu}^\frac{2}{N} \min \left\{ \| G \| K_4^{\frac{N}{2}-\frac{N}{2}}, K_5(0)^{\frac{N}{2}-\frac{N}{2}} \right\}.
\]

Proof. We follow closely the arguments in [10, 21]. Let \( \{(u_n, v_n)\} \subset (H^2_{x, T}(\Omega))^2 \) be a \((PS)_c\) sequence for \((u, v)\) with \( c < c_0^* \). Then according to (K.2), \((u_n, v_n)\) is bounded in \((H^2_{x, T}(\Omega))^2\). Consequently, up to a subsequence, we may assume that \( (u_n, v_n) \rightharpoonup (u, v) \) in \((H^2_{x, T}(\Omega))^2\).

By Lemma 3.2 there exist measures \( \eta_n^{(1)}, \eta_n^{(2)}, \sigma_n^{(1)}, \sigma_n^{(2)} \) and \( v \) such that relations (a)-(c) of this lemma hold. Let \( x_j \) be a singular point of measures \( \eta_n^{(1)}, \eta_n^{(2)}, \sigma_n^{(1)}, \sigma_n^{(2)} \) and \( v \). We define two functions \( \phi_1, \phi_2 \in C_0^\infty(\Omega) \) such that \( \phi_1 = \phi_2 = 1 \) in \( B_{\epsilon}(x_j) \), \( \phi_1 = \phi_2 = 0 \) on \( \Omega \setminus B_{2\epsilon}(x_j) \), and \( |\nabla \phi_1| \leq 2/\epsilon, |\nabla \phi_2| \leq 2/\epsilon, |\Delta \phi_1| \leq 2/\epsilon^2, |\Delta \phi_2| \leq 2/\epsilon^2 \). By Lemma 3.1, \( \lim_{n \to \infty} (u_n, v_n) = (u_0, v_0) \) in \( (H^2_{x, T}(\Omega))^2 \); hence, combining (3.2), the Hölder inequality and the Sobolev inequality, we obtain

\[
\begin{align*}
&\int_{\Omega} (\phi_1 \eta_n^{(1)}(x) + \phi_2 \eta_n^{(2)}(x)) - \int_{\Omega} K(x)(\phi_1 \sigma_n^{(1)} + \phi_2 \sigma_n^{(2)}) - \int_{\Omega} \eta_n^{(1)} \frac{K(x)}{2^{*}} (\alpha \phi_1 + \beta \phi_2) dx \\
\leq &\lim_{n \to \infty} \int_{\Omega} |\Delta u_n| |\nabla u_n| \nabla \phi_1| + \Delta v_n |\nabla v_n| \nabla \phi_2| + |u_n| \Delta u_n \Delta \phi_1 + v_n \Delta v_n \Delta \phi_2| dx \\
\leq &\sup_{n \geq 1} \left( \int_{\Omega} |\Delta u_n|^2 dx \right)^{\frac{1}{2}} \left[ \frac{2}{N} \lim_{n \to \infty} \left( \int_{\Omega} |\nabla u_n|^2 |\nabla \phi_1|^2 dx \right)^{\frac{1}{2}} + \lim_{n \to \infty} \left( \int_{\Omega} |u_n|^2 |\Delta \phi_1|^2 dx \right)^{\frac{1}{2}} \right] \\
+ &\sup_{n \geq 1} \left( \int_{\Omega} |\Delta v_n|^2 dx \right)^{\frac{1}{2}} \left[ \frac{2}{N} \lim_{n \to \infty} \left( \int_{\Omega} |\nabla v_n|^2 |\nabla \phi_2|^2 dx \right)^{\frac{1}{2}} + \lim_{n \to \infty} \left( \int_{\Omega} |v_n|^2 |\Delta \phi_2|^2 dx \right)^{\frac{1}{2}} \right] \\
\leq &C \left\{ \left( \int_{\Omega} |\nabla u|^2 |\nabla \phi_1|^2 dx \right)^{\frac{1}{2}} + \left( \int_{\Omega} |u|^2 |\Delta \phi_1|^2 dx \right)^{\frac{1}{2}} + \left( \int_{\Omega} |v|^2 |\Delta \phi_2|^2 dx \right)^{\frac{1}{2}} \right\} \\
+ &C \left\{ \left( \int_{B_{\epsilon}(x_j)} |\nabla u|^2 dx \right)^{\frac{1}{2}} + \left( \int_{B_{\epsilon}(x_j)} |\nabla \phi_1|^2 dx \right)^{\frac{1}{2}} \right\} \\
+ &C \left\{ \left( \int_{B_{\epsilon}(x_j)} |\nabla v|^2 dx \right)^{\frac{1}{2}} + \left( \int_{B_{\epsilon}(x_j)} |\nabla \phi_2|^2 dx \right)^{\frac{1}{2}} \right\} \\
\leq &C \left\{ \left( \int_{B_{\epsilon}(x_j)} |\nabla u|^2 dx \right)^{\frac{1}{2}} + \left( \int_{B_{\epsilon}(x_j)} |\nabla \phi_1|^2 dx \right)^{\frac{1}{2}} \right\} \\
+ &C \left\{ \left( \int_{B_{\epsilon}(x_j)} |\nabla v|^2 dx \right)^{\frac{1}{2}} + \left( \int_{B_{\epsilon}(x_j)} |\nabla \phi_2|^2 dx \right)^{\frac{1}{2}} \right\}.
\end{align*}
\]
Taking limits as \(\varepsilon \to 0\), we deduce from (3.8) and Lemma 3.3 that
\[
K(x_j)(\sigma_j^{(1)} + \sigma_j^{(2)} + \eta v_j) \geq \eta_j^{(1)} + \eta_j^{(2)}.
\] (3.9)

This means that the concentration of the measures \(\sigma^{(1)}, \sigma^{(2)}\) and \(\nu\) cannot occur at points where \(K(x_j) \leq 0\). More exactly, if \(K(x_j) < 0\) then \(\eta_j^{(1)} = \eta_j^{(2)} = \sigma_j^{(1)} = \sigma_j^{(2)} = v_j = 0\). Thus we conclude from (3.9) and (c) of Lemma 3.3 that either (i) \(\sigma_j^{(1)} = \sigma_j^{(2)} = v_j = 0\) or (ii) \(\sigma_j^{(1)} + \sigma_j^{(2)} + \eta v_j \geq (S_{g,\alpha,\beta}/\|K\|_\infty)^{2\nu_0}\). In the following, we show that (ii) cannot hold. For every continuous nonnegative function \(\psi\) such that \(0 \leq \psi(x) \leq 1\) on \(\Omega\), we deduce from (3.1) and (3.2) that
\[
c = \lim_{n \to \infty} \frac{1}{2} \int \left( (u_n - v_n) - \frac{1}{2} \int (f(u_n, v_n)) \right) dx = \frac{2}{N} \lim_{n \to \infty} \int \Omega (|\nabla u_n|^2 + |\nabla v_n|^2) dx
\]
\[
\geq 2 \frac{\int \Omega (|\nabla u_n|^2 + |\nabla v_n|^2) \psi(x) dx}{\int \Omega (\nabla u_n \cdot \nabla v_n)}.
\]

If (ii) occurs, then the set \(J\) must be finite because the measures \(\sigma^{(1)}, \sigma^{(2)}\) and \(\nu\) are bounded. Since functions \((u_n, v_n)\) are \(G\)-symmetric, the measures \(\sigma^{(1)}, \sigma^{(2)}\) and \(\nu\) must be \(G\)-invariant. This says that if \(x_j \neq 0\) is a singular point of \(\sigma^{(1)}, \sigma^{(2)}\) and \(\nu\), so is \(g x_j\) for each \(g \in G\), and the mass of \(\sigma^{(1)}, \sigma^{(2)}\) and \(\nu\) concentrated at \(g x_j\) is the same for each \(g \in G\). Assuming that (ii) occurs for some \(j \in J\) with \(x_j \neq 0\), we choose \(\psi\) with compact support so that \(\psi(0) = 1\) and we have
\[
c \geq 2 \frac{\int \Omega (|\nabla u_n|^2 + |\nabla v_n|^2) \psi(x) dx}{\int \Omega (\nabla u_n \cdot \nabla v_n)}.
\]
a contradiction with (3.7). Similarly, if (ii) holds for \(x_j = 0\), we choose \(\psi\) with compact support so that \(\psi(0) = 1\) and we obtain
\[
c \geq 2 \frac{\int \Omega (|\nabla u_n|^2 + |\nabla v_n|^2) \psi(x) dx}{\int \Omega (\nabla u_n \cdot \nabla v_n)}.
\]

which contradicts (3.7). Consequently, \(\sigma_j^{(1)} = \sigma_j^{(2)} = v_j = 0\) for all \(j \in J\) and this implies that
\[
\lim_{n \to \infty} \int \Omega (|u_n|^2 + |v_n|^2 + \eta |u_n|^2 |v_n|^2) dx = \int \Omega (|u|^2 + |v|^2 + \eta |u|^2 |v|^2) dx.
\]

Finally, observe that \(f(u, v) = 0\) and hence, by \(\lim_{n \to \infty} (f(u_n, v_n) - f(u, v), (u_n - u, v_n - v)) = 0\) we naturally get \((u_n, v_n) \to (u, v)\) in \((H_0^2(\Omega))^2\).

As an immediate consequence of Lemma 3.4 we obtain the following result.

**Corollary 3.1.** If \(K_+(0) = 0\) and \(|G| = +\infty\), then the functional \(J\) satisfies (PS)_\(c\) condition for every \(c \in \mathbb{R}\).

**Proof of Theorem 2.1.** Our argument is based upon the mountain pass theorem in [29]. First, we choose \(\varepsilon > 0\) so that the condition (2.7) is fulfilled, where \(V_\varepsilon = \phi_{2\varepsilon} ||\phi_{2\varepsilon}||\) satisfies (3.12)–(3.15). In view of (K.2), we deduce from (3.1) and (3.3) that
\[
J(u, v) \geq \frac{1}{2} \|u\|^2 - \frac{1}{2^\nu} \|K\|_{\infty} S_{g,\alpha,\beta}^{2\nu^2/2} \|u, v\|^{2^\nu}.
\]

Therefore, there exist constants \(\alpha_0 > 0\) and \(\rho > 0\) such that \(J(u, v) \geq \alpha_0\) for all \(\|u, v\| = \rho\). Moreover, if we set \(u = V_\varepsilon, v = \tau_{\min} V_\varepsilon\) and
\[
\Phi(t) = J(tu, tv) = J(tV_\varepsilon, t\tau_{\min} V_\varepsilon)
\]
\[
= \frac{t^2}{2} \left( 1 + \tau_{\min}^2 \right) \int \Omega |\Delta V_\varepsilon|^2 dx - \frac{t^{2^\nu}}{2^\nu} \left( 1 + \eta \tau_{\min}^{\beta} + \tau_{\min}^{2\nu} \right) \int \Omega K(x)|V_\varepsilon|^{2^\nu} dx
\]
with \(t \geq 0\), then we easily see that \(\Phi\) has a unique maximum in positive \(t\) at some \(\bar{t}\) at which \(d\Phi(t)/dt\) becomes zero. A simple calculation gives us this value
\[
\bar{t} = \left( \frac{(1 + \tau_{\min}^2 \int \Omega |\Delta V_\varepsilon|^2 dx}{(1 + \eta \tau_{\min}^{\beta} + \tau_{\min}^{2\nu} \int \Omega K(x)|V_\varepsilon|^{2^\nu} dx} \right)^{1/\nu}.
\]
Consequently, we obtain
\[
\max_{\tau \geq 0} \Phi(t) = J(\tau V_e, \tau \tau_{\min} V_e)
\]
\[
= \frac{2}{N} \left\{ \frac{(1 + \tau_{\min}^2) c_{\Omega} [\Delta V_e |^2 dx]}{\left( [1 + \eta \tau_{\min}^\beta + \tau_{\min}^\epsilon] K(x) |V_e |^{2 - \epsilon} dx \right)^{\frac{N}{2}}} \right\}^{\frac{2}{N}}
\]
\[
= \frac{2}{N} \left\{ \frac{h(\tau_{\min}) c_{\Omega} [\Delta V_e |^2 dx]}{(\int_{\Omega} K(x) |V_e |^{2 - \epsilon} dx)^{\frac{N}{2}}} \right\}^{\frac{2}{N}}.
\] (3.10)

On the other hand, since \(J(tV_e, t \tau_{\min} V_e) \to -\infty\) as \(t \to \infty\), there exists \(t_0 > 0\) such that \(J(t_0 V_e, t_0 \tau_{\min} V_e) > \rho\) and \(J(t_0 V_e, t_0 \tau_{\min} V_e) < 0\). Now we define
\[
c_0 = \inf_{\gamma \in \Gamma, \tau \in [0, 1]} \max \gamma(t),
\] (3.11)
where \(\Gamma = \{ \gamma \in C([0, 1], (H_{0, \gamma}^1(\Omega))^2); \gamma(0) = (0, 0), J(\gamma(1)) < 0, \|\gamma(1)\| \leq \rho \}.\) Combining (2.7), (3.7), (3.10), (3.11) and Lemma 3.2, we obtain that
\[
c_0 \leq J(\tau V_e, \tau \tau_{\min} V_e) = \frac{2}{N} \left\{ \frac{h(\tau_{\min}) c_{\Omega} [\Delta V_e |^2 dx]}{(\int_{\Omega} K(x) |V_e |^{2 - \epsilon} dx)^{\frac{N}{2}}} \right\}^{\frac{2}{N}}
\]
\[
\leq \frac{2}{N} \left\{ \frac{h(\tau_{\min}) c_{\Omega} [\Delta V_e |^2 dx]}{\left( \max_{\tau \in [0, 1]} \left\{ G | K | \| K \|_{\infty}, K_+ (0)^{\frac{1 - \epsilon}{4}} \right\} \right)^{\frac{N}{2}}} \right\}^{\frac{2}{N}}
\]
\[
= \frac{2}{N} S_{\Omega, \gamma} c_0 \min \left\{ \left\{ \| G | K | \right\|_{\frac{1 - \epsilon}{4}}, K_+ (0)^{\frac{1 - \epsilon}{4}} \right\} = c_0^*.
\]
If \(c_0 < c_0^*\), then by Lemma 3.4 the (PS)\(_c\) condition holds and the conclusion follows from the mountain pass theorem. If \(c_0 = c_0^*\), then \(\gamma(t) = (t_0 V_e, t_0 \tau_{\min} V_e)\), with \(0 < t \leq 1\), is a path in \(\Gamma\) such that \(\max_{t \in [0, 1]} J(\gamma(t)) = c_0\). Therefore, either \(\Phi^*(\gamma) = J(\tau V_e, \tau \tau_{\min} V_e) = 0\) and we are done, or \(\gamma\) can be deformed to a path \(\tilde{\gamma} \in \Gamma\) with \(\max_{t \in [0, 1]} J(\tilde{\gamma}(t)) < c_0\) and we get a contradiction. Consequently, we conclude that a nontrivial solution \((u_0, v_0) \in (H_{0, \gamma}^2(\Omega) \setminus \{0\})^2\) of problem (\(\mathcal{P}_\beta^\delta\)) exists. This, combined with Lemma 3.1, implies that \((u_0, v_0)\) is a nontrivial \(G\)-symmetric solution of problem (\(\mathcal{P}_{\beta}^\delta\)). \(\square\)

Proof of Corollary 2.1. Let \(\gamma_\epsilon(x)\) be the extremal functions satisfying (2.3)–(2.6). Choose \(\phi \in C_0^\infty(\Omega)\) so that \(\phi \geq 0\) on \(\Omega\), \(\phi(x) = 1\) on \(B_\rho(0)\) and \(\phi(x) = 0\) on \(\Omega\setminus B_{2\rho}(0)\), with \(\rho > 0\) to be determined. Applying the methods in [1,3], we infer from (2.3)–(2.6) that
\[
\| \phi y_\epsilon \|^2 = \int_{\Omega} [\Delta (\phi y_\epsilon)]^2 dx = 1 + O(\epsilon^{\frac{N-4}{4}}),
\] (3.12)
\[
\int_{\Omega} [\phi y_\epsilon ]^{2 - \epsilon} dx = S^{-\frac{N}{2}} + O(\epsilon^{\frac{\alpha}{2}}) = S^{\frac{\alpha}{2}} + O(\epsilon^{\frac{\alpha}{2}}),
\] (3.13)
\[
\int_{\Omega} |x|^{-\epsilon} [\phi y_\epsilon ]^q dx = \begin{cases} O \left( \epsilon^{\frac{\alpha q - 4q}{2q}} \right), & 1 < q < \frac{N - \epsilon}{N - 4}, \\ O \left( \epsilon^{\frac{\alpha q - 4q}{2q}} \right) |\ln \epsilon|, & q = \frac{N - \epsilon}{N - 4}, \\ O \left( \epsilon^{\frac{\alpha q - 4q}{2q}} \right), & \frac{N - \epsilon}{N - 4} < q < 2^*(\xi). \end{cases}
\] (3.14)

Setting \(V_\epsilon = \phi y_\epsilon / \| \phi y_\epsilon \|_{\mu}\), we deduce from (3.12) and (3.13) that
\[
\int_{\Omega} |V_\epsilon |^{2 - \epsilon} dx = \int_{\Omega} \| \phi y_\epsilon \|^{2 - \epsilon} dx = S^{\frac{\alpha}{2}} + O(\epsilon^{\frac{\alpha}{4}}).
\] (3.15)
Let us now choose \(q > 0\) so that \(K(x) \geq K(0) + c_{\Omega} q |x|^q\) for \(|x| \leq \rho\). Then we deduce from (3.15) that
\[
\int_{\Omega} K(x) |V_\epsilon |^{2 - \epsilon} dx = \int_{\Omega} (K(x) - K(0)) |V_\epsilon |^{2 - \epsilon} dx + K(0) S^{\frac{\alpha}{2}} + O(\epsilon^{\frac{\alpha}{4}}).
\]
Consequently, it is sufficient to show that
\[
\int_{\Omega} (K(x) - K(0)) |V_\epsilon |^{2 - \epsilon} dx + O(\epsilon^{\frac{\alpha}{4}}) \geq 0
\] (3.16)
for sufficiently small $\epsilon > 0$. Note that here we have
\[
\int_{\Omega} (K(x) - K(0)) |V_{\epsilon}|^{2^{**}} \, dx = \int_{|x| \leq \Omega} (K(x) - K(0)) |V_{\epsilon}|^{2^{**}} \, dx + \int_{|x| > \Omega} (K(x) - K(0)) |V_{\epsilon}|^{2^{**}} \, dx \geq \gamma_{0} \int_{|x| \leq \Omega} \frac{|x|^\theta |V_{\epsilon}|^{2^{**}}}{\|\phi_{V_{\epsilon}}\|^{2^{**}}} \, dx + \int_{|x| > \Omega} \frac{(K(x) - K(0)) |\phi_{V_{\epsilon}}|^{2^{**}}}{\|\phi_{V_{\epsilon}}\|^{2^{**}}} \, dx = J_{1} + J_{2}.
\]

For $\epsilon > 0$ small enough, we obtain from (2.3)-(2.6), (3.12) and the fact $N - 1 + \theta > -1, N - 1 + \theta - 2 \cdot \frac{N - 4}{2} < -1$ that
\[
J_{1} = \gamma_{0} \int_{|x| \leq \Omega} \frac{|x|^\theta |V_{\epsilon}|^{2^{**}}}{\|\phi_{V_{\epsilon}}\|^{2^{**}}} \, dx = \frac{C_{\epsilon} \gamma}{(1 + O(\epsilon^{\frac{N - 4}{2}}))^{\frac{N - 4}{2}}} \int_{|x| \leq \Omega} \frac{|x|^\theta}{\|\phi_{V_{\epsilon}}\|^{2^{**}}} \left[1 + \left(\frac{|x|}{\epsilon}\right)^{2}\right]^{-\frac{N - 4}{2}} \, dx \geq C_{\epsilon} \int_{0}^{1} \frac{r^{N - 1 + \theta}}{(1 + r^2)^{\frac{N - 4}{2}}} \, dr + \int_{1}^{\sqrt{\epsilon}} \frac{r^{N - 1 + \theta}}{(1 + r^2)^{\frac{N - 4}{2}}} \, dr \geq C_{1} \epsilon^{-\frac{\theta}{2}}, \quad \theta \in (0, N - 4)
\]
and
\[
|J_{2}| \leq \int_{|x| > \Omega} |K(x) - K(0)| |\phi_{V_{\epsilon}}|^{2^{**}} \, dx \leq C \int_{|x| > \Omega} \left[1 + \left(\frac{|x|}{\epsilon}\right)^{2}\right]^{-\frac{N - 4}{2}} \, dx \leq C \epsilon^{-\frac{\theta}{2}},
\]
where $C_{1} > 0$ and $C_{2} > 0$ are constants independent of $\epsilon$. In view of $0 < \frac{\theta}{2} < \frac{N - 4}{2} < \frac{N}{2}$, we remark that inequality (3.16) holds as $\epsilon > 0$ sufficiently small. Therefore, we see from (2.8), (3.15) and (3.16) that
\[
\int_{\Omega} K(x) |V_{\epsilon}|^{2^{**}} \, dx = \int_{\Omega} (K(x) - K(0)) |V_{\epsilon}|^{2^{**}} \, dx + K(0) S_{\frac{N}{2}}^{\frac{N}{2}} + O(\epsilon^{\frac{N - 4}{2}}) \geq K(0) S_{\frac{N}{2}}^{\frac{N}{2}} \geq S_{\frac{N}{2}}^{\frac{N}{2}} \max \left\{|G| \frac{N}{2} \|K_{\epsilon}\|_{\infty}, |K_{\epsilon}|(0)\right\} > 0.
\]

By Theorem 2.1 and the above inequality, we obtain the conclusion. □

To prove the existence of infinitely many $G$-symmetric solutions for problem $(\mathcal{P}_{0}^{K})$, we need the following version of the symmetric mountain pass theorem (see [30, Theorem 9.12]).

**Lemma 3.5.** Let $E$ be an infinite dimensional Banach space and let $J \in C^{1}(E, \mathbb{R})$ be an even functional satisfying (PS)$_{c}$ condition for each $c \in \mathbb{R}$ and $J(0) = 0$. Furthermore, we suppose that :

(i) there exist constants $\xi > 0$ and $\rho > 0$ such that $J(\xi) \geq \xi$ for all $\|z\| = \rho$;

(ii) there exists an increasing sequence of subspaces $(E_{n})_{n}$ of $E$, with $\dim E_{n} = m$, such that for every $m$ one can find a constant $R_{m} > 0$ such that $J(z) \leq 0$ for all $z \in E_{n}$ with $\|z\| = R_{m}$.

Then $J$ possesses a sequence of critical values $(c_{n})$ tending to $\infty$ as $m \to \infty$.

**Proof of Theorem 2.2.** Applying Lemma 3.5 with $E = (H_{0, \mathcal{C}}^{2}(\Omega))^{2}$ and $z = (u, v) \in E$, we deduce from (K.2), (3.1) and (3.3) that
\[
J(u, v) \geq \frac{1}{2} \|u, v\|^{2} - \frac{1}{2^{**}} \|K\|_{\infty}^{\frac{2^{**}}{4}} \|u, v\|^{2^{**}}.
\]
Since $2^{**} > 2$, there exist constants $\xi > 0$ and $\rho > 0$ such that $J(u, v) \geq \xi$ for all $(u, v)$ with $\|u, v\| = \rho$. To find a suitable sequence of finite dimensional subspaces of $(H_{0, \mathcal{C}}^{2}(\Omega))^{2}$, we set $\Omega_{\rho}^{2} = \{x \in \Omega; K(x) > 0\}$. Obviously, the set $\Omega_{\rho}^{2}$ is $G$-symmetric and we can define $(H_{0, \mathcal{C}}^{2}(\Omega_{\rho}^{2}))^{2}$, which is the subspace of $G$-symmetric functions of $(H_{0}^{2}(\Omega_{\rho}^{2}))^{2}$ (see Section 2). By extending functions in $(H_{0}^{2}(\Omega_{\rho}^{2}))^{2}$ by 0 outside $\Omega_{\rho}^{2}$ we can assume that $(H_{0, \mathcal{C}}^{2}(\Omega_{\rho}^{2}))^{2} \subset (H_{0, \mathcal{C}}^{2}(\Omega))^{2}$. Let $(E_{m})$ be an increasing sequence of subspaces of $(H_{0, \mathcal{C}}^{2}(\Omega_{\rho}^{2}))^{2}$ with $\dim E_{m} = m$ for each $m$. Then there exists a constant $\xi(m) > 0$ such that
\[
\int_{\Omega_{\rho}^{2}} \frac{K(x)}{2^{**}} \left(|\bar{u}|^{2^{**}} + |\bar{v}|^{2^{**}} + \eta |\bar{u}|^{\alpha} |\bar{v}|^{\beta}\right) \, dx \geq \xi(m)
\]
for all \((\tilde{u}, \tilde{v}) \in E_m\) with \(\|\tilde{u}, \tilde{v}\|_H = 1\). Therefore, if \((0, 0) \neq (u, v) \in E_m\) then we write \((u, v) = t(\tilde{u}, \tilde{v})\), with \(t = \|(u, v)\|\) and \(\|\tilde{u}, \tilde{v}\| = 1\). Hence we obtain

\[
J(u, v) = \frac{1}{2} t^2 - \frac{1}{2264} t^{2^*} \int_{\Omega^2} K(x)(|\tilde{u}|^{2^*} + |\tilde{v}|^{2^*} + \eta |\tilde{u}|^\alpha |\tilde{v}|^\beta)dx \leq \frac{1}{2} t^2 - \xi (m)^{2^*} \leq 0
\]

for \(t\) large enough. According to Lemma 3.5 and Corollary 3.1, we conclude that there exists a sequence of critical values \(c_m \to \infty\) and the results follow. □

Proof of Corollary 2.2. Since \(K(x)\) is radially symmetric, we naturally see that the corresponding group \(G\) is \(O(n)\) and \(|G| = +\infty\). This, combined with the condition \(K_0(0) = 0\) and Corollary 3.1, implies that \(J\) satisfies the \((PS)_c\) condition for every \(c \in \mathbb{R}\). Thus by Theorem 2.2 we obtain the result. □

4. Existence results for problem \((\mathcal{P}_\lambda)\)

The purpose of this section is to consider the existence of \(G\)-symmetric solutions for problem \((\mathcal{P}_\lambda)\) and prove Theorem 2.2; here \(\lambda > 0\), and \(K(x) \equiv K_0 > 0\) is a constant. We define a functional \(F: (H^1_{0, G}(\Omega))^2 \to \mathbb{R}\) given by

\[
F(u, v) = \frac{1}{2} \int_{\Omega} (|\Delta u|^2 + |\Delta v|^2)dx - \frac{K_0}{2^{2^*}} \int_{\Omega} (|u|^{2^*} + |v|^{2^*} + \eta |u|^\alpha |v|^\beta)dx
\]

\[
- \frac{\lambda}{q_1 + q_2} \int_{\Omega} \frac{|u|^{q_1}|v|^{q_2}}{|x|^s} dx,
\]

where \(0 \leq \zeta < 4, q_1, q_2 > 1\) and \(q_1 + q_2 \in (2, 2^{*}(\zeta))\). Using (2.1) and the Hölder inequality, we easily see that \(F\) is well defined and of \(C^1\). Thus there exists a one-to-one correspondence between the nontrivial weak solutions of \((\mathcal{P}_\lambda)\) and the nontrivial critical points of \(F\). Moreover, an analogously symmetric criticality principle of Lemma 3.1 clearly holds; hence the nontrivial weak solutions of problem \((\mathcal{P}_\lambda)\) are exactly the nontrivial critical points of \(F\).

Lemma 4.1. Suppose that \(\lambda > 0\), \(0 \leq \zeta < 4, q_1, q_2 > 1\), \(q_1 + q_2 \in (2, 2^{*}(\zeta))\), and \(\alpha > 1, 1 < \beta < 2\) satisfy \(\alpha + \beta = 2^{*}\). Then the \((PS)_c\) condition in \((H^2_{0, G}(\Omega))^2\) holds for \(F\), if

\[
c < \frac{2}{K_0} \frac{1}{N^{1-\frac{s}{2}} \xi^{\frac{s}{2}} \eta^{\frac{\alpha}{2}} \beta}.
\]

Proof. Suppose that \(\{(u_n, v_n)\} \subset (H^2_{0, G}(\Omega))^2\) is a \((PS)_c\) sequence of functional \(F\) with \(c\) satisfying (4.2). Then, as \(n \to \infty\), we deduce from (4.1) and the fact \(2 < q_1 + q_2 < 2^{*}(\zeta) \leq 2^{*}\) that

\[
c + o(1) = F(u_n, v_n)
\]

\[
= F_\lambda(u_n, v_n) - \frac{1}{q_1 + q_2} \langle F_\lambda'(u_n, v_n), (u_n, v_n) \rangle + \frac{1}{q_1 + q_2} \langle F_\lambda'(u_n, v_n), (u_n, v_n) \rangle
\]

\[
= \left( \frac{1}{2} - \frac{1}{q_1 + q_2} \right) \| (u_n, v_n) \|^2 + \frac{1}{q_1 + q_2} \langle F_\lambda'(u_n, v_n), (u_n, v_n) \rangle
\]

\[
+ \frac{1}{q_1 + q_2 - 2^{2^*}} K_0 \int_{\Omega} (|u|^{2^*} + |v|^{2^*} + \eta |u|^\alpha |v|^\beta)dx
\]

\[
\geq \frac{1}{2 - \frac{1}{q_1 + q_2}} \| (u_n, v_n) \|^2 + o(1)(u_n, v_n)
\]

This implies that \(\{(u_n, v_n)\}\) is bounded in \((H^2_{0, G}(\Omega))^2\). Therefore, just as in Lemma 3.4, we may assume that \(u_n \rightharpoonup u, v_n \rightharpoonup v\) in \(H^2_{0, G}(\Omega)\) and in \(L^{2^{*}}(\Omega)\); moreover, \(u_n \to u, v_n \to v\) in \(L^{2\ast + q_2}(\Omega, |x|^{-s})\) for all \(0 \leq \zeta < 4, q_1, q_2 > 1\), \(q_1 + q_2 \in (2, 2^{*}(\zeta))\) (see [25, Lemma 2.1]) and a.e. on \(\Omega\). This means

\[
\int_{\Omega} \frac{|u_n|^{q_1}|v_n|^{q_2}}{|x|^s} dx = \int_{\Omega} \frac{|u|^q|v|^{q_2}}{|x|^s} dx + o(1)
\]

A standard argument shows that \((u, v)\) is a critical point of \(F_\lambda\), and hence

\[
F_\lambda(u, v) = \frac{2K_0}{N} \int_{\Omega} (|u|^{2^*} + |v|^{2^*} + \eta |u|^\alpha |v|^\beta)dx + \frac{q_1 + q_2 - 2}{2(q_1 + q_2)} \lambda \int_{\Omega} \frac{|u|^{q_1}|v|^{q_2}}{|x|^s} dx \geq 0.
\]

Now we set \(u_n = u_n - u\) and \(v_n = v_n - v\). Then by the Brezis–Lieb lemma [31] and arguing as in [32, Lemma 2.1] we get

\[
\| (\tilde{u}, \tilde{v}) \|^2 = \| (u_n, v_n) \|^2 - \| (u, v) \|^2 + o(1).
\]

\[
\int_{\Omega} |\tilde{u}|^{2^*} dx = \int_{\Omega} |u_n|^{2^*} dx - \int_{\Omega} |u|^{2^*} dx + o(1).
\]
\[
\int_\Omega |\tilde{V}_n|^{2^*} \, dx = \int_\Omega |V_n|^{2^*} \, dx - \int_\Omega |V|^{2^*} \, dx + o(1). 
\] (4.7)

\[
\int_\Omega |\tilde{u}_n|^{\alpha'} |\tilde{V}_n|^{\beta'} \, dx = \int_\Omega |u_n|^{\alpha'} |V_n|^{\beta'} \, dx - \int_\Omega |u|^{\alpha'} |V|^{\beta'} \, dx + o(1). 
\] (4.8)

In view of \( \mathcal{F}_\lambda(u_n, v_n) = c + o(1) \) and \( \mathcal{F}_\lambda(u, v) = o(1) \), we deduce from (4.1), (4.3)–(4.8) that

\[
c + o(1) = \mathcal{F}_\lambda(u_n, v_n) = \mathcal{F}_\lambda(u, v) + \frac{1}{2} \| (\tilde{u}_n, \tilde{v}_n) \|_2^2 - \frac{K_0}{2^*} \int_\Omega \left( |\tilde{u}_n|^{2^*} + |\tilde{v}_n|^{2^*} + \eta |\tilde{u}_n|^{\alpha'} |\tilde{v}_n|^{\beta'} \right) \, dx + o(1) 
\] (4.9)

and

\[
\| (\tilde{u}_n, \tilde{v}_n) \|^2 - K_0 \int_\Omega \left( |\tilde{u}_n|^{2^*} + |\tilde{v}_n|^{2^*} + \eta |\tilde{u}_n|^{\alpha'} |\tilde{v}_n|^{\beta'} \right) \, dx = o(1). 
\] (4.10)

Therefore, for a subsequence \((\tilde{u}_n, \tilde{v}_n)\), we obtain

\[
\| (\tilde{u}_n, \tilde{v}_n) \|^2 \rightarrow l \geq 0. 
\]

It follows from (3.3) that \( S_{\eta, \alpha, \beta}(l/K_0) \frac{2^*}{2} \leq l \), which means either \( l = 0 \) or \( l \geq K_0^{1-N/4} \frac{2^*}{2} S_{\eta, \alpha, \beta} \). If \( l \geq K_0^{1-N/4} \frac{2^*}{2} S_{\eta, \alpha, \beta} \), then by (4.4), (4.9) and (4.10) we get

\[
c = \mathcal{F}_\lambda(u, v) + \left( \frac{1}{2} - \frac{1}{2^*} \right) l \geq \frac{1}{2} K_0^{1-N/4} \frac{2^*}{2} S_{\eta, \alpha, \beta},
\]

a contradiction with (4.2). Consequently, we have \( \| (\tilde{u}_n, \tilde{v}_n) \|^2 \rightarrow 0 \) as \( n \rightarrow \infty \), and thus, \((u_n, v_n) \rightarrow (u, v)\) in \((H_{0, \infty}^2(\Omega))^2\). The lemma is proved. \( \square \)

**Lemma 4.2.** Suppose that \( \lambda > 0 \), \( 0 < \xi < q < 4 \), \( q_1, q_2 > 1 \), \( q_1 + q_2 \in (2, 2^*(\xi)) \), and \( \alpha > 1, 1 < \beta < 2 \) satisfy \( \alpha + \beta = 2^* \). Then there exists a pair of functions \((\bar{u}, \bar{v}) \in (H_{0, \infty}^2(\Omega) \setminus \{0\})^2 \) such that

\[
\sup_{t \geq 0} \mathcal{F}_\lambda(t\bar{u}, t\bar{v}) = \begin{cases} 
\frac{1}{N} K_0^{1-N/4} \frac{2^*}{2} S_{\eta, \alpha, \beta}^2 & \text{if } l \geq K_0^{1-N/4} \frac{2^*}{2} S_{\eta, \alpha, \beta}, \\
0 & \text{if } l < K_0^{1-N/4} \frac{2^*}{2} S_{\eta, \alpha, \beta}.
\end{cases}
\] (4.11)

**Proof.** Recall \( V_\varepsilon = \phi y_{\varepsilon} \| \phi y_{\varepsilon} \| \) and \( \tau_{\min} > 0 \) satisfying (3.4)–(3.6). In the following, we will show that \((V_\varepsilon, \tau_{\min} V_\varepsilon)\) satisfies (4.11) for \( \varepsilon > 0 \) sufficiently small. First, we consider the functions

\[
\Psi(t) = \mathcal{F}_\lambda(tV_\varepsilon, t\tau_{\min} V_\varepsilon) = \frac{t^2}{2} \left( 1 + \tau_{\min}^2 \right) - \frac{t^{2^*}}{2^*} \left( 1 + \eta \tau_{\min}^\beta + \tau_{\min}^{2^*} \right) K_0 \int_\Omega |V_\varepsilon|^{2^*} \, dx 
\]

\[
= \frac{\lambda}{q_1 + q_2} t^{q_1 + q_2} \tau_{\min}^{q_2} \int_\Omega |x|^{-\xi} |V_\varepsilon|^{q_1 + q_2} \, dx, \quad t \geq 0
\] (4.12)

and

\[
\bar{\Psi}(t) = \frac{t^2}{2} \left( 1 + \tau_{\min}^2 \right) - \frac{t^{2^*}}{2^*} \left( 1 + \eta \tau_{\min}^\beta + \tau_{\min}^{2^*} \right) K_0 \int_\Omega |V_\varepsilon|^{2^*} \, dx, \quad t \geq 0.
\] (4.13)

Note that \( \Psi(0) = 0, \bar{\Psi}(t) > 0 \) for \( t \rightarrow 0^+ \), and \( \lim_{t \rightarrow +\infty} \Psi(t) = -\infty \). Hence \( \sup_{t \geq 0} \Psi(t) \) can be achieved at some \( t_\varepsilon > 0 \) for which we obtain

\[
(1 + \tau_{\min}^2) t_\varepsilon \leq K_0 \left( 1 + \eta \tau_{\min}^\beta + \tau_{\min}^{2^*} \right) t_\varepsilon^{2^* - 1} \int_\Omega |V_\varepsilon|^{2^*} \, dx 
\]

\[
- \lambda \tau_{\min}^{q_2} t_\varepsilon^{q_1 + q_2 - 1} \int_\Omega |x|^{-\xi} |V_\varepsilon|^{q_1 + q_2} \, dx = 0.
\] (4.14)

In view of \( V_\varepsilon = \phi y_{\varepsilon} \| \phi y_{\varepsilon} \| \), we see from (3.12) and (3.14) that

\[
\begin{cases} 
C_1 \varepsilon^{\frac{q_1}{N-4}} \leq \int_\Omega \frac{|V_\varepsilon|^q}{|x|^\xi} \, dx \leq C_2 \varepsilon^{\frac{q_2}{N-4}}, & 1 \leq q < \frac{N - \xi}{N-4}, \\
C_3 \varepsilon^{\frac{q_1}{N-4}} \ln \varepsilon \leq \int_\Omega \frac{|V_\varepsilon|^q}{|x|^\xi} \ln |V_\varepsilon| \, dx \leq C_4 \varepsilon^{\frac{q_2}{N-4}} \ln |V_\varepsilon|, & q = \frac{N - \xi}{N-4}, \\
C_5 \varepsilon^{\frac{q_1}{N-4}} \leq \int_\Omega \frac{|V_\varepsilon|^q}{|x|^\xi} \, dx \leq C_6 \varepsilon^{\frac{q_2}{N-4}}, & \frac{N - \xi}{N-4} < q < 2^*(\xi).
\end{cases}
\] (4.15)
Therefore for $\varepsilon > 0$ small enough, we conclude from (3.15), (4.14) and (4.15) and the fact $\lambda > 0$, $\tau_{\min} > 0$ that

$$0 < \overline{c}_3 \leq t_\varepsilon \leq \left\{ \frac{1 + \tau_{\min}^2}{K_0 \left( 1 + \eta \tau_{\min}^2 + \tau_{\min}^2 \right) \int_{\Omega} |V_\varepsilon|^{2^*} dx} \right\}^{\frac{1}{2^* - 2}} t_0^0 \leq \overline{c}_4,$$

(4.16)

where $\overline{c}_3 > 0$, $\overline{c}_4 > 0$ are constants independent of $\varepsilon$. On the other hand, the function $\tilde{\Psi}(t)$ defined by (4.13) attains its maximum at $t_0^0$ and is increasing in the interval $[0, t_0^0]$. Together with Lemma 3.2, (3.15) and (4.12)–(4.16), we obtain

$$\Psi(t_\varepsilon) = \tilde{\Psi}(t_\varepsilon) = \frac{\lambda \tau_{\min}}{q_1 + q_2} \int_{\Omega} \frac{|t_\varepsilon V_\varepsilon|^{q_1 + q_2}}{|x|^{\gamma}} dx \leq \tilde{\Psi}(t_0^0) = \frac{\lambda \tau_{\min}}{q_1 + q_2} \int_{\Omega} \frac{|t_0^0 V_\varepsilon|^{q_1 + q_2}}{|x|^{\gamma}} dx

= \frac{2}{N} \left\{ \frac{1 + \tau_{\min}^2}{K_0 \left( 1 + \eta \tau_{\min}^2 + \tau_{\min}^2 \right) \int_{\Omega} |V_\varepsilon|^{2^*} dx} \right\}^{\frac{q_1}{q_1 + q_2}} - \frac{\lambda \tau_{\min}}{q_1 + q_2} \int_{\Omega} \frac{|V_\varepsilon|^{q_1 + q_2}}{|x|^{\gamma}} dx

= \frac{2}{N} K_0^{1 - \frac{q_2}{q_1 + q_2}} \left( \frac{h(\tau_{\min})}{(\frac{s}{\sqrt{s^2 + o(\varepsilon^{-\frac{4}{N}}))})^{(N-4)/N}} \right)^{\frac{q_1}{q_1 + q_2}} - \frac{\lambda \tau_{\min}}{q_1 + q_2} t_0^0 \int_{\Omega} \frac{|V_\varepsilon|^{q_1 + q_2}}{|x|^{\gamma}} dx

\leq \frac{2}{N} K_0^{1 - \frac{q_2}{q_1 + q_2}} S_{\eta, \alpha, \beta}^q + O(\varepsilon^{-\frac{4}{N}}) - C \int_{\Omega} \frac{|V_\varepsilon|^{q_1 + q_2}}{|x|^{\gamma}} dx.$$  

(4.17)

Furthermore, we easily check from (2.9) that

$$\frac{N - 4}{2} > \frac{2(N - \zeta) - (q_1 + q_2)(N - 4)}{4}.$$  

(4.18)

Choosing $\varepsilon > 0$ sufficiently small, we deduce from (4.15), (4.17) and (4.18) that

$$\sup_{t_{x_0}} \mathcal{F}_\lambda(t V_\varepsilon, t \tau_{\min} V_\varepsilon) = \Psi(t_\varepsilon) < \frac{2}{N} K_0^{1 - \frac{q_2}{q_1 + q_2}} S_{\eta, \alpha, \beta}^q.$$  

Consequently, we conclude that $(V_\varepsilon, \tau_{\min} V_\varepsilon)$ satisfies (4.11) for $\varepsilon > 0$ small enough and the results follow. \qed

**Proof of Theorem 2.3.** For any $(u, v) \in (H_0^2(\Omega) \setminus \{0\})^2$, we easily get from (2.1), the Hölder inequality and the fact $q_1 + q_2 \in (2, 2^*(\zeta))$ that

$$\int_{\Omega} |x|^{-\zeta} |u|^{q_1} |v|^{q_2} dx \leq \left( \int_{\Omega} |x|^{-\zeta} |u| \right)^{\frac{q_1}{q_1 + q_2}} \left( \int_{\Omega} |x|^{-\zeta} |v| \right)^{\frac{q_2}{q_1 + q_2}} \leq C \|u\|^{q_1} \|v\|^{q_2} \leq C \|(u, v)\|^{q_1 + q_2}.$$

This, combined with (3.3) and (4.1), implies that

$$\mathcal{F}_\lambda(u, v) \geq \frac{1}{2} \|(u, v)\|^2 - \frac{K_0}{2^{2^* - q_1}} S_{\eta, \alpha, \beta}^q \|(u, v)\|^{2^*} - C \|(u, v)\|^{q_1 + q_2}.$$  

Since $2 < q_1 + q_2 < 2^*$, there exist constants $\overline{c} > 0$ and $\rho > 0$ such that $\mathcal{F}_\lambda(u, v) \geq \overline{c}$ for all $\|\|(u, v)\| = \rho$. Moreover, we conclude from $\lim_{t \to -\infty} \mathcal{F}_\lambda(tu, tv) = -\infty$ that there exists $t_0 > 0$ such that $\|(t_0 u, t_0 v)\| > \rho$ and $\mathcal{F}_\lambda(t_0 u, t_0 v) < 0$. Now we set

$$c_1 = \inf_{y \in \Gamma} \max_{t \in [0, 1]} \mathcal{F}_\lambda(y(t)).$$

where $\Gamma = \{ y \in C([0, 1], (H_0^2(\Omega))^2) ; y(0) = (0, 0) \}$. By the mountain pass theorem, we deduce that there exists a sequence $(t_n, v_n)$ such that $\mathcal{F}_\lambda(u_n, v_n) \to c_1 \geq \overline{c}$. $\mathcal{F}_\lambda(u_n, v_n) \to 0$ as $n \to \infty$. Let $(\pi, \varpi)$ be the functions obtained in Lemma 4.2. Then we have

$$0 < \overline{c} \leq c_1 \leq \sup_{t \in [0, 1]} \mathcal{F}_\lambda(t_0 \pi \overline{t}_0 \varpi) \leq \frac{2}{N} K_0^{1 - \frac{q_2}{q_1 + q_2}} S_{\eta, \alpha, \beta}^q.$$

Combining Lemma 4.1 and the above inequality, we get a critical point $(u_1, v_1)$ of $\mathcal{F}_\lambda$ satisfying $(\mathcal{F}_\lambda^{K_0^0})$. Finally, by the symmetric criticality principle, we conclude that $(u_1, v_1)$ is a nontrivial $G$-symmetric solution of problem $(\mathcal{F}_\lambda^{K_0^0})$. \qed

**References**


[28] P.L. Lions, The concentration-compactness principle in the calculus of variations, the limit case, Rev. Mat. Iberoamerican 1 (1985) 45–121. (part 1); (part 2)


