SUB AND SUPERCRITICAL STOCHASTIC QUASI-GEOSTROPHIC EQUATION

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In this paper, we study the 2D stochastic quasi-geostrophic equation on $T^2$ for general parameter $\alpha \in (0, 1)$ and multiplicative noise. We prove the existence of weak solutions and Markov selections for multiplicative noise for all $\alpha \in (0, 1)$. In the subcritical case $\alpha > 1/2$, we prove existence and uniqueness of (probabilistically) strong solutions. Moreover, we prove ergodicity for the solution of the stochastic quasi-geostrophic equations in the subcritical case driven by possibly degenerate noise. The law of large numbers for the solution of the stochastic quasi-geostrophic equations in the subcritical case is also established. In the case of nondegenerate noise and $\alpha > 2/3$ in addition exponential ergodicity is proved.

1. Introduction. Consider the following two-dimensional (2D) stochastic quasi-geostrophic equation in the periodic domain $T^2 = \mathbb{R}^2/(2\pi \mathbb{Z})^2$:

$$
\frac{\partial \theta(t, \xi)}{\partial t} = -u(t, \xi) \cdot \nabla \theta(t, \xi) - \kappa (-\Delta)^\alpha \theta(t, \xi) + (G(\theta)\eta)(t, \xi),
$$

with initial condition

$$
\theta(0, \xi) = \theta_0(\xi),
$$

where $\theta(t, \xi)$ is a real-valued function of $\xi \in T^2$ and $t \geq 0$, $0 < \alpha < 1$, $\kappa > 0$ are real numbers. $u$ is determined by $\theta$ via the following relation:

$$
u = (u_1, u_2) = (-R_2 \theta, R_1 \theta) = R_1^\perp \theta.
$$

Here, $R_j$ is the $j$th periodic Riesz transform and $\eta(t, \xi)$ is a Gaussian random field, white noise in time, subject to the restrictions imposed below. The case $\alpha = \frac{1}{2}$ is called the critical case, the case $\alpha > \frac{1}{2}$ subcritical and the case $\alpha < \frac{1}{2}$ supercritical.

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In the deterministic case ($G \equiv 0$), such equations are important models in geophysical fluid dynamics. Indeed, they are special cases of general quasigeostrophic approximations for atmospheric and oceanic fluid flows with small Rossby and Ekman numbers. These models arise under the assumptions of fast rotation, uniform stratification and uniform potential vorticity. The case $\alpha = 1/2$ exhibits similar features (singularities) as the 3D Navier–Stokes equations and can therefore serve as a model case for the latter. For more details about the geophysical background, see, for instance, [7, 46]. In the deterministic case, this equation has been intensively investigated because of both its mathematical importance and its background in geophysical fluid dynamics (see, e.g., [6, 8, 27, 28, 30, 48] and the references therein). In the deterministic case, the global existence of weak solutions has been obtained in [48] and one most remarkable result in [6] gives the existence of a classical solution for $\alpha = 1/2$. In [30], another very important result is proved, namely that solutions for $\alpha = 1/2$ with periodic $C^\infty$ data remain $C^\infty$ for all times.

There is another model considering a simplified geophysical fluid model at asymptotically high rotation rate or with small Rossby number. This geophysical model with random perturbation has been studied in [2, 26] and the references therein. The equation is of a different type compared with our equation.

In this paper, we study the 2D stochastic quasi-geostrophic equation on the torus $\mathbb{T}^2$ for general parameter $\alpha \in (0, 1)$ and for both additive as well as multiplicative noise. Here, since the dissipation term is not strong enough to control the nonlinear term, we have to work in $L^p$ and to prove appropriate $L^p$-norm estimates. This leads to considerable complications in comparison to the stochastic Navier–Stokes equation, for example, when one wants to prove $L^p$-norm estimates for the weak solutions (see Theorem 3.3), which are essential to obtain pathwise uniqueness, and the improved positivity lemma to obtain uniform $L^p$-norm estimates (see Lemma 5.5 and Proposition 5.6) which will be used to prove ergodicity.

Main results for general $\alpha \in (0, 1)$: We prove the existence of weak solutions for multiplicative noise (Theorem 3.3). In order to prove the existence of (probabilistically strong) solutions and ergodicity in subsequent sections, we need $L^p$ norm estimates for the solutions, which are obtained using the $L^p$-Itô formula proved in [33]. But these $L^p$-norm estimates we cannot prove by Galerkin approximation; instead, we use another approximation which can be seen as a piecewise linear equation on small subintervals [see (3.4)]. To piece together martingale solutions on each subinterval and to get the existence of a martingale solution for the approximation, we first use the measurable selection theorem to find a martingale solution measurable with respect to the initial condition and apply a classical theorem from [52] (see Theorem 3.2). Using an abstract result for obtaining Markov selections from [24], we prove the existence of an a.s. Markov family in Appendix C (Theorem C.5).

Main results for the subcritical case $\alpha > 1/2$: We obtain pathwise uniqueness in a larger space by using $L^p$-norm estimates (Theorem 4.2) and, therefore, get a
(probabilistically strong) solution (Theorem 4.3) by the Yamada–Watanabe theorem. In particular, it follows that the laws of the solutions form a Markov process. Subsequently, in Section 5 we use a coupling method to study the long time behavior of the solution for the 2D stochastic quasi-geostrophic equation and we obtain ergodicity, that is, the existence (Theorem 5.12) and uniqueness (Theorem 5.9) of an invariant measure, for the solution to the 2D stochastic quasi-geostrophic equation (in case \( \alpha > 1/2 \)) driven by possibly degenerate noise. Furthermore, the Markov semigroup \( P_t \) converges to the unique invariant measure polynomially fast (Theorem 5.13). Finally, we prove that a law of large numbers holds in our case, that is, the times averages \( \frac{1}{T} \int_0^T \psi(\theta_t) \, dt \) converge to a constant in probability if \( \psi : H^1 \to \mathbb{R} \) is smooth (Theorem 5.14).

We add a detailed discussion on our approach to ergodicity via coupling, in particular, on its justification and on its relation to other approaches in Remark 5.10 below. In this paper, we are inspired by [44] to construct an intermediate process \( \tilde{\theta} \) such that \( \theta - \tilde{\theta} \) has a strong dissipation term and \( \| \theta(t) - \tilde{\theta}(t) \|_{H^{-1/2}} \to 0 \) as \( t \to \infty \). Using this intermediate process, we can prove \( E \| \theta_1(t, \theta_{10}^i, \theta_{20}^j) - \theta_2(t, \theta_{10}^i, \theta_{20}^j) \|_{H^{-1/2}} \) converges to zero polynomially fast when time goes to infinity, where \( (\theta_1(t, \theta_{10}^i, \theta_{20}^j), \theta_2(t, \theta_{10}^i, \theta_{20}^j)) \) denotes a coupling of two solutions to (3.1) starting from two different initial values \( \theta_{0i} \in H^1, i = 1, 2 \). Then we can deduce the uniqueness of invariant measures (Theorem 5.9). Also by a suitable choice of the metrics the asymptotically strong Feller property of the semigroup associated with the solution to the 2D stochastic quasi-geostrophic equation is also established (Remark 5.10). Here, we want to emphasize that although we consider the semigroup in \( H^1 \), the convergence is in \( H^{-1/2} \) norm. Moreover, we obtain the existence of the invariant measure, which lives on \( H^1 \), by using the uniform \( L^p \)-estimates (Theorem 5.12), which require the improved positivity lemma (Lemma 5.5). Thus, we obtain ergodicity for the solution of the quasi-geostrophic equation in the subcritical case (Theorem 5.13).

Additional results in the subcritical case \( \alpha > 2/3 \): In Section 6, we prove the exponential convergence of the solution under a stronger condition on the noise and on \( \alpha \). In order to prove the exponential convergence (Theorem 6.13), we first show the strong Feller property of the associated semigroup (Theorem 6.3), which follows from employing the weak-strong uniqueness principle in [22] (Theorem 6.4) and the Bismut–Elworthy–Li formula. As the dynamics only exist in the (analytically) weak sense and standard tools of stochastic analysis are not available, the computations are made for an approximating cutoff dynamics, which are equal to the original dynamics on a small random time interval. Since in our case \( \alpha < 1 \), it is more difficult to use the \( H^\alpha \)-norm to control the nonlinear term even though the equation is on \( T^2 \). To prove the weak-strong uniqueness principle, we need some regularity for the trajectories of the noise. Therefore, we need conditions on \( G \) so that it is enough regularizing. However, in order to apply the Bismut–Elworthy–Li
formula, we also need $G^{-1}$ to be regularizing enough. As a result, $\alpha > 2/3$ is required (see Remark 6.2 below for details). It seems difficult to use the Kolmogorov equation method as in [11, 16] (see Remark 6.2 below).

This paper is organized as follows. In Section 2, we introduce some notation as preparation. In Section 3, we prove the existence of weak solutions for general parameter $\alpha \in (0, 1)$ and multiplicative noise. In Section 4, we prove pathwise uniqueness for all $\alpha \in (\frac{1}{2}, 1)$. Furthermore, we get the existence and uniqueness of (probabilistically strong) solutions for multiplicative noise in the subcritical case. Moreover, we prove the Markov property for this unique solution. In Section 5, we use the coupling method to prove the uniqueness of an invariant measure in the subcritical case. Moreover, we obtain that the semigroup $P_t$ converges to the invariant measure polynomially fast. The law of large numbers for the solution to the 2D stochastic quasi-geostrophic equation is also established in this section. In Section 6, for $\alpha > 2/3$, and provided the noise is nondegenerate, we prove the exponential convergence to the (unique) invariant measure. Appendix A is devoted to a measurability problem (see Theorem A.4) which arises in implementing the coupling method in Section 5. In Appendix B, we prove existence of measurable selections for the solutions to the martingale problem in Section 3, and finally Appendix C is devoted to the existence of the corresponding Markov selection.

2. Notations and preliminaries. In the following, we will restrict ourselves to flows which have zero average on the torus $\mathbb{T}^2$, that is,

$$\int_{\mathbb{T}^2} \theta \, d\xi = 0,$$

where $d\xi$ denotes the volume measure on $\mathbb{T}^2$. Thus, (1.3) can be restated as

$$u = \left(-\frac{\partial \psi}{\partial \xi_2}, \frac{\partial \psi}{\partial \xi_1}\right) \quad \text{and} \quad (-\Delta)^{1/2} \psi = -\theta.$$

Set $H = \{f \in L^2(\mathbb{T}^2) : \int_{\mathbb{T}^2} f \, d\xi = 0\}$ and let $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the norm and inner product in $H$, respectively. $L^p(\mathbb{T}^2), p \in (0, \infty]$ denote the standard $L^p$ spaces on $\mathbb{T}^2$ with norm $|\cdot|_{L^p}$. On the periodic domain $\mathbb{T}^2$, $\{\sin(k, \cdot)_{\mathbb{R}^2} \mid k \in \mathbb{Z}_+^2\} \cup \{\cos(k, \cdot)_{\mathbb{R}^2} \mid k \in \mathbb{Z}_-^2\}$ form an eigenbasis of $-\Delta$ (we denote it by $\{e_k\}$). Here, $\mathbb{Z}_+^2 = \{(k_1, k_2) \in \mathbb{Z}^2 \mid k_2 > 0\} \cup \{(k_1, 0) \in \mathbb{Z}^2 \mid k_1 > 0\}$, $\mathbb{Z}_-^2 = \{(k_1, k_2) \in \mathbb{Z}^2 \mid -k \in \mathbb{Z}_+^2\}$, $\xi \in \mathbb{T}^2$, and the corresponding eigenvalues are $|k|^2$. For $s > 0$, define

$$\|f\|_{H^s} = \sum_k |k|^{2s} \langle f, e_k \rangle^2,$$

and let $H^s$ denote the Sobolev space of all $f \in H$ for which $\|f\|_{H^s}$ is finite. For $s < 0$, define $H^s$ to be the dual of $H^{-s}$. Set $\Lambda = (-\Delta)^{1/2}$. Then

$$\|f\|_{H^s} = |f|_{\Lambda^s}.$$
STOCHASTIC QUASI-GEOSTROPHIC EQUATION

For \( s \geq 0, p \in [1, +\infty] \) we use \( H^{s,p} \) to denote a subspace of \( L^p(\mathbb{T}^2) \), consisting of all \( f \) which can be written in the form \( f = \Lambda^{-s} g, g \in L^p(\mathbb{T}^2) \) and the \( H^{s,p} \) norm of \( f \) is defined to be the \( L^p \) norm of \( g \), that is, \( \| f \|_{H^{s,p}} := \| \Lambda^s f \|_{L^p} \).

By the singular integral theory of Calderón and Zygmund (cf. [51], Chapter 3), for any \( s \geq 0, p \in (1, \infty) \), there is a constant \( C_R = C_R(s, p) \), such that

\[
\| \Lambda^s u \|_{L^p} \leq C_R(s, p) \| \Lambda^s \theta \|_{L^p}.
\] (2.1)

Fix \( \alpha \in (0, 1) \) and define the linear operator \( A_\alpha : D(A_\alpha) \subset H^2(\mathbb{T}^2) \rightarrow H \) as

\[
A_\alpha u := \kappa(\Delta u). 
\]

The operator \( A_\alpha \) is positive definite and self-adjoint with the same eigenbasis as that of \( -\Delta \) mentioned above. Denote the eigenvalues of \( A_\alpha \) by \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \), and renumber the above eigenbasis correspondingly as \( e_1, e_2, \ldots \).

First, we recall the following important product estimates (cf. [48], Lemma A.4):

**LEMMA 2.1.** Suppose that \( s > 0 \) and \( p \in (1, \infty) \). If \( f, g \in C^\infty(\mathbb{T}^2) \) then

\[
\| \Lambda^s(fg) \|_{L^p} \leq C(\| f \|_{L^{p_1}} \| \Lambda^s g \|_{L^{p_2}} + \| g \|_{L^{p_3}} \| \Lambda^s f \|_{L^{p_4}}),
\] (2.2)

with \( p_i \in (1, \infty], i = 1, \ldots, 4 \) such that

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]

We shall use as well the following standard Sobolev inequality (cf. [51], Chapter V):

**LEMMA 2.2.** Suppose that \( q > 1, p \in [q, \infty) \) and

\[
\frac{1}{p} + \frac{\sigma}{2} = \frac{1}{q}.
\]

Suppose that \( \Lambda^\sigma f \in L^q \), then \( f \in L^p \) and there is a constant \( C_S \geq 0 \) independent of \( f \) such that

\[
\| f \|_{L^p} \leq C_S \| \Lambda^\sigma f \|_{L^q}.
\]

The following commutator estimate from [27], Lemma 3.1, is very important for later use.

**LEMMA 2.3 (Commutator estimates).** Suppose that \( s > 0 \) and \( p \in (1, \infty) \). If \( f, g \in C^\infty(\mathbb{T}^2) \), then

\[
\| \Lambda^s(fg) - f \Lambda^s g \|_{L^p} \leq C(\| \nabla f \|_{L^{p_1}} \| \Lambda^{s-1} g \|_{L^{p_2}} + \| g \|_{L^{p_3}} \| \Lambda^s f \|_{L^{p_4}}),
\]

with \( p_i \in (1, \infty], i = 1, \ldots, 4 \) such that

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]
We will also use the following classical interpolation inequality (see, e.g., [10], (5.5)).

**Lemma 2.4.** For $f \in C^\infty(\mathbb{T}^2)$, we have

$$
\|f\|_{H^s} \leq C \|f\|_{H^{s_1}}^{(s_2-s)/(s_2-s_1)} \|f\|_{H^{s_2}}^{(s-s_1)/(s_2-s_1)}, \quad s_1 < s < s_2.
$$

3. **Weak solutions in the general case.** In this section, we consider the following abstract stochastic evolution equation in place of equations (1.1)–(1.3): 

$$
\begin{cases}
d\theta(t) + A_u \theta(t) \, dt + u(t) \cdot \nabla \theta(t) \, dt = G(\theta(t)) \, dW(t), \\
\theta(0) = \theta_0 \in H,
\end{cases}
$$

where $u$ satisfies (1.3) and $W(t)$, $t \in [0, T]$, is a cylindrical Wiener process in a separable Hilbert space $U$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$. Here, $G$ is a measurable mapping from $H^u$ to $L_2(U, H)$ (all Hilbert–Schmit operators from $U$ to $H$). Let $f_n, n \in \mathbb{N}$, be an ONB of $U$.

In the following, we assume the following conditions on $G$:

**Hypothesis G.1.** (i) $\|G(\theta)\|_{L_2(U, H)}^2 \leq \lambda_0 |\theta|^2 + \rho_1 |A^u \theta|^2 + \rho_2, \theta \in H^u$, for some positive real numbers $\lambda_0, \rho_2$ and $\rho_1 < 2\kappa$. Moreover, for some $\beta > 3, \|G(\theta)\|_{L_2(U, H^{-\beta})}^2 \leq \rho_3 |\theta|^2 + 1$, $\theta \in H^u$, for some positive real numbers $\rho_3$.

(ii) If $\theta, \theta_n \in H^u$ such that $\theta_n \rightarrow \theta$ in $H$, then $\lim_{n \to \infty} \|G(\theta_n)^* - G(\theta)^*(v)\|_U = 0$ for all $v \in C^\infty(\mathbb{T}^2)$, where the asterisk denotes the adjoint operator of $G(\theta)$.

First, we introduce the following definition of a weak solution.

**Definition 3.1.** We say that there exists a weak solution of equation (3.1) if there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$, a cylindrical Wiener process $W$ on the space $U$ and a progressively measurable process $\theta: [0, T] \times \Omega \rightarrow H$, such that for $P$-a.e. $\omega \in \Omega$,

$$
\theta(\cdot, \omega) \in L^\infty([0, T]; H) \cap L^2([0, T]; H^u) \cap C([0, T]; H^{-\beta}),
$$

where $\beta$ in Hypothesis G.1, and such that $P$-a.s.

$$
\begin{align*}
\langle \theta(t), \phi \rangle + \int_0^t \langle A_u^{1/2} \theta(s), A_u^{1/2} \phi \rangle \, ds - \int_0^t \langle u(s) \cdot \nabla \phi, \theta(s) \rangle \, ds &= \langle \theta_0, \phi \rangle + \int_0^t G(\theta(s)) \, dW(s), \phi \\

&= \langle \theta(0), \phi \rangle + \left\{ \int_0^t G(\theta(s)) \, dW(s), \phi \right\}
\end{align*}
$$

for $t \in [0, T]$ and all $\phi \in C^1(\mathbb{T}^2)$. 

REMARK. (i) Note that, because $\text{div } u = 0$ for smooth functions $\theta$ and $\psi$, we have

$$\langle u(s) \cdot \nabla \theta(s), \psi \rangle = -\langle u(s) \cdot \nabla \psi, \theta(s) \rangle.$$ 

Thus, the integral equation in Definition 3.1 corresponds to equation (3.1).

(ii) Note that since the solution $\theta \in L^2(0, T; H^a)$ we only need $\theta, \theta_n \in H^a$ instead of $\theta, \theta_n \in H$ in Hypothesis G.1(ii).

(iii) A typical example satisfying Hypothesis G.1 is the following: For $y \in U$,

$$G(\theta)y = \sum_{k=1}^{\infty} (c_k \Lambda^a \theta + b_k g(\theta)) \langle y, f_k \rangle_U, \quad \theta \in H^a,$$

where $g$ is continuous function on $\mathbb{R}$ of at most linear growth and $b_k, c_k \in C^\infty(T^2)$ satisfy $\sum_k c_k^2(\xi) < 2\kappa, \sum_k b_k^2(\xi) \leq M, \xi \in T^2$, and $\sum_k |\Lambda^a c_k|^2 \leq M$.

It is standard to show that under Hypothesis G.1 there exists a weak solution to (3.1) by using the Galerkin approximation. However, as mentioned in the Introduction, we also need $L^p$ norm estimates for the solutions, more precise that they belong to $L^p(\Omega_0; L^\infty([0, T]; L^p(T^2)))$, provided so do their initial values. This will be essential to the proof of pathwise uniqueness. For this, we have to use another approximation instead of the Galerkin approximation and the following theorem from [52], Theorem 6.1.2.

Let $\Omega_0 := C([0, \infty), H^1), \Omega'_0 := C([t, \infty), H^1)$ for $t > 0$ and $\mathcal{P}(\Omega_0)$ denote the set of all probability measures on $(\Omega_0, \mathcal{B})$ with $\mathcal{B}$ being the Borel $\sigma$-algebra coming from the topology of locally uniform convergence on $\Omega_0$. Define the canonical process $x : \Omega_0 \rightarrow H^1$ as

$$x_t(\omega) = \omega(t).$$

Also define the $\sigma$-algebra $\mathcal{B}_t := \sigma\{x(s), s \leq t\}$ and $\mathcal{B}' := \sigma\{x(s), s \geq t\}$.

THEOREM 3.2. Fix $t > 0$. Let $x \mapsto Q_x$ be a mapping from $\Omega_0$ to $\mathcal{P}(\Omega'_0)$ such that for any $A \in \mathcal{B}'$, $x \mapsto Q_x(A)$ is $\mathcal{B}_t$-measurable, and for any $x \in \Omega_0$

$$Q_x(y \in \Omega'_0 : y(t) = x(t)) = 1.$$

Then for any $P \in \mathcal{P}(\Omega_0)$, there exists a unique $P \otimes_t Q \in \mathcal{P}(\Omega_0)$ such that

$$(P \otimes_t Q)(A) = P(A), \quad \forall A \in \mathcal{B}_t,$$

and for $P \otimes_t Q$-almost all $x \in \Omega_0$

$$Q_x = (P \otimes_t Q)(\cdot | \mathcal{B}_t)(x).$$

Now we will prove the existence of a martingale solution under Hypothesis G.1.
THEOREM 3.3. Let $\alpha \in (0, 1)$. If $G$ satisfies Hypothesis G.1, then there exists a weak solution $(\Omega, F, \{F_t\}, P, W, \theta)$ to (3.1). Moreover, assume that $G$ satisfies the following condition:

(Gp.1) There exists some $p \in (2, \infty)$ such that for all $\theta \in H^\alpha \cap L^p(\mathbb{T}^2)$,

$$\int \left( \sum_j |G(\theta)(f_j)|^2 \right)^{p/2} d\xi \leq C \left( \int |\theta|^p d\xi + 1 \right), \quad \forall t > 0$$

for some constant $C := C(p) > 0$ and $\theta_0 \in L^p(\mathbb{T}^2)$. Then

$$E \sup_{t \in [0, T]} \|\theta(t)\|^p_{L^p} < \infty.$$ 

REMARK 3.4. Typical examples for $G$ satisfying (Gp.1) have the following form: for $\theta \in H^\alpha$

$$G(\theta)y = \sum_{k=1}^{\infty} b_k(y, f_k)g(\theta), \quad y \in U,$

where $g$ is a continuous function on $\mathbb{R}$ of at most linear growth and $b_k$ are $C^\infty$ functions on $\mathbb{T}^2$ satisfying $\sum_{k=1}^{\infty} b_k^2(\xi) \leq M$.

PROOF OF THEOREM 3.3. Step 1: We first establish the existence of martingale solutions of the following equation:

$$d\theta(t) + A_\alpha \theta(t) dt + w(t) \cdot \nabla \theta(t) dt = k_\delta \ast G(\theta) dW(t),$$

$$\theta(0) = \theta_0 \in H^3,$$

with a given smooth function $w(t)$ which satisfies $\text{div} w(t) = 0$ for all $t \in [0, T]$ and

$$\sup_{t \in [0, T]} \|w(t)\|_{C^3(\mathbb{T}^2)} \leq C.$$

Here, $k_\delta \ast G(\theta)$ means for $y \in U$, $k_\delta \ast G(\theta)(y) := k_\delta \ast (G(\theta)(y))$, where $k_\delta$ is the periodic Poisson kernel in $\mathbb{T}^2$ given by $k_\delta(\xi) = e^{-\delta |\xi|}$, $\xi \in \mathbb{Z}^2$. By [24], Theorem 4.7, this equation has a martingale solution $P \in \mathcal{P}(C([0, \infty); H^1))$ with initial value $\theta_0$ in the following sense:

(M1) $P(x(0) = \theta_0) = 1$ and for any $n \in \mathbb{N}$

$$P \left\{ x \in C([0, \infty); H^1) : \int_0^n \|A^{2\alpha} x(s) + w(s) \cdot \nabla x(s)\|_{H^1} ds + \int_0^n \|k_\delta \ast G(x(s))\|^2_{L^2(U; H^3)} ds < +\infty \right\} = 1.$$
STOCHASTIC QUASI-GEOSTROPHIC EQUATION

(M2) For every $e_i$, the process
\[ \langle x(t), e_i \rangle - \int_0^t \langle -w(s) \cdot \nabla x(s) - A_\alpha x(s), e_i \rangle ds \]
is a continuous square-integrable $\mathcal{B}_t$-martingale under $P$, whose quadratic variation process is given by
\[ \int_0^t \| (k_\delta * G)^*(x(s))(e_i) \|^2_U ds, \]
where the asterisk denotes the adjoint operator of $k_\delta * G(x(s))$.

(M3) For any $q \in \mathbb{N}$ there exists a continuous positive real function $t \mapsto C_{t, q}$ such that
\[ \mathbb{E}^P \left( \sup_{r \in [0, t]} |\Lambda^3 x(r)|^{2q} + \int_0^t \| \Lambda^3 x(r) \|^{2q-2} |\Lambda^{\alpha+3} x(r)|^2 dr \right) \leq C_{t, q} (|\Lambda^3 \theta_0|^{2q} + 1), \]
where $\mathbb{E}^P$ denotes the expectation under $P$.

Indeed, we only need to check conditions (C1)–(C3) in [24]. The demi-continuity condition (C1) is obvious by Hypothesis G.1(ii) and the linearity of the equation. For (C2), we have that for $x \in H^4$
\[ \langle -w \cdot \nabla x - A_\alpha x, x \rangle_{H^3} \leq -\kappa |\Lambda^{3+\alpha} x|^2 + |\Lambda^{3}(w \cdot \nabla x), \Lambda^3 x|. \]
By Lemma 2.3 and because $\langle w \cdot \nabla \Lambda^3 x, \Lambda^3 x \rangle = 0$ for $x \in H^4$, we have that for $x \in H^4$
\[ \|\Lambda^3(w \cdot \nabla x), \Lambda^3 x\| = \|\Lambda^3(w \cdot \nabla x) - w \cdot \nabla \Lambda^3 x, \Lambda^3 x\| \leq \|w\|_{C^1(\mathbb{T}^2)} \|\Lambda^3 x\| |\Lambda^{3+\alpha} x|. \]
Thus, the coercivity condition (C2) follows from the above two inequalities and Young’s inequality. Also by Hypothesis G.1, we have for $x \in H^4$
\[ \|k_\delta * G(x)\|^2_{L^2(U, H^3)} \leq C(\delta) \|G(x)\|^2_{L^2(U, H)} \leq C \|\Lambda^\alpha x\|^2 + 1, \]
and by Lemma 2.1
\[ \|w \cdot \nabla x + A_\alpha x\|^2_{H^1} \leq 2\|A_\alpha x\|^2_{H^1} + C \|w\|^2_{C^1(\mathbb{T}^2)} |\Lambda^3 x|^2 \leq C |\Lambda^3 x|^2, \]
which implies the growth condition (C3).

Step 2: Now we construct an approximation of (3.1).
We pick a smooth $\phi \geq 0$, with $\text{supp} \phi \subset [1, 2]$, $\int_0^\infty \phi = 1$, and for $\delta > 0$ let
\[ U_\delta[\theta](t) := \int_0^\infty \phi(\tau)(k_\delta * R^3 \theta)(t - \delta \tau) d\tau, \]
where $k_3$ is the periodic Poisson kernel in $\mathbb{T}^2$ given by $\hat{k}_3(\xi) = e^{-|\xi|}$, $\xi \in \mathbb{Z}^2$, and we set $\theta(t) = 0$, $t < 0$. We take a sequence $\delta_n \to 0$ and consider the equation:

\[(3.4) \quad d\theta_n(t) + A_\alpha \theta_n(t) \, dt + u_n(t) \cdot \nabla \theta_n(t) \, dt = k_3 \ast G(\theta_n) \, dW(t),\]

with initial data $\theta_n(0) = k_3 \ast \theta_0$ and $u_n = U_{\delta_n}[\theta_n]$. For a fixed $n$, this is a linear equation in $\theta_n$ on each subinterval $[t_k^n, t_{k+1}^n]$ with $t_k^n = k \delta_n$, since $u_n$ is determined by the values of $\theta_n$ on the two previous subintervals. By Step 1, we obtain the existence of a martingale solution to (3.4) for fixed $n$. Indeed, we obtain the martingale solution $P_{t_0}^n \in \mathcal{P}(C([0, \infty), H^1))$ with initial condition $k_3 \ast \theta_0$ on the subinterval $[0, t_0]$ by Step 1. Also, by Step 1, we get that for $x_0 \in B_0$ with $B_0 := \{x \in \Omega_0 : \sup_{0 \leq t \leq t_1} \|x(t)\|_{H^3} < \infty\}$, there exists a $Q_{x_0} \in \mathcal{P}(C([t_0^n, t_1^n], H^1))$ satisfying the following:

(M1) $Q_{x_0}(x(t_1^n) = x_0(t_1^n)) = 1$

\[
Q_{x_0}\left\{ x \in C([t_0^n, t_1^n]; H^1) : \int_{t_0^n}^{t_1^n} \| \Lambda^{2\alpha} x(s) + U_{\delta_n}[x_0](s) \cdot \nabla x(s) \|_{H^1} \, ds + \int_{t_0^n}^{t_1^n} \| k_3 \ast G(x(s)) \|_{L_2(U; H^3)} \, ds < +\infty \right\} = 1.
\]

(M2) For every $e_i, i \in \mathbb{N}$, the process

\[
M_i(t \wedge t_2^n, x) := \{x(t \wedge t_2^n), e_i\} - \langle x_0(t_1^n), e_i \rangle
\]

\[
- \int_{t_0^n}^{t \wedge t_2^n} [-U_{\delta_n}[x_0](s) \cdot \nabla x(s) - A_\alpha x, e_i] \, ds, \quad t \geq t_1^n
\]

is a continuous square-integrable $B_r$-martingale under $Q_{x_0}$, whose quadratic variation process is given by

\[
\langle M_i \rangle(t \wedge t_2^n, x) := \int_{t_0^n}^{t \wedge t_2^n} \| (k_3 \ast G)(x(s))(e_i) \|_{U_2}^2 \, ds,
\]

where the asterisk denotes the adjoint operator of $k_3 \ast G(x(s))$.

(M3) For any $q \in \mathbb{N}$, there exists a constant $C_q$ depending on $\sup_{t \in [t_1^n, t_2^n]} \|x_0(t)\|_{H^1}$ such that

\[
\mathbb{E}^{Q_{x_0}} \left( \sup_{r \in [t_1^n, t_2^n]} |A^3 x(r)|^{2q} + \int_{t_1^n}^{t_2^n} |A^3 x(r)|^{2q-2} |A^3 x(r)|^2 \, dr \right) \leq C_q \left( |A^3 x_0(t_1^n)|^{2q} + 1 \right).
\]

Now we extend $Q_{x_0}$ to a probability measure on $C([t_0^n, +\infty), H^1)$ by $Q_{x_0} \circ \psi^{-1}$ with $\psi : C([t_0^n, t_1^n], H^1) \to C([t_0^n, +\infty), H^1)$ by $\psi(x(s)) := x(s \wedge t_2^n)$, $s \in [t_1^n, +\infty)$. The set of all such martingale solutions is denoted by $Q_{x_0}$. Now we can find $Q_{x_0} \in Q_{x_0}$ satisfying (M1)–(M3) such that the map $x_0 \mapsto Q_{x_0}$ from $B_0$
to $\mathcal{P}(\Omega_1^1)$ is measurable with respect to $\mathcal{B}_1^1$. This will be proved in Lemma B.1 in Appendix B.

For $x_0 \in B_0^1$, define $Q_{x_0} := \delta_{x_0}$. Thus, by Theorem 3.2 we get that there exists $P_n^1 \otimes \iota_1^1 Q \in \mathcal{P}(C([0, \infty), H^1))$ such that

$$ (P_n^1 \otimes \iota_1^1 Q)(A) = P_n^1(A), \quad \forall A \in \mathcal{B}_1^1,$$

and for $P_n^1 \otimes \iota_1^1 Q$-almost all $x \in \Omega_0$

$$Q_x = (P_n^1 \otimes \iota_1^1 Q)(\cdot | B_1^1)(x).$$

Here, $Q_{x_0}$ extends to a probability measure on $C([0, \infty), H^1)$ by the following:

Let $\delta_{x_0}$ be the point-mass on $C([0, t_1^1], H^1)$ at $x_0 \in [0, t_1^1]$, that is,

$$\delta_{x_0}(x \in C([0, t_1^1], H^1) : x(t) = x_0(t), 0 \leq t \leq t_1^1) = 1.$$  

Define $\tilde{Q} := \delta_{x_0} \times Q_{x_0}$ on $\tilde{X} := C([0, t_1^1], H^1) \times C([t_1^1, \infty), H^1)$ and set $X := \{(x_1, x_2) \in C([0, t_1^1], H^1) \times C([t_1^1, \infty), H^1) : x_1(t_1^1) = x_2(t_1^1)\}$. Then $X$ is a measurable subset of $\tilde{X}$ and $\tilde{Q}(X) = 1$. Then $\tilde{Q}$ can be restricted to $X$. Finally, $\Psi : X \rightarrow C([0, \infty), H^1)$ defined by $\Psi((x_1, x_2))(t) := x_1(t)$, if $0 \leq t \leq t_1^1$, $\Psi((x_1, x_2))(t) := x_2(t)$, if $t > t_1^1$, is a measurable map from $X$ onto $C([0, \infty), H^1)$. Then $\tilde{Q}|_X \circ \Psi^{-1}$ is the desired measure, which still be denoted $Q_{x_0}$.

By (M2), we have for every $e_i, i \in \mathbb{N}$, that the process

$$M_i(t \wedge t_2^1, x) = \langle x(t \wedge t_2^1), e_i \rangle - \langle x_0(t_1^1), e_i \rangle$$

$$- \int_{t_1^1}^{t \wedge t_2^1} \langle -U_{\alpha x}(x)(s) \cdot \nabla x(s) - A_{\alpha x}, e_i \rangle ds$$

$$= \langle x(t \wedge t_2^1), e_i \rangle - \langle x_0(t_1^1), e_i \rangle$$

$$- \int_{t_1^1}^{t \wedge t_2^1} \langle -U_{\alpha x}(x)(s) \cdot \nabla x(s) - A_{\alpha x}, e_i \rangle ds$$

is a continuous square-integrable $\mathcal{B}_t$-martingale under $Q_{x_0}$. Thus, by [52], Theorem 1.2.10, we obtain for every $e_i, i \in \mathbb{N}$, that the process

$$\langle x(t \wedge t_2^1), e_i \rangle - \int_{0}^{t \wedge t_2^1} \langle -U_{\alpha x}(x)(s) \cdot \nabla x(s) - A_{\alpha x}, e_i \rangle ds$$

is a continuous square-integrable $\mathcal{B}_t$-martingale under $P_n^1 \otimes \iota_1^1 Q$, whose quadratic variation process is given by

$$\int_{0}^{t \wedge t_2^1} \| (k_{\alpha x} \ast G)^*(x(s))(e_i) \|_{L^2}^2 ds.$$
Thus, we construct a martingale solution \( P_n^1 \otimes Q \in \mathcal{P}(C([0, \infty), H^1)) \) of (3.4) on \([0, t_n^2]\). Then step by step we can construct a martingale solution \( P_n \in \mathcal{P}(C([0, \infty), H^1)) \) of (3.4) on \([0, T]\) for any given \( T \) in the following sense:

\[
(M1') \quad P_n(x(0) = k_{\delta_n} \ast \theta_0) = 1 \quad \text{and} \quad P_n \left\{ x \in C([0, +\infty); H^1) : \int_0^T \left\| \Lambda^2 x(s) + U_{\delta_n} [x](s) \cdot \nabla x(s) \right\|_{H^1} ds \right. \\
+ \left. \int_0^T \left\| k_{\delta_n} \ast G(x(s)) \right\|_{L^2(U; H^2)}^2 ds < +\infty \right\} = 1.
\]

\[
(M2') \quad \text{For every } e_i, \text{ the process} \\
\left\{ x(t \wedge T), e_i \right\} - \int_0^{t \wedge T} \left\{ -U_{\delta_n} [x](s) \cdot \nabla x(s) - A_{\alpha} x, e_i \right\} ds
\]

is a continuous square-integrable \( B_t \)-martingale under \( P_n \), whose quadratic variation process is given by

\[
\int_0^{t \wedge T} \left\| (k_{\delta_n} \ast G)(x(s))(e_i) \right\|_{U}^2 ds,
\]

where the asterisk denotes the adjoint operator of \( k_{\delta_n} \ast G(x(s)) \).

\[
(M3') \quad P_n(L^\infty_{\text{loc}}([0, +\infty), H^3) \cap \Omega_0) = 1.
\]

Then by the martingale representation theorem (cf. [45], Theorem 2, [12], Theorem 8.2) we can find a new probability space \((\Omega^n, P^n, W_n)\) and \( \theta_n \) such that \( (\theta_n, W_n) \) is a weak solution of (3.4) and \( \theta_n \) has the same law as \( P_n \).

Step 3: Now we show that \( \theta_n \) converge to the solution of (3.1). Since we have

\[
\langle u_n(t) \cdot \nabla \theta_n(t), \theta_n(t) \rangle = 0,
\]

by Itô’s formula we have

\[
d|\theta_n|^p + p\kappa |\theta_n|^{p-2} |\Lambda^\alpha \theta_n|^2 \, dt \leq p|\theta_n|^{p-2}|k_{\delta_n} \ast G(\theta_n)| \, dW_n \quad \text{and} \\
+ \frac{p(p-1)}{2} |\theta_n|^{p-2} \left\| k_{\delta_n} \ast G(\theta_n) \right\|_{L^2(U; H)}^2 \, dt.
\]

By classical arguments, we easily show that there exist positive constants \( C_1, C_2 \) independent of \( n \), such that (cf. [20], Appendix 1) for \( 2 \leq p < 1 + \frac{2\kappa}{\rho_1} \) if \( \rho_1 > 0 \) and for \( 2 \leq p < \infty \) if \( \rho_1 = 0 \), the following are satisfied:

\[
(E) \quad \mathbb{E}^{P_n} \left( \sup_{0 \leq s \leq T} |\theta_n(s)|^p \right) \leq C_1
\]

and

\[
(E) \quad \mathbb{E}^{P_n} \int_0^T \|\theta_n(s)\|^2_{H^\alpha} \, ds \leq C_2.
\]
Now we prove that the family $D(\theta_n), n \in \mathbb{N}$, is tight in $C([0, T]; H^{-\beta})$, for all $\beta > 3$. Here, $D(\theta_n)$ means the law of $\theta_n$. By (3.5) for each $t \in [0, T]$, $D(\theta_n(t))$ is tight on $H^{-\beta}$. Then by Aldous’ criterion in [1], it suffices to check that for all stopping times $\tau_n \leq T$ and $\eta_n \to 0$,

$$\lim_{n \to \infty} E^n_\eta \|\theta_n(\tau_n + \eta_n) - \theta_n(\tau_n)\|_{H^{-\beta}} = 0.$$  (3.7)

We have $P^n$-a.s.

$$\theta_n(\tau_n + \eta_n) - \theta_n(\tau_n) = - \int_{\tau_n}^{\tau_n + \eta_n} A_\alpha \theta_n(s) \, ds - \int_{\tau_n}^{\tau_n + \eta_n} u_n(s) \cdot \nabla \theta_n(s) \, ds$$
$$+ \int_{\tau_n}^{\tau_n + \eta_n} k_{\delta_n} \ast G(\theta_n(s)) \, dW_n(s).$$

It is easy to obtain the following:

$$E^n_\eta \left\| \int_{\tau_n}^{\tau_n + \eta_n} A_\alpha \theta_n(s) \, ds \right\|_{H^{-\beta}} \leq C \eta_n E^n_\eta \sup_{t \in [0, T]} |\theta_n(t)|.$$  (3.8)

And since $H^2 \subset L^\infty$, we obtain that for $v \in H^3$,

$$|\langle u_n \cdot \nabla \theta_n, v \rangle| = |\langle u_n \cdot \nabla v, \theta_n \rangle| \leq |\theta_n| \|u_n\| \|\nabla v\|_{L^\infty} \leq |\theta_n| \|u_n\| \|v\|_{H^3}.$$  (3.9)

Since $\sup_{[0, t]} |u_n| \leq C \sup_{[0, t]} |\theta_n|$, we get that

$$E^n_\eta \left\| \int_{\tau_n}^{\tau_n + \eta_n} u_n(s) \cdot \nabla \theta_n(s) \, ds \right\|_{H^{-\beta}} \leq C \eta_n E^n_\eta \sup_{t \in [0, T]} |\theta_n(t)|^2.$$  (3.10)

In addition by Hypothesis G.1, we have

$$E^n_\eta \left\| \int_{\tau_n}^{\tau_n + \eta_n} k_{\delta_n} \ast G(\theta_n(s)) \, dW_n(s) \right\|_{H^{-\beta}}^2$$
$$\leq C E^n_\eta \int_{\tau_n}^{\tau_n + \eta_n} \|G(\theta_n(s))\|^2_{L^2(U, H^{-\beta})} \, ds$$
$$\leq C \eta_n \left( E^n_\eta \sup_{t \in [0, T]} |\theta_n(t)|^2 + 1 \right) \to 0 \quad \text{as } \eta_n \to 0.$$

Thus, (3.7) follows by (3.8), (3.9) and (3.10), which implies the tightness of $D(\theta_n)$ in $C([0, T], H^{-\beta})$. This yields that for each $\eta > 0$

$$\limsup_{\delta \to 0} n \sup_{|s-t| \leq \delta, s, t \leq T} |\theta_n(t) - \theta_n(s)|_{H^{-\beta}} > \eta = 0.$$  (3.11)

By this and (3.5), (3.6), it is easy to get that $D(\theta_n)$ is tight in $L^2([0, T]; H) \cap C([0, T], H^{-\beta})$ (cf. [41], Lemma 2.7). Therefore, we find a subsequence, still denoted by $\theta_n$, such that $D(\theta_n)$ converges weakly in

$$L^2([0, T]; H) \cap C([0, T], H^{-\beta}).$$
By Skorohod’s representation theorem, there exist a stochastic basis \((\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}, \tilde{\mathbb{P})}\) and, on this basis, \(L^2([0,T]; H) \cap C([0,T], H^{-\beta})\)-valued random variables \(\tilde{\theta}, \tilde{\theta}_n, n \geq 1\), such that \(\tilde{\theta}_n\) has the same law as \(\tilde{\theta}_n\) on \(L^2([0,T]; H) \cap C([0,T], H^{-\beta})\), and \(\tilde{\theta}_n \to \tilde{\theta}\) in \(L^2([0,T]; H) \cap C([0,T], H^{-\beta})\) \(\tilde{\mathbb{P}}\)-a.s. For \(\tilde{\theta}_n\) we also have (3.5) and (3.6). Hence, it follows that
\[
\tilde{\theta}(\cdot, \omega) \in L^2([0,T]; H^\alpha) \cap L^\infty([0,T]; H) \quad \text{for } \tilde{\mathbb{P}}\text{-a.e. } \omega \in \Omega.
\]
For each \(\tilde{\theta}_n\) we define \(\tilde{u}_n := U_{\tilde{b}_n}[\tilde{\theta}_n]\) and for each \(n \geq 1\) we define the process
\[
\tilde{M}_n(t) := \tilde{\theta}_n(t) - k_{\tilde{b}_n} \ast \theta_0 + \int_0^t A_0 \tilde{\theta}_n(s) \, ds + \int_0^t \tilde{u}_n(s) \cdot \nabla \tilde{\theta}_n(s) \, ds.
\]
In fact \(\tilde{M}_n\) is a square integrable martingale with respect to the filtration
\[
\{\mathcal{G}_n\} = \sigma\{\tilde{\theta}_n(s), s \leq t\}.
\]
For all \(r \leq t \in [0,T]\), all bounded continuous functions \(\phi\) on \(C([0,r]; H^{-\beta}) \cap L^2([0,r]; H)\), and all \(v \in C^\infty(\mathbb{T}^2)\), we have
\[
\tilde{E}(\langle \tilde{M}_n(r) - \tilde{M}_n(r), v \rangle) \phi(\tilde{\theta}_n|_{[0,r]})) = 0
\]
and
\[
\tilde{E}\left(\langle \tilde{M}_n(t), v \rangle^2 - \langle \tilde{M}_n(r), v \rangle^2 - \int_r^t \Vert k_{\tilde{b}_n} \ast G \ast (\tilde{\theta}_n) \vert_U \Vert^2 \, ds \vert_{U} \big) \phi(\tilde{\theta}_n|_{[0,r]}) \right) = 0.
\]
By the BDG inequality, we have for \(1 < p < 1 + \frac{\kappa}{\rho_1}\) if \(\rho_1 > 0\) and \(1 < p < \infty\) if \(\rho_1 = 0\), that
\[
\sup_n \tilde{E}(|\tilde{M}_n(t), v|^p) \leq C \sup_n \tilde{E}\left(\int_0^t \Vert k_{\tilde{b}_n} \ast G \ast (\tilde{\theta}_n) \vert_U \Vert^2 \, ds \vert_{U} \big)^p < \infty.
\]
Since \(\tilde{\theta}_n \to \tilde{\theta}\) in \(L^2(0,T; H) \cap C(0,T, H^{-\beta})\), we also have
\[
\lim_{n \to \infty} \tilde{E}(\langle \tilde{M}_n(t) - M(t), v \rangle) = 0
\]
and
\[
\lim_{n \to \infty} \tilde{E}(\langle \tilde{M}_n(t) - M(t), v \rangle^2) = 0,
\]
where
\[
M(t) := \tilde{\theta}(t) - \theta_0 + \int_0^t \tilde{u} \cdot \nabla \tilde{\theta} + A_0 \tilde{\theta} \, ds.
\]
Here, \(\tilde{u}\) is defined by (1.3) with \(\theta\) replaced by \(\tilde{\theta}\). Taking the limit, we obtain that for all \(r \leq t \in [0,T]\), all bounded continuous functions \(\phi\) on \(C([0,r]; H^{-\beta}) \cap L^2([0,r]; H)\), and \(v \in C^\infty(\mathbb{T}^2)\),
\[
\tilde{E}(\langle M(t) - M(r), v \rangle) \phi(\tilde{\theta}|_{[0,r]}) = 0
\]
and
\[ \hat{E}\left( \left( [M(t), v]^2 - [M(r), v]^2 - \int_r^t \| G(\theta)^* v \|^2_{L^2} ds \right) \phi(\theta)_{|t,r]} \right) = 0. \]

Thus, the existence of a weak solution for (3.1) follows by the martingale representation theorem (cf. [12], Theorem 8.2, [O05], Theorem 2).

**Step 4:** Now we prove the last statement. It is sufficient to prove that
\[ E^P \sup_{t \in [0,T]} \| \theta_n(t) \|^p_{L^p} \leq C, \]
where \( C \) is a constant independent of \( n \). We write for simplicity \( \theta(t) = \theta_n(t) \), \( u(t) = u_n(t) \), \( W(t) = W_n(t) \), \( P = P_n \).

By [33], Lemma 5.1, or [5], Theorem 2.4, we have
\[ \| \theta(t) \|^p_{L^p} = \| k_{\delta_n} \ast \theta_0 \|^p_{L^p} \]
\[ + \int_0^t \left[ -p \int_{\Omega^2} |\theta(s)|^{p-2} \theta(s) (\Lambda^{2\alpha} \theta(s) + u(s) \cdot \nabla \theta(s)) d\xi \right. \]
\[ + \frac{1}{2} p(p-1) \int_{\Omega^2} |\theta(s)|^{p-2} \left( \sum_j |k_{\delta_n} \ast G(\theta(s))(f_j)|^2 \right) d\xi \right] ds \]
\[ + p \int_0^t \int_{\Omega^2} |\theta(s)|^{p-2} \theta(s) k_{\delta_n} \ast G(\theta(s)) d\xi dW(s) \]
\[ \leq \| k_{\delta_n} \ast \theta_0 \|^p_{L^p} \]
\[ + \int_0^t \frac{1}{2} p(p-1) \int_{\Omega^2} |\theta(s)|^{p-2} \left( \sum_j |k_{\delta_n} \ast G(\theta(s))(f_j)|^2 \right) d\xi ds \]
\[ + p \int_0^t \int_{\Omega^2} |\theta(s)|^{p-2} \theta(s) k_{\delta_n} \ast G(\theta(s)) d\xi dW(s) \]
\[ \leq \| k_{\delta_n} \ast \theta_0 \|^p_{L^p} \]
\[ + \int_0^t \left( \varepsilon \int_{\Omega^2} |\theta(s)|^p d\xi \right. \]
\[ + C(\varepsilon) \int \left( \sum_j |k_{\delta_n} \ast G(\theta(s))(f_j)|^2 \right)^{p/2} d\xi \right] ds \]
\[ + p \int_0^t \int_{\Omega^2} |\theta(s)|^{p-2} \theta(s) k_{\delta_n} \ast G(\theta(s)) d\xi dW(s), \]
where in the first inequality we used \( \text{div} u = 0 \) and \( \int |\theta|^{p-2} \theta \Lambda^{2\alpha} \theta \geq 0 \) (cf. [48], Lemma 3.2) as well as Young’s inequality in the second inequality. Then by the Burkholder–Davis–Gundy inequality and Minkowski’s inequality, we ob-
tain

\[ E \sup_{s \in [0, t]} \| \theta(s) \|_{L^p} \]
\[
\leq E \| \theta_0 \|_{L^p}^p + E \int_0^t \left( \epsilon \int_{\mathbb{T}^2} |\theta(s)|^p d\xi \right) + C \int \left( \sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2 \right)^{p/2} d\xi \right) ds
\]
\[
+ pE \left( \int_0^t \left( \int_{\mathbb{T}^2} |\theta(s)|^{p-1} \left( \sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2 \right)^{p/2} d\xi \right)^{1/2} \right) ds
\]
\[
\leq E \| \theta_0 \|_{L^p}^p
\]
\[
+ E \int_0^t \left( \epsilon \int_{\mathbb{T}^2} |\theta(s)|^p d\xi \right) + C \int \left( \sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2 \right)^{p/2} d\xi \right) ds
\]
\[
+ C(T)E \sup_{s \in [0, t]} \| \theta(s) \|_{L^p}^{p-1} \left( \int_0^t \left( \int_{\mathbb{T}^2} \left( \sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2 \right)^{p/2} d\xi \right) d\sigma \right)^{1/p}
\]
\[
\leq E \| \theta_0 \|_{L^p}^p + \epsilon E \sup_{s \in [0, t]} \| \theta(s) \|_{L^p}^p + C_1 E \int_0^t \| \theta(s) \|_{L^p}^p d\sigma + C_2
\]
\[
\leq E \| \theta_0 \|_{L^p}^p + \epsilon E \sup_{s \in [0, t]} \| \theta(s) \|_{L^p}^p + C_1 \int_0^t E \sup_{s \in [0, \sigma]} \| \theta(s) \|_{L^p}^p d\sigma + C_2.
\]

Here, in the fourth inequality, we used (Gp.1) and Young’s inequality. By Gronwall’s lemma, the assertion follows. \( \Box \)

4. Existence and uniqueness of probabilistically (strong) solutions in the subcritical case. In this section, we assume \( \alpha > 1/2 \) and prove pathwise uniqueness for equation (3.1), and hence by the Yamada–Watanabe theorem the existence of a unique (probabilistically) strong solution to (3.1) in the subcritical case. Let us first give the definition of a (probabilistically) strong solution to (3.1).
DEFINITION 4.1. We say that there exists a (probabilistically) strong solution to (3.1) over the time interval \([0, T]\) if for every probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)\) with an \(\mathcal{F}_t\)-Wiener process \(W\), there exists an \(\mathcal{F}_t\)-adapted process \(\theta : [0, T] \times \Omega \to H\) such that for \(P\)-a.e. \(\omega \in \Omega\)

\[
\theta(\cdot, \omega) \in L^\infty(0, T; H) \cap L^2(0, T; H^\alpha) \cap C([0, T]; H^{-\beta})
\]

and \(P\)-a.e.

\[
|\theta(t), \varphi| + \int_0^t |A_\alpha^{1/2} \theta(s), A_\alpha^{1/2} \varphi| ds - \int_0^t (u(s) \cdot \nabla \varphi, \theta(s)) ds
\]

\[
= (\theta_0, \varphi) + \left( \int_0^t G(\theta(s)) dW(s), \varphi \right)
\]

for all \(t \in [0, T]\) and all \(\varphi \in C^1(T^2)\) (assuming also that all integrals in the equation are defined).

THEOREM 4.2. Assume \(\alpha > \frac{1}{2}\). If \(G\) satisfies the following condition:

\[
\|\Lambda^{-1/2}(G(u) - G(v))\|_{L^2(U, H)}^2 \leq \beta |\Lambda^{-1/2}(u - v)|^2 + \beta_1 |\Lambda^{\alpha - 1/2}(u - v)|^2
\]

(GL.1)

for all \(u, v \in H^\alpha\), for some \(\beta \in \mathbb{R}\) independent of \(u, v\), and \(\beta_1 < 2\kappa\), then (3.1) admits at most one probabilistically strong solution in the sense of Definition 4.1 such that

\[
\sup_{t \in [0, T]} \|\theta(t)\|_{L^p} < \infty, \quad P\text{-a.s.}
\]

for some \(p \in ((\alpha - \frac{1}{2})^{-1}, \infty)\), and

\[
E \sup_{t \in [0, T]} |\Lambda^{-1/2} \theta(t)|^2 < \infty.
\]

REMARK. The examples in Remark 3.4 with \(g\) being a Lipschitz function on \(\mathbb{R}\) satisfy (GL.1) since

\[
\|\Lambda^{-1/2}(G(u) - G(v))\|_{L^2(K, H)}^2 = \sum_k |\Lambda^{-1/2}(b_k (g(u) - g(v)))|^2
\]

\[
\leq \int_{\mathbb{T}^2} \sum_k b_k^2 (g(u) - g(v))^2 d\xi
\]

\[
\leq C |u - v|^2
\]

\[
\leq C |\Lambda^{-1/2}(u - v)|^2 + \varepsilon |\Lambda^{\alpha - 1/2}(u - v)|^2.
\]
Proof of Theorem 4.2. Let \( \theta_1, \theta_2 \) be two solutions of (3.1), and let \( \{e_k\}_{k \in \mathbb{N}} \) be the eigenbasis of \( A_\alpha \) from above. Then their difference \( \theta = \theta_1 - \theta_2 \) satisfies for \( \psi \in C^1(T^2) \)

\[
\langle \psi, \theta(t) \rangle - \int_0^t \langle u \cdot \nabla \psi, \theta_1 \rangle \, ds - \int_0^t \langle u_2 \cdot \nabla \psi, \theta \rangle \, ds + \kappa \int_0^t \langle \theta, \Lambda^{2\alpha} \psi \rangle \, ds
\]

\[
= \int_0^t \langle \psi, (G(\theta_1) - G(\theta_2)) \, dW \rangle.
\]

Here, \( u_1, u_2, u \) satisfy (1.3) with \( \theta \) replaced by \( \theta_1, \theta_2, \theta \), respectively. Now set \( \phi_k = \langle e_k, \theta(t) \rangle, \varphi_k = \langle \Lambda^{-1} e_k, \theta(t) \rangle \). Itô's formula and (4.2) yield

\[
\phi_k \varphi_k = \int_0^t \phi_k \, d\varphi_k + \int_0^t \varphi_k \, d\phi_k + \langle \varphi_k, \phi_k \rangle(t)
\]

\[
= 2 \int_0^t \langle u \cdot \nabla e_k, \theta_1 \rangle \langle \Lambda^{-1} \theta, e_k \rangle + \langle u_2 \cdot \nabla e_k, \theta \rangle \langle \Lambda^{-1} \theta, e_k \rangle \, ds
\]

\[
- \kappa \langle \Lambda^{2\alpha} e_k, \theta \rangle \langle \Lambda^{-1} \theta, e_k \rangle \, ds
\]

\[
+ 2 \int_0^t \langle \Lambda^{-1} \theta, e_k \rangle \langle e_k, (G(\theta_1) - G(\theta_2)) \rangle \, dW(s)
\]

\[
+ \int_0^t \langle (G(\theta_1) - G(\theta_2))^* e_k, (G(\theta_1) - G(\theta_2))^* \Lambda^{-1} e_k \rangle \, ds.
\]

Here, \( \langle \varphi_k, \phi_k \rangle(t) \) denotes the covariation process of \( \varphi_k, \phi_k \). The dominated convergence theorem implies

\[
\sum_{k \leq N} \int_0^t \langle u \cdot \nabla e_k, \theta_1 \rangle \langle \Lambda^{-1} \theta, e_k \rangle \, ds \to \int_0^t H^{-1} \langle u \cdot \nabla \Lambda^{-1} \theta, \Lambda^{-1} \theta \rangle \, ds, \quad N \to \infty,
\]

\[
\sum_{k \leq N} \int_0^t \langle u_2 \cdot \nabla e_k, \theta \rangle \langle \Lambda^{-1} \theta, e_k \rangle \, ds \to \int_0^t H^{-1} \langle u_2 \cdot \nabla \Lambda^{-1} \theta, \Lambda^{-1} \theta \rangle \, ds, \quad N \to \infty
\]

and

\[
\sum_{k \leq N} \int_0^t \langle \Lambda^{2\alpha} e_k, \theta \rangle \langle \Lambda^{-1} \theta, e_k \rangle \, ds \to \int_0^t \langle \theta, \Lambda^{2\alpha-1} \theta \rangle \, ds, \quad N \to \infty.
\]

Furthermore, since

\[
\int_0^t \langle \Lambda^{-1/2} \theta \rangle^2 \langle \Lambda^{-1/2} (G(\theta_1) - G(\theta_2)) \rangle^2 \, ds
\]

\[
\leq C \sup_{s \leq t} \langle \theta(s) \rangle^2 \int_0^t \| \Lambda^{-1/2} (G(\theta_1) - G(\theta_2)) \|_{L_2(U,H)}^2 \, ds < \infty,
\]
we obtain
\[\sum_{k \leq N} \int_0^t \langle \Lambda^{-1} \theta, e_k \rangle \langle e_k, (G(\theta_1) - G(\theta_2)) \rangle \, dW(s)\]
\[\rightarrow M_t := \int_0^t \langle \Lambda^{-1/2} \theta, \Lambda^{-1/2}(G(\theta_1) - G(\theta_2)) \rangle \, dW(s), \quad N \to \infty,\]
in probability. Finally, the following inequality holds:
\[\sum_{k \leq N} \int_0^t \langle (G(\theta_1) - G(\theta_2))^* e_k, (G(\theta_1) - G(\theta_2))^* \Lambda^{-1} e_k \rangle_U \, ds\]
\[\leq \int_0^t \| \Lambda^{-1/2}(G(\theta_1) - G(\theta_2)) \|^2_{L^2(U, H)} \, ds.\]

Thus, summing up over \(k \leq N\) in (4.3) and letting \(N \to \infty\), we obtain
\[|\Lambda^{-1/2} \theta|^2 + 2\kappa \int_0^t |\Lambda^{\alpha-1/2} \theta|^2 \, ds\]
\[\leq 2M(t) + 2 \int_0^t \langle u_1 \cdot \nabla \theta_1, \Lambda^{-1} \theta \rangle_{H^1} + \int_0^t \langle u_2 \cdot \nabla \theta, \Lambda^{-1} \theta \rangle_{H^1} \, ds\]
\[+ \int_0^t \| \Lambda^{-1/2}(G(\theta_1) - G(\theta_2)) \|^2_{L^2(U, H)} \, ds.\]

By [48], we have
\[H^{-1} \langle u_2 \cdot \nabla \theta_1, \Lambda^{-1} \theta \rangle_{H^1} = 0\]
and
\[|H^{-1} \langle u_2 \cdot \nabla \theta, \Lambda^{-1} \theta \rangle_{H^1}|\]
\[\leq \| u_2 \|_{L^p} \| \theta \|_{L^p} \| \nabla \Lambda^{-1} \theta \|_{L^1} \leq C \| u_2 \|_{L^p} \| \theta \|_{H^{1/p}} \| \nabla \Lambda^{-1} \theta \|_{H^{1/p}}\]
\[\leq C \| u_2 \|_{L^p} \| \Lambda^{-1} \theta \|^2_{H^{s+1/p}} \leq C \| u_2 \|_{L^p} \| \Lambda^{-1} \theta \|^2_{H^{s/r}} \| \Lambda^{-1} \theta \|^2_{H^{1/s/r}}\]
\[\leq \varepsilon |\Lambda^{\alpha-1/2} \theta|^2 + C \| u_2 \|_{L^p} |\Lambda^{-1/2} \theta|^2,\]
where \(\frac{1}{p} + \frac{2}{p_1} = 1\) for \(p \in ((\alpha - \frac{1}{2})^{-1}, +\infty), r = \frac{\alpha}{\alpha - 1/2 - 1/p}\). Here we use div \(u_2 = 0\) in the first inequality, that \(H^{s/r} \hookrightarrow L^{p_1}\) continuously in the second inequality, the interpolation inequality (2.3) in the fourth inequality and Young’s inequality in the last equality.

Now by (GL.1) we have
\[|\Lambda^{-1/2} \theta|^2 \leq 2M(t) + \int_0^t C \| u_2 \|_{L^p} |\Lambda^{-1/2} \theta|^2 \, ds + \beta \int_0^t |\Lambda^{-1/2}(\theta_1 - \theta_2)|^2 \, ds.\]

Let
\[\tau_n := \inf \{t > 0, \| \theta_2(t) \|_{L^p} > n \}.\]
Then by the weak continuity of $\theta_2$, $\tau^1_n$ are stopping times with respect to $\mathcal{F}_{t+}$, $(\mathcal{F}_{t+} := \bigcap_{s \geq t} \mathcal{F}_s)$ and $\|\theta_2(t \wedge \tau^1_n)\|_{L^p} \leq n$ for large $n$. Furthermore, let $\tau_2^n$ be a localizing sequence of stopping times for $M$ and $\tau_n := \tau^1_n \wedge \tau_2^n$. Then, since $M(t \wedge \tau_n)$ is a martingale with respect to $\mathcal{F}_{t+}$, we get

$$E|\Lambda^{-1/2}\theta(t \wedge \tau_n)|^2 \leq Cn^r E \int_0^{t \wedge \tau_n} |\Lambda^{-1/2}\theta|^2 ds + \beta E \int_0^{t \wedge \tau_n} |\Lambda^{-1/2}\theta|^2 ds$$

$$= C(n) \int_0^t E|\Lambda^{-1/2}\theta(s \wedge \tau_n)|^2 ds + \beta \int_0^t E|\Lambda^{-1/2}\theta(s \wedge \tau_n)|^2 ds.$$

By Gronwall’s inequality, we get $|\Lambda^{-1/2}\theta(t \wedge \tau_n)|^2 \equiv 0$ $P$-a.s., and recalling that $\tau_n \to T$ $P$-a.s. as $n \to \infty$, we obtain that $\theta(t) = 0$ $P$-a.s. for $t \leq T$. By the weak continuity of $\theta$, we obtain the zero set does not depend on $t$, thus completing the proof. □

**Remark.** From the proof of Theorem 4.2, we immediately obtain that if there exists a probabilistically strong solution $\tilde{\theta}$ in the sense of Definition 3.1 satisfying

$$E\sup_{t \in [0,T]} \|\theta(t)\|_{L^p} < \infty, \quad P\text{-a.s.}$$

for some $p \in ((\alpha - \frac{1}{2})^{-1}, +\infty)$ and $G$ satisfies (GL.1), then for any other solution $\tilde{\theta}$ such that

$$E\sup_{t \in [0,T]} |\Lambda^{-1/2}\tilde{\theta}(t)|^2 < \infty,$$

it follows that $\tilde{\theta} = \theta$, which implies that

$$\sup_{t \in [0,T]} \|\tilde{\theta}(t)\|_{L^p} < \infty.$$

**Theorem 4.3.** Assume $\alpha > \frac{1}{2}$ and that $G$ satisfies Hypothesis G.1, (GL.1) and (Gp.1) for some $p \in ((\alpha - \frac{1}{2})^{-1}, +\infty)$. Then for each initial condition $\theta_0 \in L^p$, there exists a pathwise unique probabilistically strong solution $\theta$ of equation (3.1) over $[0, T]$ with initial condition $\theta(0) = \theta_0$ such that

$$E\sup_{t \in [0,T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty.$$

Moreover, the solution satisfies

$$E\sup_{t \in [0,T]} \|\theta(t)\|_{L^p}^p + E \int_0^T |\Lambda^\alpha \theta(t)|^2 dt < \infty.$$
PROOF. By Theorem 4.2, Theorem 3.3 and the Yamada–Watanabe theorem (cf. [49]), we get that for each initial condition $\theta_0 \in L^p$, there exists a pathwise unique probabilistically strong solution $\theta$ of equation (3.1) over $[0, T]$ with initial condition $\theta(0) = \theta_0$ such that
\[
\sup_{t \in [0, T]} \| \theta(t) \|_{L^p} < \infty, \quad P\text{-a.s.,}
\]
and
\[
E \sup_{t \in [0, T]} |\Lambda^{-1/2} \theta(t)|^2 < \infty.
\]
By the remark before Theorem 4.3, the first result follows. By Theorem 3.3 and (3.6), the last part of the assertion follows. □

THEOREM 4.4 (Markov property). Assume $\alpha > \frac{1}{2}$ and that $G$ satisfies Hypothesis G.1, (GL.1) and (Gp.1) for some $p \in (\frac{\alpha - 1}{2}, +\infty)$. If $\theta_0 \in L^p$, then for every bounded, $\mathcal{B}(H)$-measurable $F : H \to \mathbb{R}$, and all $s, t \in [0, T], s \leq t$
\[
E(F(\theta(t))|\mathcal{F}_s)(\omega) = E(F(\theta(t), s, \theta(s)(\omega))) \quad \text{for } P\text{-a.s. } \omega \in \Omega.
\]
Here, $\theta(t, s, \theta(s)(\omega))$ denotes the solution to (3.1) starting from $\theta(s)$ at time $s$ satisfying
\[
E \sup_{t \in [s, T]} |\Lambda^{-1/2} \theta(t)|^2 < \infty.
\]
PROOF. By Theorem 4.3, we have $\theta(t) = \theta(t, s, \theta(s))$ $P$-a.s. Then by the Yamada–Watanabe theorem in [49], we have $P$-a.s.
\[
E(F(\theta(t))|\mathcal{F}_s)(\omega) = E(F(\theta(t), s, \theta(s))|\mathcal{F}_s)(\omega)
\]
\[
= E(F(H(\theta(s), W(\cdot + s) - W(s)))|\mathcal{F}_s)(\omega)
\]
\[
= E(F(H(\theta(s)(\omega), W(\cdot + s) - W(s))))
\]
\[
= E(F(\theta(t, s, \theta(s)(\omega))).
\]
where $H$ is the functional obtained by the Yamada–Watanabe theorem such that $H(\theta(0), W)$ is a strong solution to (3.1). □
5. Ergodicity in the subcritical case. Now fix $\alpha > \frac{1}{2}$ and we assume $U = H$, $W(t)$ is a cylindrical Wiener process in $H$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. We make the following assumptions on $G$.

**Hypothesis E.1.** $G$ does not depend on $\theta$ and there exists $\sigma > 0$ such that $G \in L_2(H; H^{2-\alpha+\sigma})$ that is,

$$\mathcal{E}_0 := \text{Tr}(\Lambda^{4-2\alpha+2\sigma} GG^*) < \infty.$$  

**Hypothesis E.2.** There exist $N \in \mathbb{N}$ and $g \in L(H)$ such that $Gg = P_N$.

For $\varepsilon_0 > 0$ and any $\mathcal{W} \in C(\mathbb{R}^+, H^{-1-\varepsilon_0})$, we define

$$z(\mathcal{W})(t) := \sum_{i,j=1}^{\infty} \left( g_{ij} \beta_i(t) - \lambda_j \int_0^t e^{-\lambda_j(t-s)} g_{ij}(s) ds \right) e_j,$$

if the convergence of the sum is uniformly with respect to $t$ in every bounded time interval, otherwise set $z(\mathcal{W}) := +\infty$. Here, $\beta_i(t) := H^{t+\varepsilon_0} (e_i, \mathcal{W}(t)) H^{1+\varepsilon_0}$, $g_{ij} = \langle Ge_i, e_j \rangle$. Under Hypothesis E.1, there exists $\Omega \subset \Omega$ such that $P(\Omega) = 1$ and for $\omega \in \Omega$, $z(W(\omega)) \in C([0, \infty), H^{2+\varepsilon})$ for some $0 < \varepsilon < \sigma$, and on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $z(W)$ is the mild solution of the equation: $dz + A_{a}z = G dW$ with initial condition $z(0) = 0$.

Now for $v_0 \in H^1$, $\mathcal{W} \in C(\mathbb{R}^+, H^{-1-\varepsilon_0})$ we define

$$v(t, \mathcal{W}, v_0) := \begin{cases} v(t, v_0, z(\mathcal{W})), & \text{if } z(\mathcal{W}) \in C(\mathbb{R}^+, H^m) \text{ for } m < 2 + \sigma, \\ 0, & \text{otherwise}, \end{cases}$$

where $v(t, v_0, z(\mathcal{W}))$ is the solution to (A.1) we obtained in Theorem A.1. Then by Theorem A.4 in Appendix A, $v$ is a measurable mapping from $\mathbb{R}^+ \times C(\mathbb{R}^+, H^{-1-\varepsilon_0}) \times H^1 \rightarrow H^1$, $(t, \mathcal{W}, \theta_0) \mapsto v(t, \mathcal{W}, \theta_0)$. We can now define

$$\theta(t, \mathcal{W}, \theta_0) := v(t, \mathcal{W}, \theta_0) + z(t, \mathcal{W}),$$

which is a measurable map from $\mathbb{R}^+ \times C(\mathbb{R}^+, H^{-1-\varepsilon_0}) \times H^1 \rightarrow H^1$. Then for the cylindrical Wiener process $W$, $\theta(t, W, \theta_0)$ is a solution to (3.1), whose laws $P_{\theta_0}, \theta_0 \in H^1$ form a Markov process on $H^1$, since $H^1$ is an invariant space for (3.1) under assumption Hypothesis E.1. Let $(P_t)_{t \geq 0}$ be the associated transition semigroup on $B_b(H^1)$. Now we want to study the long time behavior of the semigroup $P_t$.

**Remark 5.1.** (i) Hypothesis E.1 obviously implies Hypothesis G.1, (Gp.1) for all $p \in ((\alpha - \frac{1}{2})^{-1}, \infty)$ and (GL.1). For $x := \theta_0 \in L^p$, let $P_x$ denote the law of the corresponding solution $\theta$ to (3.1). Then by Theorems 4.3 and 4.4, the measures $P_x, x \in L^p$ form a Markov process.
(ii) The existence of a map $g$ such that $Gg = P_N$ is equivalent to the following property:

$$P_N H \subset \text{Im}(G).$$

(iii) Hypothesis E.1 is to make sure that the associated O–U process has a version $z \in C([0, \infty); H^{1,\infty}(\mathbb{T}^2))$ (see, e.g., [12], the proof of Theorem 5.16, and use Sobolev embedding). If we consider the stochastic integral taking values in a Banach space [e.g., $L^p(\mathbb{T}^2)$, $p > 1$] and use the theory developed in [4], we can change Hypothesis E.1 to the following condition: $G \in L_2(H; H^{1-\alpha+\varepsilon_1/2})$ and for some $\varepsilon_1, q$ satisfying $\varepsilon_1q > 2$

$$\left\| \sum_k (\Lambda^{1-\alpha+\varepsilon_1} G_{E_k})^2 \right\|^{1/2}_{L^q} + \left\| \sum_k (G_{E_k})^2 \right\|^{1/2}_{L^{(\alpha+1)/(\alpha-1/2)}} < \infty.$$  

By this and similar arguments as in [4], we obtain for $\varepsilon < \varepsilon_1$ and $\varepsilon q > 2$ that the O–U process has a version $z \in C([0, \infty); H^{1+\varepsilon,\infty'}) \subset C([0, \infty); H^{1,\infty}(\mathbb{T}^2))$, but in this paper we stay in the Hilbert space framework for simplicity.

(iv) For more general noise, we do not know how to obtain Proposition 5.7 since we cannot control $E \exp \left\| \theta \right\|^p_{L^p}$. Therefore, we restrict ourselves to additive noise.

5.1. Preliminaries and some useful estimates. First, we want to collect some useful and fundamental results about coupling from [38] and [40] which we will use later. Let $(\Lambda_1, \Lambda_2)$ be two probability measures on a Polish space $E$. Let $(Z_1, Z_2)$ be a couple of random variables $(\Omega, \mathcal{F}) \to E \times E$. We say that $(Z_1, Z_2)$ is a coupling of $(\Lambda_1, \Lambda_2)$ if $\Lambda_i = D(Z_i)$ for $i = 1, 2$, where we use $D(Z_i)$ to denote the distribution of $Z_i$.

**Lemma 5.2.** Let $(\Lambda_1, \Lambda_2)$ be two probability measures on a Polish space $(E, \mathcal{B}(E))$. Then

$$\|\Lambda_1 - \Lambda_2\|_{\text{var}} = \min P(Z_1 \neq Z_2),$$

where the minimum is taken over all couplings $(Z_1, Z_2)$ of $(\Lambda_1, \Lambda_2)$. There exists a coupling for which the minimum value is attained and it is called a maximal coupling. Moreover, the maximal coupling has the following property:

$$P(Z_1 = Z_2, Z_1 \in \Gamma) = (\Lambda_1 \wedge \Lambda_2)(\Gamma), \quad \Gamma \in \mathcal{B}(E).$$

**Lemma 5.3 (Cf. [40], Lemma C.1).** Let $\Lambda_1$ and $\Lambda_2$ be two equivalent probability measures on $E$. Then for any $p > 1$ and any measurable subset $A \subset E$

$$I_p(A) := \int_A \left( \frac{d\Lambda_1}{d\Lambda_2} \right)^p d\Lambda_1 < \infty$$

implies

$$\left( \Lambda_1 \wedge \Lambda_2 \right)(A) \geq \left( 1 - \frac{1}{p} \right) \left( \frac{\Lambda_1(A)^p}{pI_p(A)} \right)^{1/(p-1)}.$$
PROPOSITION 5.4 (Cf. [44], Proposition 1.4). Let $E$ and $F$ be two Polish spaces, $f_0: E \to F$ be a measurable map and $(\Lambda_1, \Lambda_2)$ be two probability measures on $E$. Set $\lambda_i = f_0^* \Lambda_i$, $i = 1, 2$. Then there exists a coupling $(V_1, V_2)$ of $(\Lambda_1, \Lambda_2)$ such that $(f_0(V_1), f_0(V_2))$ is a maximal coupling of $(\lambda_1, \lambda_2)$.

Now we give some useful estimates which will be used in the next two subsections. Let $\theta_n$ denote the approximation in the proof of Theorem 3.3. As will be seen below, we shall need uniform $L^p$-estimates, and a crucial ingredient to prove them is the following improved version of the “positivity lemma,” that is, Lemma 3.2 in [48].

**Lemma 5.5 (Improved positivity lemma).** For $\alpha \in (0, 1)$, and $\theta \in L^p$ with $\Lambda_2 \alpha \theta \in L^p$, for some $2 < p < \infty$,

$$\int |\theta|^{p-2}\theta \left(\kappa \Lambda_2^\alpha - \frac{2\lambda_1}{p}\right)\theta \geq 0.$$

**Proof.** Denote the semigroup with respect to $-\kappa \Lambda_2^\alpha + \frac{2\lambda_1}{p}$ and $-\kappa \Lambda_2^\alpha$ in $L^2$ by $P^0_t$ and $P^1_t$, respectively. Then we have $P^0_t f = e^{2\lambda_1/p} P^1_t f$. Since $\|P^1_t f\|_{L^2} \leq e^{-\lambda_1 t} \|f\|_{L^2}$ and $\|P^1_t f\|_{L^\infty} \leq \|f\|_{L^\infty}$, by the interpolation theorem, we have $\|P^1_t f\|_{L^p} \leq e^{-2\lambda_1 t/p} \|f\|_{L^p}$, which implies that $\|P^0_t f\|_{L^p} \leq \|f\|_{L^p}$.

Then we get that

$$\frac{d}{dt} \|P^0_t \theta\|_{L^p}^p = \int |P^0_t \theta|^{p-2} (P^0_t \theta) \left(\kappa \Lambda_2^\alpha + \frac{2\lambda_1}{p}\right) \theta \, dx \leq 0.$$

Letting $t \to 0$, we obtain the result. □

**Proposition 5.6.** Let $\alpha > \frac{1}{2}$. Suppose Hypothesis E.1 holds. For $x \in L^p$, let $\theta$ denote the solution of equation (3.1) with the initial value $x$. Then for $2 < p < \infty$

$$E \|\theta(t)\|_{L^p}^p \leq \|x\|_{L^p}^p e^{-\lambda_1 t} + C_S \left[\frac{1}{2} p(p - 1)\right]^{p/2} \lambda_1^{-p/2} \epsilon_0^{p/2} (1 - e^{-\lambda_1 t}),$$

where $C_S$ is the constant for the Sobolev embedding.
STOCHASTIC QUASI-GEOSTROPHIC EQUATION

PROOF. Using [33], Lemma 5.1, or [5], Theorem 2.4, for \( \theta_n \), we obtain

\[
\| \theta(t) \|_{L^p}^p = \| \theta(s) \|_{L^p}^p + \int_s^t \left[ -p \int_{\mathbb{T}^2} |\theta(l)|^{p-2} \theta(l) (\kappa \Lambda^{2\alpha} \theta(l) + u(l) \cdot \nabla \theta(l)) \, d\xi \right. \\
+ \left. \frac{1}{2} p(p-1) \int_{\mathbb{T}^2} |\theta(l)|^{p-2} \left( \sum_j |k_{\delta_n} * G(e_j)|^2 \right) \, d\xi \right] \, dl \\
+ p \int_s^t \int_{\mathbb{T}^2} |\theta(l)|^{p-2} \theta(l) k_{\delta_n} * G \, d\xi \, dW(l)
\]

Here, we use Lemma 5.5 to get the first inequality and Young’s inequality to get the last inequality. Then Gronwall’s lemma yields that

\[
E \| \theta_n(t) \|_{L^p}^p \leq E \| \theta_n(s) \|_{L^p}^p + E \lambda_1 \int_s^t \int_{\mathbb{T}^2} |\theta_n(l)|^p \, d\xi \, dl + C_S^p \left[ \frac{1}{2} p(p-1) \right]^{p/2} \lambda_1^{-(p-2)/2} \mathcal{E}_0^{p/2}(t-s).
\]

Here, we use \( \int_{\mathbb{T}^2} (\sum_j |G(e_j)|^2)^{p/2} \, d\xi \leq (\sum_j (\int_{\mathbb{T}^2} |G(e_j)|^p \, d\xi)^{2/p})^{p/2} \leq C_S^p \mathcal{E}_0^{p/2} \). Then Gronwall’s lemma yields that

\[
E \| \theta_n(t) \|_{L^p}^p \leq E \| \theta_n(0) \|_{L^p}^p e^{-\lambda_1 t} + C_S^p \left[ \frac{1}{2} p(p-1) \right]^{p/2} \lambda_1^{-p/2} \mathcal{E}_0^{p/2} (1 - e^{-\lambda_1 t}).
\]

Letting \( n \to \infty \) in the above inequality, we deduce

\[
E \| \theta(t) \|_{L^p}^p \leq \| x \|_{L^p}^p e^{-\lambda_1 t} + C_S^p \left[ \frac{1}{2} p(p-1) \right]^{p/2} \lambda_1^{-p/2} \mathcal{E}_0^{p/2} (1 - e^{-\lambda_1 t}).
\]
5.2. Uniqueness of the invariant measure. In this subsection, we assume conditions Hypotheses E.1 and E.2 to hold. To prove uniqueness of invariant measure is much harder in this section we first concrete on proving this. Existence will be shown in the next subsection. In addition, we shall prove polynomial convergence of the semigroup to the invariant measure in Section 5.3 below. If the dissipation term is strong enough (i.e., $\alpha > \frac{2}{3}$) we actually obtain exponential convergence (see Section 6).

Now we build an auxiliary process $\tilde{\theta}$. The aim is to find a shift $h$ belonging to Cameron–Martin space of the driving process such that $E\|\theta(t) - \tilde{\theta}(t)\|_{H^{-1/2}} \to 0$ as $t \to \infty$. Fix $\theta$, and consider

$$
\begin{align*}
\left\{ \begin{array}{ll}
d\tilde{\theta}(t) + A_\alpha \tilde{\theta}(t) dt + \tilde{u}(t) \cdot \nabla \tilde{\theta}(t) dt + K_0 P_N (\tilde{\theta} - \theta(t,W,\theta_0)) dt \\
\tilde{\theta}(0) = \tilde{\theta}_0 \in H^1,
\end{array} \right.
\end{align*}
$$

(5.2)

$$
\text{where } \tilde{u} \text{ satisfies } (1.3) \text{ with } \theta \text{ replaced by } \tilde{\theta} \text{ and } K_0 \text{ is a constant to be determined later. Since } \|P_N \tilde{\theta}\|_{L^p} \leq C_N \|\tilde{\theta}\|_{L^p} \text{ for } p \geq 2, \text{ by a similar argument as in the proof of Theorems A.4 in Appendix A we obtain that there exists a measurable mapping from } \mathbb{R}^+ \times \mathcal{C}(\mathbb{R}^+, H^{-1}) \times H^1 \times H^1 \text{ into } H^1, (t, W, \theta_0, \tilde{\theta}_0) \mapsto \tilde{\theta}(t, W, \theta_0, \tilde{\theta}_0), \text{ such that } \tilde{\theta}(t, W, \theta_0, \tilde{\theta}_0) \text{ is the solution of (5.2). Moreover, by the } \omega \text{-wise uniqueness of (3.1) and (5.2) (which can be easily checked by a similar argument as the proof of Theorem 4.2), we have}
$$

$$(\theta(t, \theta_0), \tilde{\theta}(t, \theta_0, \tilde{\theta}_0)) = (\theta(t, s, \theta(s)), \tilde{\theta}(t, s, \theta(s, \theta_0), \tilde{\theta}(s, \theta_0, \tilde{\theta}_0))) \quad P\text{-a.s.,}
$$

which implies that $(\theta(t), \tilde{\theta}(t)) = (\theta(t, W, \theta_0), \tilde{\theta}(t, W, \theta_0, \tilde{\theta}_0))$ defines a Markov process. Here, for simplicity, we omit $W$ and $\theta(t, s, \theta(s))$, $\tilde{\theta}(t, s, \theta(s, \theta_0), \tilde{\theta}(s, \theta_0, \tilde{\theta}_0))$ denote the solutions to (3.1), (5.2) starting from $\theta(s), \tilde{\theta}(s)$ at time $s$, respectively.

Now we derive a uniform $|\cdot|^4$ estimate for $\tilde{\theta}$. Here, we give formal calculations which can be made rigorous by using Galerkin approximations:

$$
\begin{align*}
&d|\tilde{\theta}(t)|^4 + 4\kappa |\tilde{\theta}(t)|^2 \|\tilde{\theta}\|_{H^\alpha}^2 dt + 4K_0 |\tilde{\theta}(t)|^2 P_N |\tilde{\theta}|^2 dt \\
&\leq 4|\tilde{\theta}(t)|^2 \|G dW(t, \tilde{\theta}) + 4K_0 |\tilde{\theta}(t)|^2 P_N \tilde{\theta}|^2 dt + 6|\tilde{\theta}|^2 \|G\|_{L^2(H, H)}^2 dt \\
&\leq 4|\tilde{\theta}(t)|^2 \|G dW(t, \tilde{\theta}) + \varepsilon |\tilde{\theta}(t)|^4 dt + C(\varepsilon) |(\tilde{\theta})^4 + 1| dt.
\end{align*}
$$

(5.3)

Taking expectation and by Proposition 5.6, we obtain

$$
E|\tilde{\theta}(t)|^4 \leq C, \quad \forall t \geq 0,
$$

where $C$ is a constant independent of $t$.

Define $h(\theta, \tilde{\theta}) := -gK_0 P_N (\tilde{\theta} - \theta)$ for $g$ in Hypothesis E.2. Then for any $(t, \theta_0, \tilde{\theta}_0) \in \mathbb{R}^+ \times H^1 \times H^1$ and the cylindrical Wiener process $W$ we have for
STOCHASTIC QUASI-GEOSTROPHIC EQUATION

1. \( \omega \in \Omega' \), that \( z(W(\omega)) = (W(\omega) + \int_0^\infty \theta(s, W(\omega), \theta_0) \, ds) \in C([0, \infty), H^{2+\varepsilon}), \varepsilon < \sigma \). Then for \( \omega \in \Omega' \),

\[
\theta(t, W(\omega) + \int_0^t \theta(s, W(\omega), \theta_0), \tilde{\theta}(s, W(\omega), \theta_0, \tilde{\theta}_0) \, ds, \tilde{\theta}_0) - z(W(\omega))
\]

is a solution to the following equation:

\[
d\tilde{u}(t) + A_\alpha \tilde{v}(t) \, dt + u \tilde{v}+z(t) \cdot \nabla (\tilde{u} + z)(t) \, dt + K_0 p_N(\tilde{u} - v(t, W(\theta_0)) \, dt = 0,
\]

where \( u \tilde{v}+z \) satisfies (1.3) with \( \theta \) replaced by \( \tilde{v} + z \). Since for every \( \omega \in \Omega' \), the above equation admits at most one solution, for \( \omega \in \Omega' \) we have

\[
\tilde{\theta}(t, W(\omega), \theta_0, \tilde{\theta}_0)
\]

(5.4)

\[
= \theta(t, W(\omega) + \int_0^t \theta(s, W(\omega), \theta_0), \tilde{\theta}(s, W(\omega), \theta_0, \tilde{\theta}_0) \, ds, \tilde{\theta}_0).
\]

Now for \( \rho = \tilde{\theta}(t, W, \theta_0, \tilde{\theta}_0) - \theta(t, W, \theta_0) \), we have the following results. Here, we want to emphasize that although the initial value \( \theta_0 \in H^1 \), we can only obtain that \( \rho \) converges to 0 in \( H^{-1/2} \) norm.

**PROPOSITION 5.7.** Fix \( \alpha > 1/2 \). Let \( \delta_0 := \lambda_{N+1} - 2p/2C_R C_S \kappa^{1-p} [p(p-1)]^{p/2} \lambda_1^{-p/2} \tilde{E}_0^{p/2} > 0 \) for \( p = \frac{\alpha+1}{\alpha-1/2} \), where \( N \) is as in Hypothesis E.2, and \( C_S, C_R \) are the constants for the Sobolev embedding and Riesz transform, respectively. Then for \( \|\theta_0\|_{L^{2m(p-1)}} + \|\tilde{\theta}_0\|_{L^{2m(p-1)}} \leq 2C_0 \) for some \( m > 5 \), \( K_0 > \lambda_{N+1} \) and \( 1 < \tilde{q} < \frac{m-1}{4} \), there exists a positive constant \( \overline{C} \) such that for any \( t > 0 \)

\[
\|\Lambda^{-1/2} \rho(t)\|^2 \leq \frac{\overline{C}}{(t+1)^{2\tilde{q}}}
\]

(where we can choose \( C_0 \) large enough such that \( C_0 > 4C_S \frac{1}{2} p(p-1) \lambda^{p/2} \lambda_1^{-p/2} \times \tilde{E}_0^{p/2} \).

**REMARK 5.8.** From the condition \( \lambda_{N+1} - 2p/2C_R C_S \kappa^{1-p} [p(p-1)]^{p/2} \lambda_1^{-p/2} \tilde{E}_0^{p/2} > 0 \), which also appears in the main theorem, we know that if the viscosity constant \( \kappa \) is large enough or \( \tilde{E}_0 \) is small enough we could even take \( N = 0 \).

**PROOF OF PROPOSITION 5.7.** In the proof, we omit \( W \) for simplicity. From (3.1) and (5.2), we obtain that \( \rho \) satisfies the following equation in the weak sense:

\[
\frac{d\rho(t)}{dt} = -A_\alpha \rho - K_0 PN \rho - \tilde{u} \cdot \nabla \tilde{\theta} + u \cdot \nabla \theta
\]

\[
= -A_\alpha \rho - K_0 PN \rho - u \cdot \nabla \rho - u_\rho \cdot \nabla \tilde{\theta},
\]
where \( u_\rho \) satisfies (1.3) with \( \theta \) replaced by \( \rho \). Taking the inner product with \( \Lambda^{-1} \rho \) in \( H \), and using that
\[
H^{-1}(u_\rho \cdot \nabla \tilde{\theta}, \Lambda^{-1} \rho)_{H^1} = 0
\]
(cf. [48]), we obtain
\[
\frac{1}{2} \frac{d}{dt} |\Lambda^{-1/2} \rho|^2 = -\kappa |\Lambda^{a-1/2} \rho|^2 - K_0 |P_N \Lambda^{-1/2} \rho|^2 - H^{-1}(u \cdot \nabla \rho, \Lambda^{-1} \rho)_{H^1}.
\]
We have
\[
|H^{-1}(u \cdot \nabla \rho, \Lambda^{-1} \rho)_{H^1}|
\]
\[
\leq \|u\|_{L^p} \|\rho\|_{{L^p}} \|\nabla \Lambda^{-1} \rho\|_{{L^p}} 
\leq C_S \|u\|_{L^p} \|\rho\|_{{H^{1/p}}} \|\nabla \Lambda^{-1} \rho\|_{{H^{1/p}}}
\]
\[
\leq C_S C_R \|\theta\|_{L^p} \|\Lambda^{-1} \rho\|_{{H^{1/p}}} \leq C_S C_R \|\theta\|_{L^p} \|\Lambda^{-1} \rho\|_{{L^p}} \|\Lambda^{-1} \rho\|_{{H^{1/2}}} \|\Lambda^{-1} \rho\|_{{H^{1/2 + \alpha}}},
\]
\[
\leq \kappa |\Lambda^{a-1/2} \rho|^2 + C'_1 \left( \frac{\kappa}{2} \right)^{1-r} \|\theta\|_{L^p} \|\Lambda^{-1} \rho\|^2,
\]
where \( C_S, C_R \) are the constants for Sobolev embedding and Riesz transform, respectively, and \( C_1 = C_S C_R \). Here, \( \frac{1}{p} + \frac{2}{p'} = 1 \) for \( p > \frac{1}{a-1/2}, \frac{r}{a-1/2-1/p} \) and we use Hölder’s inequality and that \( \text{div} u = 0 \) in the first inequality and \( H^{1/p} \hookrightarrow L^p \) continuously in the second inequality, the interpolation inequality (2.3) in the fourth inequality and Young’s inequality in the last equality. Then we obtain
\[
\frac{d}{dt} |\Lambda^{-1/2} \rho|^2 \leq -\kappa |\Lambda^{a-1/2} \rho|^2 - K_0 |P_N \Lambda^{-1/2} \rho|^2
\]
\[
+ 2C'_1 \left( \frac{\kappa}{2} \right)^{1-r} \|\theta\|_{L^p} \|\Lambda^{-1} \rho\|^2.
\]
Since, because \( K_0 > \lambda_{N+1} \), we have
\[
\lambda_{N+1} |\Lambda^{-1/2} \rho|^2 \leq \kappa |Q_N \Lambda^{a-1/2} \rho|^2 + K_0 |P_N \Lambda^{-1/2} \rho|^2
\]
\[
\leq \kappa |\Lambda^{a-1/2} \rho|^2 + K_0 |P_N \Lambda^{-1/2} \rho|^2,
\]
it follows that
\[
\frac{d}{dt} |\Lambda^{-1/2} \rho|^2 + \left( \lambda_{N+1} - 2C'_1 \left( \frac{\kappa}{2} \right)^{1-r} \|\theta\|_{L^p} \right) |\Lambda^{-1/2} \rho|^2 \leq 0.
\]
Thus, by Gronwall’s lemma, we obtain
\[
|\Lambda^{-1/2} \rho(t)|^2 \leq e^{\Gamma(t, \theta_0)} |\Lambda^{-1/2} \rho(0)|^2
\]
where
\[
\Gamma(t, \theta_0) = -\lambda_{N+1} + 2C'_1 \left( \frac{\kappa}{2} \right)^{1-r} \frac{1}{t} \int_0^t \|\theta(s)\|_{L^p} ds.
\]
By the same arguments as in the proof of Theorem A.1 in Appendix A, we have \(\theta_n \rightarrow \theta\) in \(L^2([0, T], H^1)\) a.s. Letting \(n \rightarrow \infty\) in (5.1), by (3.11) we obtain

\[
\|\theta(t)\|_{L_p}^p + \lambda_1 \int_0^t \int_{T^2} |\theta(l)|^p d\xi dl \\
\leq \|\theta_0\|_{L_p}^p + C_{p, m} \left[ \frac{1}{2} p (p - 1) \right] \lambda_1^{-(p-2)/2} \mathcal{E}_0^{p/2} t \\
+ p \int_0^t \int_{T^2} |\theta(l)|^{p-2} \theta(l) G d\xi dW(l).
\]

Here, we use that \(\int_{T^2} (\sum_j |G(e_j)|^2)^{p/2} d\xi \leq (\int_{T^2} |G(e_j)|^p d\xi)^{2/p} \leq C_S^{p} \mathcal{E}_0^{p/2}\).

Since \(p = \frac{a+1}{a-1/2}\) implies \(p = r\), we get

\[
\Gamma(t, \theta_0) \leq -\lambda_{N+1} + 2 C_1 \left( \frac{\kappa}{2} \right)^{1-p} \frac{1}{t} \int_0^t \|\theta(s)\|_{L_p}^p ds \\
\leq -\lambda_{N+1} + 2 C_1 \left( \frac{\kappa}{2} \right)^{1-p} \frac{1}{t \lambda_1} \|\theta_0\|_{L_p}^p \\
+ 2^{p/2} C_1^{p/2} C_{S}^{p/2} \kappa^{1-p} \left[ p (p - 1) \right] \lambda_1^{-p/2} \mathcal{E}_0^{p/2} \\
+ 2 C_1 \left( \frac{\kappa}{2} \right)^{1-p} \frac{1}{t \lambda_1} \int_0^t \int_{T^2} |\theta(l)|^{p-2} \theta(l) G d\xi dW(l).
\]

For \(M(t) := p \int_0^t \int_{T^2} |\theta(l)|^{p-2} \theta(l) G d\xi dW(l)\), we have

\[
\langle M \rangle_t \leq p^2 \mathcal{E}_0 C_S^2 \int_0^t \left( \int_{T^2} |\theta(s)|^{p-1} d\xi \right)^2 ds,
\]

where we use that \(\sum_j |G(e_j)|^2(\xi) \leq \sum_j |G(e_j)|^2 \leq C_S^2 \mathcal{E}_0\). Then for any \(m > 1\)

\[
\langle M \rangle_t^m \leq C_S^{2m} p^{2m} C_{m'}^{m'} m' \left( \int_0^t \left( \int_{T^2} |\theta(s)|^{p-1} d\xi \right)^2 ds \right)^m \\
\leq C_S^{2m} p^{2m} C_{m'}^{m'} m'^{-1} \int_0^t \left( \int_{T^2} |\theta(s)|^{2m(p-1)} d\xi \right) ds.
\]

Since \(\|\theta_0\|_{L^{2m(p-1)}} \leq C_0\) by Proposition 5.6 there exists a constant \(C_{p,m}(C_0)\) independent of \(t\) such that \(E \|\theta(t)\|_{L^{2m(p-1)}}^{2m(p-1)} \leq C_{p,m}\) for \(t \geq 0\). Thus, for \(M_n = \sup_{0 \leq t \leq n} M(t)\), we have

\[
P\left( |M_n| > \frac{\varepsilon \lambda_1}{4 C_1^{p} \left( \frac{\kappa}{2} \right)^{1-p}} \right) \leq \frac{p^{2m} C_{m,n} C_m C_S^{2m}}{(\varepsilon \left( \frac{p-1}{p+1} \right) C_{C_1^{p} \left( \frac{\kappa}{2} \right)^{1-p}})} \frac{1}{n^{2m}}.
\]
Now define the following random times:

\[ T_{\text{bound}} := \sup \left\{ n : |M_n| > \frac{\varepsilon \lambda_1}{4C_1^p (\kappa/2)^{1-p}} n \right\}. \]

By [39], Lemma 5, we have that if \( m > 1 \), then \( T_{\text{bound}} \) is finite almost surely. Set

\[ \tau := \max \left( T_{\text{bound}}, \frac{2^{p+1} C_0^p (p/2m(p-1))^p}{\kappa^{p-1} \lambda_1^p} \right), \]

then we have

\[ t > \tau \implies \Gamma(t, \theta_0) - (-\delta_0) < \varepsilon, \]

where \( \delta_0 = \lambda N + 1 - 2^{p/2} C_1^p C_0^p \kappa^{1-p} [p(p-1)]^{p/2} \lambda_1^{1-p} \delta_0^{p/2} \), which implies that for \( \delta \in (0, \delta_0) \) and \( t > \tau \),

\[ |\Lambda^{-1/2} \rho(t)|^2 \leq |\Lambda^{-1/2} (\theta_0 - \tilde{\theta}_0)|^2 e^{-\delta t}. \]

For \( p_0 \in (0, m - 1) \), by [39], Lemma 5, \( E \tau^{p_0} \) is finite. Moreover, we obtain that for \( 1 < q < \frac{m-1}{4} \), there exists \( C > 0 \) such that for any \( t > 0 \)

\[ E|\Lambda^{-1/2} \rho(t)|^2 \leq C e^{-\delta t} + (E|\Lambda^{-1/2} \rho(t)|^4)^{1/2} P(\tau > t)^{1/2} \]

\[ \leq \frac{C}{(t + 1)^{2q}}, \]

where we used (5.3) in the last inequality. \( \square \)

Now we fix \( m > 35 \) and \( 8 < q < \frac{m-3}{4} \). Proposition 5.7 still holds for such \( m, q \). Moreover, we also have for any \( t_0 \geq 0 \):

\[ P\left( \int_{t_0}^{\infty} |h(t)|^2 \, dt \geq \frac{1}{(t_0 + 1)^q} \right) \leq C \frac{(t_0 + 1)^q}{C} \int_{t_0}^{\infty} E|\Lambda^{-1/2} \rho(t)|^2 \, dt \]

\[ \leq \frac{1}{(t_0 + 1)^q}, \]

where \( h(t) = h(\theta(t, W, \theta_0), \tilde{\theta}(t, W, \theta_0, \tilde{\theta}_0)) \) and we used Proposition 5.7 in the last inequality. Moreover, by Theorem 5.9, we obtain that there exists \( p_2 > 0 \) such that

\[ P\left( \int_{0}^{\infty} |h(t)|^2 \geq \frac{C_1^p}{C} \right) \leq \frac{C_1^p}{C} E \int_{0}^{\infty} \left| \Lambda^{-1/2} \rho(t) \right|^2 \, dt \]

\[ \leq 1 - p_2, \]

where \( C \) can be chosen large enough such that (5.5), (5.6) and (5.7) are satisfied.
STOCHASTIC QUASI-GEOSTROPHIC EQUATION

Theorem 5.9. Fix $\alpha > 1/2$. Assume Hypotheses E.1 and E.2 hold. Let $\delta_0 := \lambda_{N+1} - 2^{-p/2} C_R^p C_S^p K^{-1} - p [p(p - 1)]^{p/2} \lambda_1^{-p/2} \lambda_0^{p/2} > 0$ for $p = \frac{\alpha + 1}{\alpha - 1/2}$, where $N$ is as in Hypothesis E.2, and $C_S, C_R$ are the constants for Sobolev embedding and Riesz transform, respectively. Then there exists at most one invariant measure for the Markov semigroup $P_t$ on $H^1$.


For $\theta_0, \theta_2 \in H^1$ and $T > 0$, we apply [44], Corollary 1.5, to $(\theta(\cdot, W, \theta_0), \theta(\cdot, W, \theta_1), \theta(\cdot, W, \theta_2))$ on $[0, T]$ and obtain $(\theta_0(\cdot, \theta_0^1, \theta_0^2), \theta_1(\cdot, \theta_0^1, \theta_0^2), \theta_2(\cdot, \theta_0^1, \theta_0^2))$ on $[0, T]$ such that the law of $(\theta_0(\cdot, \theta_0^1, \theta_0^2), \theta(\cdot, \theta_0^1, \theta_0^2))$ is the same as $(\theta(\cdot, W, \theta_0), \theta(\cdot, W, \theta_1), \theta(\cdot, W, \theta_2))$ and $(\theta_0(\cdot, \theta_0^1, \theta_0^2), \theta(\cdot, \theta_0^1, \theta_0^2))$ is a maximal coupling of $(D(\theta(\cdot, W, \theta_0)), D(\theta(\cdot, W, \theta_1), \theta(\cdot, W, \theta_2)))$ on $[0, T]$.

Then we obtain a sequence of independent versions of the mapping

$$(\theta_0^1, \theta_0^2) \rightarrow (\theta_0(\cdot, \theta_0^1, \theta_0^2), \theta_1(\cdot, \theta_0^1, \theta_0^2), \theta_2(\cdot, \theta_0^1, \theta_0^2)).$$

We denote this sequence by $(\theta_n^1, \theta_n^2, \bar{\theta}_n)$ and define recursively

$$\begin{align*}
\theta_1(nT + \cdot, \theta_0^1, \theta_0^2) &= \theta_1(\cdot, \theta_1(nT), \theta_2(nT)), \\
\theta_2(nT + \cdot, \theta_0^1, \theta_0^2) &= \theta_2(\cdot, \theta_1(nT), \theta_2(nT)), \\
\bar{\theta}(nT + \cdot, \theta_0^1, \theta_0^2) &= \bar{\theta}(\cdot, \theta_1(nT), \theta_2(nT)).
\end{align*}$$

Then $\theta_1(t, \theta_0^1, \theta_0^2), \theta_2(t, \theta_0^1, \theta_0^2), \bar{\theta}(t, \theta_0^1, \theta_0^2)$ is defined for all $t \in [0, \infty)$ such that $(\theta_1(\cdot, \theta_0^1, \theta_0^2), \theta_2(\cdot, \theta_0^1, \theta_0^2))$ is a coupling of $(D(\theta(\cdot, W, \theta_0)), D(\theta(\cdot, W, \theta_1), \theta(\cdot, W, \theta_2)))$. We denote the associated probability space by $(\Omega, F, \mathcal{F}, P)$. Moreover, $(\theta_1(nT, \theta_0^1, \theta_0^2), \theta_2(nT, \theta_0^1, \theta_0^2), \bar{\theta}(nT, \theta_0^1, \theta_0^2))$ is a Markov chain and $\theta_1(\cdot, \theta_0^1, \theta_0^2), \theta_2(\cdot, \theta_0^1, \theta_0^2), \bar{\theta}(\cdot, \theta_0^1, \theta_0^2)$ satisfy the following property:

$$E^{(\theta_0^1, \theta_0^2)}[f(\theta_1, \theta_2, \bar{\theta}) | \mathcal{F}_{kT}] = E^{(\theta_1(kT), \theta_2(kT))} f(\theta_1, \theta_2, \bar{\theta}),$$

where $\Phi_{kT}$ is the shift operator.

Step 2. Introduction of $l_0$.

We set

$$l_0(k) = \min\{l \leq k | P_{l,k}\},$$

where $\min \varnothing = \infty$ and

$$\begin{align*}
(P_{l,k}) \begin{cases} 
\bar{\theta}(\cdot, \theta_0^1, \theta_0^2) = \theta_2(\cdot, \theta_0^1, \theta_0^2) & \text{on } (lT, kT), \\
\|\theta_1(lT, \theta_0^1, \theta_0^2)\|_{L^{2m(p-1)}} + \|\theta_2(lT, \theta_0^1, \theta_0^2)\|_{L^{2m(p-1)}} \leq 2C_0.
\end{cases}
\end{align*}$$
Then by (5.5) and the Markov property of $\theta_1(\cdot, \theta^1_0, \theta^2_0)$, $\theta_2(\cdot, \theta^1_0, \theta^2_0)$, $\tilde{\theta}(\cdot, \theta^1_0, \theta^2_0)$ we have for $t > lT$

$$E\left( |A^{-1/2}(\theta_2(t, \theta^1_0, \theta^2_0) - \theta_1(t, \theta^1_0, \theta^2_0))| 1_{l_0(\infty) \leq t} \right)$$

$$= \sum_{k=0}^{l} E\left( |A^{-1/2}(\theta_2(t, \theta^1_0, \theta^2_0) - \theta_1(t, \theta^1_0, \theta^2_0))| 1_{l_0(\infty) = k} \right)$$

$$= \sum_{k=0}^{l} E\left[ E\left( |A^{-1/2}(\theta_2(t-kT + kT, \theta^1_0, \theta^2_0) - \theta_1(t-kT + kT, \theta^1_0, \theta^2_0))| \right)^2 \right] 1_{l_0(\infty) = k} \mathcal{F}_{kT}$$

$$\leq C \sum_{k=0}^{l} (t-kT + 1)^{-q} \leq C(t - lT + 1)^{-q+1},$$

where we used $\theta_1(kT)$ to denote $\theta_1(kT, \theta^1_0, \theta^2_0)$ for simplicity.

**Step 3. Construction of Wiener processes.**

Now we want to estimate $P(l_0(k + 1) = 0 | l_0(k) = 0)$. As in most papers using coupling methods for SPDEs, our tool is the Girsanov transform. Set

$$\begin{cases}
    h(t, W) = h(\theta(t - kT, W, \theta_1(kT, \theta^1_0, \theta^2_0)), \\
    \tilde{\theta}(t - kT, W, \theta_1(kT, \theta^1_0, \theta^2_0)) = \tilde{\theta}(t - kT, \theta_1(kT, \theta^1_0, \theta^2_0)), \\
    \tau_1(W) = \inf \left\{ t \in (kT, (k+1)T) \left| \int_{kT}^{t} |h(t', W)|^2 dt' > C(kT + 1)^{-q} \right. \right\}.
\end{cases}$$

Then by Proposition 5.4, we obtain cylindrical Wiener processes $W_1, W_2$ on $(\mathcal{F}, \mathcal{F}, \tilde{\mathcal{P}})$ such that

$$\left( W_2, W_1 + \int_{kT}^{\tau_1(W_1)^{\vee}} h(t, W_1) dt \right),$$
is a maximal coupling of \( (\mathcal{D}(W), \mathcal{D}(W + \int_{kT}^{\tau_1(W)} h(t, W) \, dt)) \) on \([kT, (k+1)T]\).

If \( l_0(k) = 0 \), by construction in Step 1, we have

\[ P(l_0(k+1) = 0 | \mathcal{F}_k T) \]

\[ = P(\tilde{\theta}(t, \theta^1_0, \theta^2_0) = \theta_2(t, \theta^1_0, \theta^2_0) \text{ for } t \in [kT, (k+1)T] | \mathcal{F}_k T) \]

(5.8)

\[ \geq \tilde{P}(\tilde{\theta}(\cdot - kT, W_1, \theta_1(kT, \theta^1_0, \theta^2_0), \theta_2(kT, \theta^1_0, \theta^2_0)) \text{ for } t \in [kT, (k+1)T] \]

\[ \geq \tilde{P} \left( W_2 = W_1 + \int_{kT}^{\tau_1(W_1)} h(t, W_1) \, dt \text{ and } \tau_1(W_1) = (k+1)T \right) , \]

where we used that \( (\tilde{\theta}(\cdot, \theta^1_0, \theta^2_0), \theta_2(\cdot, \theta^1_0, \theta^2_0)) \) is a maximal coupling of \( (\tilde{\theta}(\cdot - kT, W_1, \theta_1(kT, \theta^1_0, \theta^2_0), \theta_2(kT, \theta^1_0, \theta^2_0)), \theta_2(\cdot - kT, W_2, \theta_2(kT, \theta^1_0, \theta^2_0))) \) in the first inequality and (5.4) in the last inequality.

Now set \( A := \{ W | \tau_1(W) = (k+1)T \}, \Lambda_1 := \mathcal{D}(W), \Lambda_2 := \mathcal{D}(W + \int_{kT}^{\tau_1(W)} h(t, W) \, dt) \). Then the Novikov condition is satisfied for \( \Lambda_1 \) and \( \Lambda_2 \), which by the Girsanov transform implies that

\[ \left( \frac{d\Lambda_1}{d\Lambda_2} \right)(W) = \exp \left( -\int_{kT}^{\tau_1(W)} h(t, W) \, dW(t) - \frac{1}{2} \int_{kT}^{\tau_1(W)} |h(t, W)|^2 \, dt \right). \]

Thus, we have

\[ \int \left( \frac{d\Lambda_1}{d\Lambda_2} \right)^2 d\Lambda_1 \leq E(M_2 e^{\int_{kT}^{\tau_1(W)} |h(t, W)|^2 \, dt}) \leq e^{\mathcal{E}(kT + 1)^{-q}}, \]

where \( M_2 = \exp(-2 \int_{kT}^{\tau_1(W)} h(t, W) \, dW(t) - 2 \int_{kT}^{\tau_1(W)} |h(t, W)|^2 \, dt) \) and \( EM_2 \leq 1 \).

By this, (5.7), (5.8) and Lemmas 5.2 and 5.3, we obtain

(5.9)

\[ P(l_0(1) = 0) \geq (\Lambda_1 \wedge \Lambda_2)(A) \]

\[ \geq \frac{1}{4} \left( \int \left( \frac{d\Lambda_1}{d\Lambda_2} \right)^2 d\Lambda_1 \right)^{-1} \Lambda_1(A)^2 \geq \frac{p_2}{4} e^{-\mathcal{F}}. \]

Step 4. Estimate for \( P(l_0(k+1) \neq 0, l_0(k) = 0) \).

By (5.8), we obtain

\[ P(l_0(k+1) \neq 0 | \mathcal{F}_k T) \]

\[ \leq \tilde{P} \left( W_2 = W_1 + \int_{kT}^{\tau_1(W)} h(t, W_1) \, dt \text{ and } \tau_1(W_1) < (k+1)T \right) \]

\[ + \tilde{P} \left( W_2 \neq W_1 + \int_{kT}^{\tau_1(W)} h(t, W_1) \, dt \right). \]
Since \((W_2, W_1 + \int_0^{\mathcal{T}_1(W_1)} h(t, W_1) \, dt)\) is a maximal coupling, it follows from Lemma 5.2 and the construction of \(\tau_1\) that
\[
\tilde{P}(W_2 \neq W_1 + \int_{kT}^{\mathcal{T}_1(W_1)} h(t, W_1) \, dt)
\]
\[
= \|\Lambda_1 - \Lambda_2\|_{\text{var}}
\]
\[
\leq \frac{1}{2} \sqrt{\int \left(\frac{d\Lambda_1}{d\Lambda_2}\right)^2 \, d\Lambda_2 - 1} \leq \frac{1}{2} \sqrt{\int \left(\left(\frac{d\Lambda_1}{d\Lambda_2}\right)^2 \, d\Lambda_1\right)^{1/2} - 1} \leq e^{C/8} (kT + 1)^{-q/2}.
\]
(5.10)

Since by the Markov property of \((\theta_1(\cdot, \theta_0^1, \theta_0^2), \tilde{\theta}(\cdot, \theta_0^1, \theta_0^2))\), we have
\[
\tilde{P}(W_2 = W_1 + \int_{kT}^{\mathcal{T}_1(W_1)} h(t, W_1) \, dt \text{ and } \mathcal{T}_1(W_1) < (k + 1)T)
\]
\[
\leq \tilde{P}(\theta(\cdot, kT, W_2, \theta_0^1, \theta_0^2)) = \tilde{\theta}(\cdot, kT, W_1, \theta_0^1, \theta_0^2, \theta_0^1, \theta_0^2))
\]
\[
\text{and } \mathcal{T}_1(W_1) < (k + 1)T)
\]
\[
\leq \tilde{P}
\left(\int_{kT}^{(k+1)T} |h(t, W_1)|^2 \, dt > \mathcal{C}(kT + 1)^{-q}\right)
\]
\[
\leq \mathcal{P}
\left(\int_{kT}^{(k+1)T} \left|h(\theta(t, W, \theta_0^1), \tilde{\theta}(t, W, \theta_0^1, \theta_0^2))\right|^2 \, dt > \mathcal{C}(kT + 1)^{-q}\right),
\]
by (5.6) and (5.10), we obtain
(5.11)
\[
\mathcal{P}(l_0(k + 1) \neq 0 \text{ and } l_0(k) = 0) \leq \mathcal{C}(kT + 1)^{-q/2},
\]
where \(\mathcal{C}\) depends on \(\mathcal{C}\).

Step 5. Estimate for \(El_0(\infty)^q\).
Since \(l_0(k) = 0\) implies \(l_0(l) = 0\) for any \(0 \leq l \leq k \leq \infty\),
\[
\mathcal{P}(l_0(\infty) \neq 0) \leq \sum_{k=0}^{\infty} \mathcal{P}(l_0(k + 1) \neq 0 \text{ and } l_0(k) = 0).
\]
(5.12)
By (5.9) and (5.11), we obtain
\[
\mathcal{P}(l_0(\infty) \neq 0) \leq 1 - \frac{P_0^2}{4} e^{-\mathcal{C}} + \mathcal{C} \sum_{k=1}^{\infty} (kT + 1)^{-q/2}.
\]
Then there exists \(T_0\) such that for \(T \geq T_0\) we have
(5.12)
\[
\mathcal{P}(l_0(\infty) = 0) \geq p_0 = \frac{P_0^2}{8} e^{-\mathcal{C}}.
\]
Now fix $T = T_0$. Define
\[
\sigma := \inf \{ n \in \mathbb{N} | l_0(n) > 0 \}.
\]
It follows from (5.11) that
\[
P(\sigma = k + 1) \leq C(kT + 1)^{-q/2}.
\]
Now for $1 < q_1 < \frac{3}{2} - 1$,
\[
\text{(5.13)} \quad E\sigma^{q_1} 1_{\sigma < \infty} \leq K_1,
\]
where $K_1$ is a constant. For $\delta := \min\{n \in \mathbb{N} | \|\theta_1(nT)\|_{L^{2m(p-1)}}^2, \|\theta_2(nT)\|_{L^{2m(p-1)}}^2 \leq 2C_0\}$, by Proposition 5.6 we obtain that there exist $\gamma > 0$ and $c > 0$ such that
\[
\text{(5.14)} \quad E(e^r \delta) \leq c(1 + \|\theta_1^0\|_{L^{2m(p-1)}}^2 + \|\theta_2^0\|_{L^{2m(p-1)}}^2)
\]
(cf. [40], [42], (1.56)), where we used $C_0 > 4C^p\left(\frac{1}{2} p(p - 1)\right)^{p/2} \lambda_1^{-p/2} \varepsilon_0^{p/2}$. Set
\[
\left\{
\begin{array}{ll}
\delta_0 := \delta, \\
\sigma_k + 1 := \infty & \text{if } \delta_k = \infty; \\
\sigma_k := \sigma \circ \Phi_{\delta k} + \delta_k & \text{else,}
\end{array}
\right.
\]
where $\Phi_t$ is the shift operator. Set $\eta := \sigma + \delta \circ \Phi_{\delta T}$. If $l_0(0) = 0$, by the Markov property, (5.13) and (5.14)
\[
E(\eta^{q_1} 1_{\eta < \infty}) \leq C(E(\sigma^{q_1} 1_{\sigma < \infty}) + E((\delta \circ \Phi_{\delta T})^{q_1} 1_{\delta \circ \Phi_{\sigma} < \infty} 1_{\sigma < \infty}))
\]
\[
\leq C(E(\sigma^{q_1} 1_{\sigma < \infty})
\]
\[
+ cE(1 + \|\theta_1^0(\sigma T)\|_{L^{2m(p-1)}}^2 + \|\theta_2^0(\sigma T)\|_{L^{2m(p-1)}}^2) 1_{\sigma < \infty})
\]
\[
\leq C(1 + \|\theta_1^0\|_{L^{2m(p-1)}}^2 + \|\theta_2^0\|_{L^{2m(p-1)}}^2),
\]
where we used Proposition 5.6 in the last inequality. Since $\delta_k = \delta_{k-1} + \eta \circ \Phi_{\delta k-1} T$, we obtain for $1 < q_1 < \frac{3}{2} - 1$,
\[
E(\delta_k^{q_1} 1_{\delta_k < \infty}) \leq (k + 1)^{q_1 - 1} \left(E\delta^{q_1} + \sum_{n=0}^{k-1} E(\eta \circ \Phi_{\delta n} T)^{q_1} 1_{\eta \circ \Phi_{\delta n} T < \infty}\right)
\]
\[
\leq C(k + 1)^{q_1} (1 + \|\theta_1\|_{L^{2m(p-1)}}^2 + \|\theta_2\|_{L^{2m(p-1)}}^2).
\]
Moreover, if $\delta_k < \infty$, then $\sigma_{k+1} = \infty$ deduces that $l_0(\infty) = \delta_k$. Define
\[
k_0 := \inf\{k \in \mathbb{Z}^+ | \sigma_{k+1} = \infty\}.
\]
Then (5.12) implies that
\[
P(k_0 \geq n) \leq (1 - p_0)^n.
\]
By (5.16), we obtain $k_0 < \infty$ a.s., which implies $l_0(\infty) < \infty$ a.s. Moreover, we have for $1 < q_2 < \frac{q}{2} - 1$,

$$E(l_0(\infty)^{q_2}) \leq \sum_{n=0}^{\infty} E(\delta_{n}^{q_2} 1_{\delta_n} \leq 1_{k_0 = n}).$$

Then by Hölder’s inequality, we have for $\frac{1}{p_1} + \frac{1}{p_1'} = 1$, $p_1, p_1' > 1$, satisfying $p_1 q_2 < \frac{q}{2} - 1$

$$E(l_0(\infty)^{q_2}) \leq \sum_{n=0}^{\infty} (E\delta_{n}^{q_2} 1_{\delta_n} \leq \infty)^{1/p_1} P(k_0 = n)^{1/p_1'}.$$

By (5.15) and (5.16), we obtain

$$E(l_0(\infty)^{q_2}) \leq C \left( \sum_{n=0}^{\infty} (n + 1)^{q_2} (1 - p_0)^{n/p_1'} \right) (1 + \|\theta_1^0\|_{L^{2n(p-1)}} + \|\theta_2^0\|_{L^{2n(p-1)}})^{1/p_1} < \infty.$$  

**Step 6. Conclusion.**

By Step 2 and Step 5, we have for $t > 0$ and $1 < q_2 < \frac{q}{2} - 1$

$$E(l_0(\infty)^{q_2}) \leq C \left( \sum_{n=0}^{\infty} (n + 1)^{q_2} (1 - p_0)^{n/p_1'} \right) (1 + \|\theta_1^0\|_{L^{2n(p-1)}} + \|\theta_2^0\|_{L^{2n(p-1)}})^{1/p_1} < \infty,$$

where we used Proposition 5.6 in the first inequality. Choosing $l = \left[t + \frac{1}{2}\right]$, we obtain for $1 < q_3 < \frac{q}{4} - 1$

$$E(l_0(\infty)^{q_2}) \leq C \left( \sum_{n=0}^{\infty} (n + 1)^{q_2} (1 - p_0)^{n/p_1'} \right) (1 + \|\theta_1^0\|_{L^{2n(p-1)}} + \|\theta_2^0\|_{L^{2n(p-1)}})^{1/p_1} < \infty.$$  

Thus, for $\psi \in C(H^1)$ with $C_\psi := \sup_{x, y \in H^1} |\phi(x) - \psi(y)| / |\lambda^{-1/2}(x - y)| < \infty$, we have

$$|P_\psi \phi(x) - P_\psi \psi(y)| \leq C \left( \sum_{n=0}^{\infty} (n + 1)^{q_2} (1 - p_0)^{n/p_1'} \right) (1 + \|\theta_1^0\|_{L^{2n(p-1)}} + \|\theta_2^0\|_{L^{2n(p-1)}})^{1/p_1} < \infty.$$  

By Proposition 5.6, we obtain that for $2 < p_2 < \infty$

$$E(\|\theta(t)\|_{L^{p_2}}^{p_2}) \leq \|x\|_{L^{p_2}}^{p_2} e^{-\lambda_1 t} + C \left[ \frac{1}{2} p_2(p_2 - 1) \right]^{p_2/2} \lambda_1^{-p_2/2} E_0^{p_2/2} (1 - e^{-\lambda_1 t}).$$
STOCHASTIC QUASI-GEOSTROPHIC EQUATION

Since for any invariant measure \( \mu \) on \( H^1 \) and any \( \varepsilon > 0 \), there exists \( b_\varepsilon > 0 \) such that \( \mu(x \in H^1 : \|x\|_{L^2} > b_\varepsilon) \leq \varepsilon \), we obtain that for any \( L > 0 \)

\[
\int (\|x\|_{L^2}^p + L) \, d\mu \leq \int (\|x\|_{L^2}^p \wedge b_\varepsilon) \, (E^x \|\theta(t)\|_{L^2}^p + L) \, d\mu + L \varepsilon
\]

\[
\leq b_\varepsilon e^{-\lambda_1 t} + C_S^p \left[ \frac{1}{2} p_2(p_2 - 1) \right]^{p_2/2} \lambda_1^{-p_2/2} C_0^{p_2/2} (1 - e^{-\lambda_1 t}) + L \varepsilon.
\]

Letting \( t \to \infty, \varepsilon \to 0 \) and \( L \to \infty \), we obtain that for any invariant measure \( \mu \)

\[
(5.19) \quad \int \|x\|_{L^2}^p \, d\mu(x) \leq C_S^p \left[ \frac{1}{2} p_2(p_2 - 1) \right]^{p_2/2} \lambda_1^{-p_2/2} C_0^{p_2/2}.
\]

Then by (5.18), (5.19) for any invariant measures \( \mu_1, \mu_2 \) we obtain for \( \psi \in C(H^1) \) with \( C_\psi < +\infty \) and \( 1 < q < \frac{q}{2} - 1 \),

\[
\left| \int \psi(x) \mu_1(dx) - \int \psi(x) \mu_2(dx) \right|
\]

\[
\leq CC_\psi \left[ 1 + \int \|x\|^{2m(p-1)}_{L^2} \mu_1(dx) + \int \|x\|^{2m(p-1)}_{L^2} \mu_2(dx) \right](t + 1)^{-q}.
\]

Letting \( t \to \infty \), we get that \( \mu_1 = \mu_2 \). \( \square \)

REMARK 5.10. (i) The coupling method has been introduced, for example, in [34–36, 40] to study ergodicity for stochastic partial differential equations. In these papers, they decompose the process into the sum of a strongly dissipative process \( h \) and another finite dimensional dynamics \( l \) driven by a nondegenerate noise. The process is uniquely determined by the nondegenerate part \( l \) which can be treated by probabilistic arguments. However, in our case, we cannot decompose the process into the two desired parts since the uniqueness of the process \( h \) depends on the \( L^p \)-norm estimate, which cannot be obtained for \( h \).

(ii) It is not clear how to directly use the results in [44] for the following two reasons: Although we consider the semigroup in \( H^1 \), the convergence we used in Theorem 5.9 is in \( H^{-1/2} \). In [44], only one state space has been considered. If we choose the general Hilbert space in [44] as \( H^1 \), we cannot get the estimate (5.5) for the \( H^1 \)-norm. If we choose the general Hilbert space in [44] as \( H^{-1/2} \), the estimate (5.5) does also not hold for rough initial values in \( H^{-1/2} \). The second reason is that, since Theorem 5.9 depends on the \( L^p \)-norm estimate, we can only prove \( E\|\theta_1(t, \theta_0^1, \theta_0^2) - \theta_2(t, \theta_0^1, \theta_0^2)\|_{H^{-1/2}} \) converges to zero polynomially fast instead of exponentially fast, when time goes to infinity, where \( \theta_1(t, \theta_0^1, \theta_0^2), \theta_2(t, \theta_0^1, \theta_0^2) \) denotes a coupling of two solutions to (3.1) with different initial values \( \theta_0^1 \in H^1 \), \( i = 1, 2 \).
In the situation of Theorem 5.9, we also obtain that \( P_t \) on \( H^1 \) is asymptotically strong Feller. In fact, for \( x, y \in H^1 \), define \( d_n(x, y) := 1 \wedge n|\Lambda^{-1/2}(x - y)| \). For any two probabilities on \( H^1 \times H^1 \) with marginals \( \mu_1 \) and \( \mu_2 \) by \( C(\mu_1, \mu_2) \). Define the Wasserstein distance

\[
\|\mu_1 - \mu_2\|_d := \inf_{\mu \in C(\mu_1, \mu_2)} \int_{H^1 \times H^1} d(x, y) \mu(dx, dy)
\]

By definition and (5.17), we obtain

\[
\|P_n(x, \cdot) - P_n(y, \cdot)\|_{d_n} \leq nE|\Lambda^{-1/2}(\theta_2(n, x, y) - \theta_1(n, x, y))|
\]

\[
\leq C(\|x\|_{L^{2m(p-1)}}, \|y\|_{L^{2m(p-1)}}) \mu_n^{-q_3}.
\]

Then we have

\[
\lim_{\gamma \to 0} \lim_{n \to \infty} \sup_{y \in B(x, \gamma)} \|P_n(x, \cdot) - P_n(y, \cdot)\|_{d_n} = 0,
\]

where \( B(x, \gamma) \) denotes the ball in \( H^1 \) with center \( x \) and radius \( \gamma \), which implies that \( P_t \) on \( H^1 \) is asymptotically strong Feller.

(iv) It seems difficult to directly verify the gradient estimate for the semigroup as [25] did for the 2D Navier–Stokes equation. By their method, we need to consider an infinitesimal perturbation to the initial condition and to estimate the derivative of the solution \( D\theta \) with respect to the initial value, which requires a good estimate for \( E|\theta|^p \). However, this cannot be obtained for \( \alpha > \frac{1}{2} \). Even if the noise is nondegenerate and we use the Bismut–Elworthy–Li formula to compute the gradient of the semigroup, the ergodicity results only holds for \( \alpha > \frac{3}{2} \) by delicate estimates (see Section 6). We cannot directly use the criterion in [31], since it is not clear how to verify the e-property in [31] for the semigroup associated with the 2D stochastic quasi-geostrophic equation.

5.3. Existence of invariant measures for \( \alpha > \frac{1}{2} \). Assume that \( G \) satisfies condition Hypothesis E.1.

**Lemma 5.11.** Let \( \alpha > \frac{1}{2} \). If \( \theta_0 \in H^1, t > 0 \), then

(i) \( E(|\theta(t)|^2) + E\int_0^t |\Lambda^\alpha \theta(r)|^2 dr \leq |\theta_0|^2 + t Tr[GG^*] \),

(ii) for \( \delta \leq 1 \) and \( q \geq \frac{2\alpha + 2}{2\alpha - 1} \), \( p \geq 1 \), we have

\[
E\int_0^t \frac{|\Lambda^{\delta + \alpha} \theta(r)|^2}{(1 + |\Lambda^\delta \theta(r)|^2)^{p+1}} dr \leq C\left( \int_0^t E\|\theta(r)\|_{L^q}^q dr + 1 \right) \leq Ct(\|\theta_0\|_{L^q}^q + 1),
\]

(iii) for \( q \geq \frac{2\alpha + 2}{2\alpha - 1} \), there exist \( 0 < \delta_1 < 1 - \alpha \) and \( 0 < \gamma_0 < 1 \) such that

\[
E\left[ \int_0^t |A_\Delta^{\delta_1} \theta(r)|^2_{H^1} dr \right] \leq C(1 + t)(\|\theta_0\|_{L^q}^q + 1).
\]
Then we apply Itô's formula to the function $(1 + |\Lambda^\delta \theta|^2)^{-p}$ and get
\[
\frac{1}{(1 + |\Lambda^\delta \theta(t)|^2)^p} - \frac{1}{(1 + |\Lambda^\delta \theta_0|^2)^p} = 2p \int_0^t \frac{|\Lambda^{\delta + \alpha} \theta|^2}{(1 + |\Lambda^\delta \theta|^2)^{p+1}} \, dt + 2p \int_0^t \frac{\langle \Lambda^{\delta - \alpha} (u \cdot \nabla \theta), \Lambda^{\delta + \alpha} \theta \rangle}{(1 + |\Lambda^\delta \theta|^2)^{p+1}} \, dt
\]
\[
- 2p \int_0^t \frac{\langle \Lambda^{\alpha} \theta, \Lambda^\delta G \, dW_t \rangle}{(1 + |\Lambda^\delta \theta|^2)^{p+1}} - p \int_0^t \text{Tr}(GG^\ast \Lambda^{2\delta}) \, dt
\]
\[
+ 2p(p + 1) \int_0^t \frac{|G^\ast \Lambda^{2\delta} \theta|^2}{(1 + |\Lambda^\delta \theta|^2)^{p+2}} \, dt,
\]
where the last term is meaningful since $|G^\ast \Lambda^{2\delta} \theta|^2 \leq |\Lambda^\delta \theta|^2 \|\Lambda^\delta G\|_{L^2(H,H)}^2$. For $q \geq \frac{2\alpha + 2}{2\alpha - 1}$ and $\sigma := \frac{2}{q} < 2\alpha - 1$, we have
\[
||\Lambda^{\delta - \alpha} (u \cdot \nabla \theta), \Lambda^{\delta + \alpha} \theta || = |\langle \Lambda^{\delta - \alpha} \nabla \cdot (u \theta), \Lambda^{\delta + \alpha} \theta \rangle|
\]
\[
\leq C |\Lambda^{\delta - \alpha + 1 + \sigma} \theta| \cdot \|\theta\|_{L^q} \|\Lambda^{\delta + \alpha} \theta\|
\]
\[
\leq C \|\theta\|^{2\alpha/(2\alpha - 1 - \sigma)} \|\Lambda^\delta \theta\|^2 + \kappa |\Lambda^{\delta + \alpha} \theta|^2,
\]
where we used $\text{div} u = 0$ in the first equality and Lemmas 2.1 and 2.2 in the first inequality and Young’s together with the interpolation inequality (2.3) in the last inequality.

Hence, we obtain
\[
E \int_0^t \frac{|\Lambda^{\delta + \alpha} \theta|^2}{(1 + |\Lambda^\delta \theta|^2)^{p+1}} \, dt \leq C \left( \int_0^t E \|\theta\|^q_{L^q} \, dt + t \right) \leq C t (\|\theta_0\|^q_{L^q} + 1),
\]
where we used Proposition 5.6 in the last step.

(iii) Since by Young’s inequality for some $\gamma_0 > 0$, we have
\[
|\Lambda^{\delta + \alpha} \theta|^{2\gamma_0} \leq C \left[ \frac{|\Lambda^{\delta + \alpha} \theta|^2}{(1 + |\Lambda^\delta \theta|^2)^{p+1}} + 1 + |\Lambda^\delta \theta|^2 \right],
\]
we obtain for $\delta + \alpha > 1$
\[
E \left[ \int_0^t |\Lambda^{\delta + \alpha} \theta|^{2\gamma_0} \, dt \right] \leq C (1 + t)(\|\theta_0\|^q_{L^q} + 1).
\]
THEOREM 5.12. Let $\alpha > \frac{1}{2}$ and suppose Hypothesis E.1 holds. Then $(P_t)_{t \geq 0}$ is $H^1$-Feller, that is, for every $t > 0$ and $\psi \in \mathcal{C}_b(H^1)$, $P_t \psi \in \mathcal{C}_b(H^1)$. Furthermore, there exists an invariant measure $\nu$ on $H^1$ of the transition semigroup $(P_t)_{t \geq 0}$. Moreover, there are $0 < \delta_1 < 1 - \alpha$ and $0 < \gamma_0 < 1$ such that

$$
\int |A^{\delta_1}_\alpha x|^{2\gamma_0}_{H^1} \, d\nu < \infty.
$$

Proof. Choose $x_0 \in H^1$ and define for $t > 0$

$$
\mu_t := \frac{1}{t} \int_0^t P_r^* \delta_{x_0} \, dr.
$$

By Lemma 5.11(iii), we have for $t > 1$ that

$$
\int |A^{\delta_1}_\alpha x|^{2\gamma_0}_{H^1} \mu_t(dx) \leq C.
$$

This implies that $(\mu_t | t > 0)$ is tight on $H^1$. By Theorem A.3 in Appendix A, we obtain that $(P_t)_{t \geq 0}$ is $H^1$-Feller. Hence, any limit point of $\mu_t$ is an invariant measure for $(P_t)_{t \geq 0}$. □

Combining Theorem 5.2.4 and Theorem 5.12, we obtain the following results.

THEOREM 5.13. Fix $\alpha > 1/2$. Assume Hypotheses E.1 and E.2 hold. Let $\delta_0 = \lambda_{N+1} - 2^{p/2} C_P^q C_R^{p+1} \kappa_1^{-p} [p(p-1)]^{p/2} L_2^{p/2} \mathcal{E}^{p/2} > 0$ for $p = \frac{\alpha+1}{\alpha+1/2}$, where $N$ is as in Hypothesis E.2, $C_S$, $C_R$ are the constants for Sobolev embedding and Riesz transform, respectively. Then there exists exactly one invariant probability measure $\nu$ for $P_t$.

Moreover, for $\psi \in C(H^1)$ with $C_\psi := \sup_{x,y \in H^1} \frac{|\psi(x) - \psi(y)|}{|x-y|^{2(p-1)}} < \infty$ and any initial distribution $\mu_0$ on $H^1$ with $\int \|x\|^{2m(p-1)} d\mu_0 < \infty$ for some $m > 35$, the following polynomial bound is satisfied for $1 < q_3 < \frac{m-19}{16}$:

$$
\left| \int P_t \psi(x) \mu_0(dx) - \int \psi(x) \nu(dx) \right| \leq CC_\psi \left( 1 + \int \|x\|^{2m(p-1)} \mu_0(dx) \right) (t + 1)^{-q_3}.
$$

Proof. (5.20) can be easily deduced from (5.18) and (5.19). □

5.4. Law of large numbers. In this section, we establish the law of large numbers for the solution of the stochastic quasi-geostrophic equation. The proof is mainly inspired by the approach used in [32].
THEOREM 5.14. Fix \( \alpha > 1/2 \). Assume Hypotheses E.1 and E.2 hold. Set
\[ \delta_0 := \lambda_{N+1} - 2^{p/2} C_R^p C_0^{2p} \kappa^{p-1}[p(p-1)]^{p/2} \lambda_1^{p/2} \xi_0^{p/2} > 0 \] for \( p = \frac{\alpha+1}{\alpha-1/2} \), where \( N \) is as in Hypothesis E.2, \( C_R, C_0 \) are the constants for the Sobolev embedding and Riesz transform, respectively. Then for \( \psi \in C(H^1) \) with \( C_\psi := \sup_{x,y \in H^1} \frac{||\psi(x)-\psi(y)||}{|\lambda_{1/2}(x-y)|} < \infty \) and any initial distribution \( \mu_0 \) on \( H^1 \) with
\[ \int \|x\|^{2m(p-1)} d\mu_0 < \infty \] for some \( m > 35 \),
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \psi(\theta(s)) ds = \int \psi d\nu \] in probability.

PROOF. (5.20) implies that for \( \psi \in C(H^1) \) with \( C_\psi < \infty \)
\[ \lim_{T \to \infty} \left| \frac{1}{T} \int_0^T E\left[\psi(\theta(t)) dt - \int \psi(x) \nu(dx)\right] \right| = 0. \] Now we want to prove that for bounded \( \psi \in C(H^1) \) with \( C_\psi < \infty \)
\[ \lim_{T \to \infty} \left| \frac{1}{T^2} E\left(\int_0^T \psi(\theta(t)) dt\right)^2 - \left(\int \psi(x) \nu(dx)\right)^2 \right| = 0. \]

We have
\[ \frac{1}{T^2} E\left(\int_0^T \psi(\theta(t)) dt\right)^2 = \frac{1}{T^2} E\left(\int_0^T \psi(\theta(t)) dt \int_0^T \psi(\theta(s)) ds\right) \]
\[ = \frac{2}{T^2} \int_0^T \int_0^T E\left[\psi(\theta(t))\psi(\theta(s))\right] dt ds \]
\[ = \frac{2}{T^2} \int_0^T \int_0^T \langle \mu_0 P_s, \psi P_{t-s} \psi \rangle dt ds. \]

Moreover, we have that for \( B := \{\|x\|^{2m(p-1)} \leq R\}, \)
\[ \left| \frac{2}{T^2} \int_0^T \int_0^T \left( \mu_0 P_s, \psi P_{t-s} \psi - \int \psi(x) \nu(dx) \right) \right| dt ds \]
\[ \leq \left| \frac{2}{T^2} \int_0^T \int_0^T \left( \mu_0 P_s, 1_B \psi P_{t-s} \psi - \int \psi(x) \nu(dx) \right) \right| dt ds \]
\[ + \left| \frac{2}{T^2} \int_0^T \int_0^T \left( \mu_0 P_s, 1_B \psi P_{t-s} \psi - \int \psi(x) \nu(dx) \right) \right| dt ds \]
\[ := I_T + II_T. \]

By (5.20), we obtain that there exists \( T_1 > 0 \) such that for any \( T > T_1 \)
\[ \sup_{x \in B} \left| \frac{1}{T} \int_0^T P_t \psi(x) - \int \psi d\nu \right| < \varepsilon. \]
Thus, for the first term we have the following:

\[
I_T = \left| \frac{2}{T^2} \int_0^T (T-s) \left( \mu_0 P_s, 1_B \psi \left[ \frac{1}{T-s} \int_0^{T-s} \left( P_t \psi - \int \psi(x) \nu(dx) \right) dt \right] \right) ds \right|
\]

\[
\leq \left| \frac{2}{T^2} \int_0^{T_1} s \left( \mu_0 P_{T-s}, 1_B \psi \left[ \frac{1}{s} \int_0^s \left( P_t \psi - \int \psi(x) \nu(dx) \right) dt \right] \right) ds \right|
\]

\[
+ \left| \frac{2}{T^2} \int_{T_1}^T s \left( \mu_0 P_{T-s}, 1_B \psi \left[ \frac{1}{s} \int_0^s \left( P_t \psi - \int \psi(x) \nu(dx) \right) dt \right] \right) ds \right|
\]

\[
\leq 4 \| \psi \|_{L^\infty}^2 \left( \frac{T_1}{T} \right)^2 + \varepsilon \| \psi \|_{L^\infty},
\]

where we used (5.23) in the last step. For the second term by Proposition 5.6, we have

\[
II_T \leq 4 \| \psi \|_{L^\infty}^2 \int_0^T \int_0^t \mu_0 P_s (B^c) ds \, dt
\]

\[
\leq \| \psi \|_{L^\infty}^2 \frac{C}{R}.
\]

Choosing \( R \) large enough, we obtain for any \( \varepsilon > 0 \) that there exists \( T_0 \) such that for \( T \geq T_0 \)

\[
\left| \frac{2}{T^2} \int_0^T \int_0^t \left( \mu_0 P_s, \psi \left( P_t \psi - \int \psi(x) \nu(dx) \right) \right) ds \, dt \right| \leq \varepsilon.
\]

The latter implies

\[
\lim_{T \to \infty} \left| \frac{1}{T^2} E \left( \int_0^T \psi(\theta(t)) dt \right)^2 - \left( \int \psi(x) \nu(dx) \right)^2 \right|
\]

\[
\leq \lim_{T \to \infty} \left| \frac{2}{T^2} \int \psi(x) \nu(dx) \int_0^T \int_0^t \left( \mu_0 P_s, \psi \right) dt \, ds - \left( \int \psi(x) \nu(dx) \right)^2 \right|
\]

\[
= \left| \int \psi(x) \nu(dx) \right| \lim_{T \to \infty} \left| \frac{2}{T^2} \int_0^T t \, dt \left[ \frac{1}{t} \int_0^t \left( \mu_0 P_s, \psi \right) ds - \int \psi(x) \nu(dx) \right] \right|
\]

\[
= 0.
\]

Now by (5.21) and (5.22) we obtain for bounded \( \psi \) with \( C_\psi < \infty \),

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \psi(\theta(s)) ds = \int \psi \, d\nu \quad \text{in probability.}
\]

In general, we can remove the restriction of the boundedness of \( \psi \) by defining

\[
\psi_L = \psi \wedge L \vee (-L) \text{ for } L \in \mathbb{R}^+. \text{ Since for } x, y \in H^1
\]

\[
|\psi_L(x) - \psi_L(y)| \leq |\psi(x) - \psi(y)| \leq C_\psi |x - y|,
\]
we have
\begin{equation}
\lim_{T \to \infty} \frac{1}{T} \int_0^T \psi_L(\theta(s)) \, ds = \int \psi_L \, d\nu \quad \text{in probability.}
\end{equation}
\label{eq:5.24}

Since \( \int |\psi| \, d\nu < \infty \), it is clear that
\begin{equation}
\lim_{L \to \infty} \int \psi_L \, d\nu = \int \psi \, d\nu.
\end{equation}

Applying \((5.21)\) for \(|\psi - \psi_L|\), we have
\begin{equation}
\lim_{L \to \infty} \lim_{T \to \infty} E \left[ \frac{1}{T} \int_0^T |\psi_L(\theta(s)) - \psi(\theta(s))| \, ds \right] = \lim_{L \to \infty} \int |\psi_L - \psi| \, d\nu = 0.
\end{equation}

Now the result follows by taking the limit on both sides of (5.24). \(\square\)

6. Exponential convergence for \(\alpha > \frac{2}{3}\). Under the conditions (Hypotheses E.1, E.2) on \(G\) we only obtain the semigroup converges to the invariant measure polynomially fast \cite[(5.20)]{[12]}). In this section, we prove that the convergence is exponentially fast, however, under stronger conditions for \(\alpha\) and \(G\). We assume that \(\alpha > \frac{2}{3}\), and that \(G\) satisfies:

**HYPOTHESIS E.3.** There are an isomorphism \(Q_0\) of \(H\) and a number \(s \geq 1\) such that \(G = A^{1/2}_\alpha Q_0^{1/2}\), and furthermore, \(G\) satisfies \((Gp.1)\) for some fixed \(p \in ((\alpha - \frac{1}{2})^{-1}, \infty)\) (which is, e.g., always the case if \(Q_0 = I\)).

For \(x := \theta_0 \in L^p\), let \(P_x\) denote the law of the corresponding solution \(\theta(\cdot, x)\) to (3.1). Since Hypothesis G.1, \((Gp.1)\) and \((GL.1)\) are satisfied under Hypothesis E.3, by Theorems 4.3 and 4.4 the measures \(P_x, x \in L^p\), form a Markov process. Let \((P_t)_{t \geq 0}\) be the associated transition semigroup on \(B_b(H)\), defined as
\begin{equation}
P_t(\varphi)(x) := E[\varphi(\theta(t, x))], \quad x \in L^p, \varphi \in B_b(H).
\end{equation}

**REMARK 6.1.** If Hypothesis E.3 is satisfied with \(s > 3 - 2\alpha\), then Hypotheses E.1, E.2 hold for \(G\) and \((Gp.1)\) holds for any \(p \in (0, \infty)\).

6.1. **The strong Feller property for \(\alpha > \frac{2}{3}\).** In this subsection, we prove that its transition semigroup has the strong Feller property under Hypothesis E.3.

**REMARK 6.2.** (i) Since in our case \(\alpha < 1\), the linear part \((-\Delta)^\alpha\) in (1.1) is less regularizing. As \(G = A^{1/2}_\alpha Q_0^{1/2}\), we get the trajectories \(z\) of the associated O–U process to be in \(C([0, \infty), H^{s+2\alpha-1-\delta_0})\) for every \(\delta_0 > 0\) \cite[(cf. [12], Theorem 5.16, [16], Proposition 3.1)]{[12]}. However, in order to prove the weak-strong uniqueness principle (see Theorem 6.4 below) and the strong Feller property of the semigroup associated with the solution of the cutoff equation (see Proposition 6.5...
below), we need \( z \in C([0, \infty), H^{s+1-\alpha+\sigma_1}) \) for some \( \sigma_1 > 0 \). Therefore, we need \( s + 2\alpha - 1 > s + 1 - \alpha \), that is, \( \alpha > 2/3 \). The situation of the 3D Navier–Stokes equation is different. While in our case the needed regularity of \( z \) is higher than the regularity of our solution space \( C([0, \infty), H^s) \) for the cutoff equation (6.2), for the 3D Navier–Stokes equation the needed regularity of \( z \) is the same as for the solution of the cutoff equation.

(ii) Since \( \alpha < 1 \), we cannot apply the same type of estimate as in [22] (cf. [22], Lemma D.2). Instead, we use Lemma 2.1 and choose suitable parameters \((s, \sigma_1, \sigma_2)\) such that the approach in [22] can be modified to apply here [see (6.6)–(6.10) and so on].

(iii) It seems difficult to use the Kolmogorov equation method as in [11, 16] or a coupling approach as in [43] in our situation. In fact, to get a uniform \( H^s \)-norm estimate for the solutions of the Galerkin approximations of equation (1.1) for some \( s > 0 \), the regularity, needed for the trajectories of the associated Ornstein–Uhlenbeck (O–U) process \( z \) is higher than \( H^s \), which is entirely different from the situation of the 3D Navier–Stokes equation. According to the method in [11, 16] and [43], we should use the solutions’ \( H^{s+\alpha} \)-norm to control the \( H^{s+\alpha} \)-norm of the derivative of the solutions as required for the Bismut–Elworthy–Li formula. In particular, the associated O–U process \( z \) should be also in \( H^{s+\alpha} \). However, under Hypothesis E.3 for the noise, the O–U process \( z \) is only in \( L^2([0, T], H^{s+2\alpha-1}) \).

As a consequence, for their method to apply here, we need even \( \alpha \geq 1 \).

Fix \( s > 1 \) as in Hypothesis E.3 and set \( \mathcal{W} := H^s \) and \( |x|_{\mathcal{W}} := \|x\|_{H^s} \). In this subsection, we choose

\[
\Omega := C([0, \infty); H^{-\beta})
\]

for some \( \beta > 3 \) and let \( \mathcal{B} \) denote the Borel \( \sigma \)-algebra on \( \Omega \).

Now we state the main result of this section.

**THEOREM 6.3.** Fix \( \alpha > \frac{2}{3} \). Under Hypothesis E.3, \((P_t)_{t \geq 0}\) is \( \mathcal{W} \)-strong Feller, that is, for every \( t > 0 \) and \( \psi \in \mathcal{B}_b(H) \), \( P_t \psi \in C_b(\mathcal{W}) \).

We shall use [22], Theorem 5.4, which is an abstract result to prove the strong Feller property. In order to use [22], Theorem 5.4, we follow the idea of [22], Theorem 5.11, to construct \( P^R \). We introduce an equation which differs from the original one by a cut-off only, so that with large probability they have the same trajectories on a small random time interval [see (6.3) below]. We consider the equation

\[
(6.2) \quad d\theta(t) + A\alpha \theta(t) \, dt + \chi_R(|\theta|^2_{\mathcal{W}^2}) u(t) \cdot \nabla \theta(t) \, dt = G \, dW(t),
\]

where \( \chi_R : \mathbb{R} \to [0, 1] \) is of class \( C^\infty \) such that \( \chi_R(|\theta|) = 1 \) if \( |\theta| \leq R \), \( \chi_R(|\theta|) = 0 \) if \( |\theta| > R + 1 \) and with its first derivative bounded by 1. Then, if we can prove the following Theorem 6.4 and Proposition 6.5, Theorem 6.3 follows.
THEOREM 6.4 (Weak–strong uniqueness). Fix \( \alpha > \frac{2}{3} \). Suppose Hypothesis E.3 holds. Then for every \( x \in \mathcal{W} \), equation (6.2) has a unique martingale solution \( P_\alpha^{(R)} \), with

\[
P_\alpha^{(R)}[C([0, \infty); \mathcal{W})] = 1.
\]

Let \( \tau_R : \Omega \to [0, \infty] \) be defined by

\[
\tau_R(\omega) := \inf\{t \geq 0 : |\omega(t)|_{\mathcal{W}}^2 \geq R\},
\]

and \( \tau_R(\omega) := \infty \) if this set is empty. If \( x \in \mathcal{W} \) and \( |x|_{\mathcal{W}}^2 < R \), then

\[
\lim_{{\varepsilon \to 0}} P_\alpha^{(x+h)}[\tau_R \geq \varepsilon] = 1, \quad \text{uniformly in } h \in \mathcal{W}, |h|_{\mathcal{W}} < 1.
\]

Moreover

\[
E P_\alpha^{(R)}[\varphi(\omega_t) 1_{[\tau_R \geq t]}] = E P_\alpha^{(R)}[\varphi(\omega_t) 1_{[\tau_R \geq t]}]
\]

for every \( t \geq 0 \) and \( \varphi \in \mathcal{B}_b(H) \).

PROOF. Let \( \Omega^* \) denote the solution to

\[
dz(t) + A_\alpha z(t) \, dt = G \, dW(t),
\]

with initial data \( z(0) = 0 \) and let \( v^{(R)}_\alpha \) be the solution to the auxiliary problem

\[
\left(\frac{dv^{(R)}_\alpha}{dt} + A_\alpha v^{(R)}_\alpha(t) + u^{(R)}_\alpha(t) \cdot \nabla (v^{(R)}_\alpha(t) + z(t))) \chi_R(|v^{(R)}_\alpha + z|^2_{\mathcal{W}}) = 0, \right.
\]

with \( v^{(R)}_\alpha(0) = x \). Here, \( u^{(R)}_\alpha(t) = u^{(R)}_\alpha(t) + u^{(R)}_\alpha(t) \), \( u^{(R)}_\alpha \) and \( u^{(R)}_\alpha \) satisfy (1.3) with \( \theta \) replaced by \( v^{(R)}_\alpha \) and \( z \), respectively. Moreover, define \( \theta^{(R)} := v^{(R)}_\alpha + z \), which is a weak solution to equation (6.2). We denote its law on \( \Omega \) by \( P_\alpha^{(R)} \). By Hypothesis E.3, the trajectories of the noise belong to

\[
\Omega^* := \bigcap_{\beta \in (0, 1/2), \eta \in [0, (s+\alpha)/(2\alpha) - 1/(2\alpha))] C^\beta([0, \infty); D(A_\alpha^\eta))
\]

with probability one. Hence, the analyticity of the semigroup generated by \( A_\alpha \) implies that for each \( \omega \in \Omega^* \), \( z(\omega) \in C([0, \infty), H^{s+2\alpha-1/\varepsilon_0}) \) for every \( \varepsilon_0 > 0 \).

Now, for \( \omega \in \Omega^* \) we prove that equation (6.5) with \( z(\omega) \) replacing \( z \) has a unique global weak solution in the space \( C([0, \infty); \mathcal{W}) \). First, we obtain the following a priori estimate for suitable \( \sigma_1, \sigma_2 > 0 \) with \( \sigma_2 \leq s, \sigma_2 + \sigma_1 = 1, s + \sigma_1 - \alpha + 1 < s + 2\alpha - 1 < s + \alpha \), where we used that \( \alpha > \frac{2}{3} \) since \( 0 < \sigma_1 < 3\alpha - 2 \):

\[
\frac{1}{2} \frac{d}{dt} |\Lambda^{s} v^{(R)}_\alpha|^2 + \kappa |\Lambda^{s+\alpha} v^{(R)}_\alpha|^2
\]

\[
= \chi_R(|\theta^{(R)}_\alpha|^2_{\mathcal{W}}) |\Lambda^{s-\alpha} \nabla \cdot (u^{(R)}_\alpha \theta^{(R)}_\alpha) + \Lambda^{s+\alpha} v^{(R)}_\alpha|
\]
\[
\leq C \chi_R (| \theta (R) |^2_{L^\infty}) \left| \Lambda^{s-\alpha+1} \left( u (R) \theta (R) \right) \right| \cdot | \Lambda^{s+\alpha} v (R) |
\]
\[
\leq C \chi_R (| \theta (R) |^2_{L^\infty}) \left| \Lambda^{s-\alpha+1+\sigma_1} \theta (R) \right| \left| \Lambda^{\sigma_2} \theta (R) \right| \cdot | \Lambda^{s+\alpha} v (R) |
\]
\[
\leq C \chi_R (| \theta (R) |^2_{L^\infty}) \left| \Lambda^{s-\alpha+1+\sigma_1} v (R) \right| \left| \Lambda^{s+\alpha} v (R) \right| \cdot | \Lambda^{s+\alpha} v (R) |
\]
\[
\leq C \chi_R (| \theta (R) |^2_{L^\infty}) (| \Lambda^s v (R) |^2 + | \Lambda^{s-\alpha+1+\sigma_1} z |^2) + \frac{K}{2} | \Lambda^{s+\alpha} v (R) |^2
\]
\[
\leq C \chi_R (| \theta (R) |^2_{L^\infty}) (C + | \Lambda^{s-\alpha+1+\sigma_1} z |^2) + \frac{K}{2} | \Lambda^{s+\alpha} v (R) |^2,
\]

where \( r_1 := \frac{1-\alpha+\sigma_1}{\alpha} \). Here, in the first equality, we used \( \text{div} u = 0 \), and in the second inequality we used Lemmas 2.1 and 2.2, and in the fourth inequality we used the interpolation inequality (2.3) and that \( s-\alpha+1+\sigma_1 < s+2\alpha-1 \), and in the fifth inequality we used Young’s inequality and in the last inequality we used \( \| \Lambda^s v (R) \| \leq | \Lambda^s \theta (R) | + | \Lambda^{s-\alpha+1+\sigma_1} z | \). Then as in the proof of Theorem A.1 in Appendix A, we prove (6.5) has a weak solution in \( L^\infty ([0, T], \mathcal{W}) \cap L^2 ([0, T], H^{s+\alpha}) \).

\textbf{Continuity.} For each \( \omega \in \Omega^\ast \), \( \sigma_1 \) and \( \sigma_2 \) as in (6.6), since \( s-\alpha+1+\sigma_1 < s+2\alpha-1 \), we have \( z \in C ([0, \infty); H^{s-\alpha+1+\sigma_1}) \). Since \( s > 3-3\alpha \), multiplying the equations (6.5) by \( \frac{d}{dt} \Lambda^{2(s-\alpha)} u (R) \), we obtain
\[
\frac{\kappa}{2} \frac{d}{dt} | \Lambda^s v (R) |^2 + | \Lambda^{s-\alpha} \dot{v} (R) |^2
\]
\[
= C \chi_R (| \theta (R) |^2_{L^\infty}) | \Lambda^{s-\alpha} \nabla \cdot (u (R) \theta (R), \Lambda^{s-\alpha} \dot{v} (R)) |
\]
\[
\leq C \chi_R (| \theta (R) |^2_{L^\infty}) | \Lambda^{s-\alpha+1} (u (R) \theta (R)) | \cdot | \Lambda^{s-\alpha} \dot{v} (R) |
\]
\[
\leq C \chi_R (| \theta (R) |^2_{L^\infty}) | \Lambda^{s-\alpha+1+\sigma_1} \theta (R) | | \Lambda^{\sigma_2} \theta (R) | \cdot | \Lambda^{s-\alpha} \dot{v} (R) |
\]
\[
\leq C \chi_R (| \theta (R) |^2_{L^\infty}) (| \Lambda^{s+\alpha} v (R) |^2 + | \Lambda^s v (R) |^2 + | \Lambda^{s-\alpha+1+\sigma_1} z |^2)
\]
\[
+ \frac{1}{2} | \Lambda^{s-\alpha} \dot{v} (R) |^2.
\]
Here, \( \dot{v} (R) = \frac{dv (R)}{dt} \) and in the first equality we used \( \text{div} u = 0 \), in the second inequality we used Lemmas 2.1 and 2.2, and in the third inequality we used the interpolation inequality (2.3), that \( s-\alpha+1+\sigma_1 \leq s+\alpha \) and Young’s inequality.

As \( \int_0^T | \Lambda^{s+\alpha} v (R) (t) |^2 \, dt \) can be dominated by (6.6), we get an a priori estimate for the time derivative \( \frac{d}{dt} v (R) \) in \( L^2 (0, T; H^{s-\alpha}) \). Then by [53], we obtain \( v (R) \in C ([0, T], \mathcal{W}) \).
Uniqueness. Let $v_1, v_2$ be two solutions of equation (6.5) in $C([0, \infty); \mathcal{W})$ and set $w := v_1 - v_2$ and $u_w := u_1 - u_2$, where $u_1, u_2$ satisfy (1.3) with $\theta$ replaced by $\theta_1 = v_1 + z, \theta_2 = v_2 + z$. Then by a similar argument as in the proof of Theorem 4.2, we have for small $0 < \varepsilon_1 < (2\alpha - 1 - \sigma_1) \wedge \sigma_1$ with $\sigma_1$ as in (6.6)

$$\frac{1}{2} \frac{d}{dt} |\Lambda^{s-\alpha} w|^2 + \kappa |\Lambda^s w|^2 \leq - (\chi_R(|\theta_1|^2_{\mathcal{W}}) - \chi_R(|\theta_2|^2_{\mathcal{W}}))(\Lambda^{s+\varepsilon_1 - 2\alpha} (u_1 \cdot \nabla \theta_1), \Lambda^{s-\varepsilon_1} w)$$

$$- \chi_R(|\theta_2|^2_{\mathcal{W}})(\Lambda^{s-2\alpha} (u_1 \cdot \nabla w), \Lambda^s w)$$

$$- \chi_R(|\theta_2|^2_{\mathcal{W}})(\Lambda^{s-2\alpha} (u_w \cdot \nabla \theta_2), \Lambda^s w)$$

$$= I + II + III.$$ 

As

$$|\chi_R(|\theta_1|^2_{\mathcal{W}}) - \chi_R(|\theta_2|^2_{\mathcal{W}})| \leq C(R) |w|_{\mathcal{W}} [1_{[0,R+1]}(|\theta_1|^2_{\mathcal{W}}) + 1_{[0,R+1]}(|\theta_2|^2_{\mathcal{W}})],$$

we get for $\sigma_1, \sigma_2$ as in (6.6),

$$I \leq - (\chi_R(|\theta_1|^2_{\mathcal{W}}) - \chi_R(|\theta_2|^2_{\mathcal{W}}))(\Lambda^{s+\varepsilon_1 - 2\alpha} \nabla \cdot (u_1 \theta_1), \Lambda^{s-\varepsilon_1} w)$$

$$\leq C[1_{[0,R+1]}(|\theta_1|^2_{\mathcal{W}}) + 1_{[0,R+1]}(|\theta_2|^2_{\mathcal{W}})]$$

$$\times |w|_{\mathcal{W}}|\Lambda^{s-2\alpha+\varepsilon_1+1+\sigma_1 \theta_1}| |\Lambda^{\sigma_2 \theta_1}| |\Lambda^{s-\varepsilon_1} w|$$

$$\leq C(R, |\theta_1|_{\mathcal{W}}, |\theta_2|_{\mathcal{W}}) |w|_{\mathcal{W}} |\Lambda^{s-\varepsilon_1} w|$$

$$\leq C(R, |\theta_1|_{\mathcal{W}}, |\theta_2|_{\mathcal{W}}) |\Lambda^{s-\alpha} w|^2 + \frac{\kappa}{4} |w|^2_{\mathcal{W}},$$

where in the first equality we used $\text{div} u_1 = 0$ and in the first inequality we used Lemmas 2.1 and 2.2, in the second inequality we used that $s - 2\alpha + \varepsilon_1 + 1 + \sigma_1 < s$, that is, $\varepsilon_1 < 2\alpha - 1 - \sigma_1$ and in the third inequality we used the interpolation inequality (2.3) and Young’s inequality. In a similar way, we obtain

$$II \leq |\Lambda^s w| |\Lambda^{s-2\alpha+1}(u_1 w)|$$

$$\leq C |\Lambda^s w| (|\Lambda^{s-2\alpha+1+\sigma_1 \theta_1}| |\Lambda^{s-\varepsilon_1} w| + |\Lambda^{s-2\alpha+1+\sigma_1} w||\Lambda^{s-\varepsilon_1} w|)$$

$$\leq C(R, |\theta_1|_{\mathcal{W}}) |\Lambda^{s-\alpha} w|^2 + \frac{\kappa}{4} |w|^2_{\mathcal{W}},$$

where in the first inequality we used $\text{div} u_1 = 0$ and in the second inequality we used Lemmas 2.1 and 2.2 and $s - \varepsilon_1 \geq 1 - \sigma_1$, and in the third inequality we used the interpolation inequality (2.3) and Young’s inequality. Similarly,

$$III \leq C(R, |\theta_2|_{\mathcal{W}}) |\Lambda^{s-\alpha} w|^2 + \frac{\kappa}{4} |w|^2_{\mathcal{W}}.$$
Then we obtain

\[
\frac{1}{2} \frac{d}{dt} |\Lambda^{s-\alpha} w|^2 + \kappa |\Lambda^s w|^2 \leq C \left( R, \sup_{t \in [0,T]} |\theta_1(t)|_{W}, \sup_{t \in [0,T]} |\theta_2(t)|_{W} \right) |\Lambda^{s-\alpha} w|^2 + \frac{3\kappa}{4} |w|^2_{W}.
\]

Gronwall’s lemma now yields that \(|\Lambda^{s-\alpha} w| = 0\), which implies \(w = 0\).

So, equation (6.5) has a unique global weak solution in the space \(C([0, \infty); W)\).

Next, we prove (6.3). In order to do so, it is sufficient to show that \(P(R) x[\tau_R < \varepsilon] \leq C(\varepsilon, R)\) for all \(x \in W\), with \(|x|^2_{W} \leq R^8\). So, fix \(\varepsilon > 0\) small enough, let \(\Theta_{\varepsilon, R} := \sup_{t \in [0,\varepsilon]} |\Lambda^{s-\alpha+1+\sigma_1} - \alpha z(t)|\) and assume that \(\Theta_{\varepsilon, R}^2 \leq \frac{R}{8}\).

Setting \(\phi(t) := |v^{(R)}|^2_{W} + \Theta_{\varepsilon, R}^2\), by (6.6) we get \(\dot{\phi} \leq C(R)\). This implies, together with the bounds on \(x\) and \(\Theta_{\varepsilon, R}\), that \(\sup_{t \in [0,\varepsilon]} |\theta^{(R)}(t)|^2_{W} \leq R\) for \(\varepsilon\) small enough. It follows that \(\tau_R \geq \varepsilon\). Hence,

\[
P^{(R)}(\tau_R < \varepsilon) \leq P^{(R)} \left[ \sup_{t \in [0,\varepsilon]} |\Lambda^{s+1+\sigma_1} - \alpha z(t)|^2 > \frac{R}{8} \right].
\]

Letting \(\varepsilon \downarrow 0\), we have \(P^{(R)}(\tau_R < \varepsilon) \to 0\), and the claim is proved, since the probability above is independent of \(x\).

Finally, the same arguments as in the proof of Theorem 4.2 imply that

\[
\theta(t \wedge \tau_R(\theta^{(R)})) = \theta^{(R)}(t \wedge \tau_R(\theta^{(R)})) \quad \forall t, P\text{-a.s.}
\]

Moreover, since \(\theta\) is \(H\)-valued weakly continuous, we obtain \(\tau_R(\theta^{(R)}) = \tau_R(\theta)\).

In order to apply [22], Theorem 5.4, we now only need the following result.

**Proposition 6.5.** Fix \(\alpha > \frac{2}{3}\). Suppose Hypothesis E.3 holds. For every \(R > 0\), the transition semigroup \((P^{(R)}_t)_{t \geq 0}\) associated to equation (6.2) is \(W\)-strong Feller.

**Proof.** We shall provide formal estimates that can, however, be made rigorous through Galerkin approximations. Let \((\Sigma, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space, \((W_t)_{t \geq 0}\) a cylindrical Wiener process on \(H\) and, for every \(x \in W\), let \(\theta^{(R)}_x\) be the solution to equation (6.2) with initial value \(x \in W\). By the Bismut–Elworthy–Li formula,

\[
D_x(P^{(R)} \psi)(x) = \frac{1}{t} E^\mathbb{P} \left[ \psi(\theta^{(R)}_x(t)) \int_0^t (G^{-1} D_x \theta^{(R)}_x(l), dW(l)) \right].
\]
where $D_y(P_t^R \psi)$ denotes $\langle D(P_t^R \psi), y \rangle$ for $y \in H$, $D_y \theta_x^R = D \theta_x^R \cdot y$ and $D \theta_x^R$ denotes the derivative of $\theta_x^R$ with respect to the initial value. Then for $\| \psi \|_\infty \leq 1$, by the B–D–G inequality

$$\left| \langle P_t^R \psi \rangle(x_0 + h) - \langle P_t^R \psi \rangle(x_0) \right| \leq \frac{C}{t} \sup_{\eta \in [0,1]} E^\theta \left[ \left( \int_0^t |G^{-1} D_h \theta_x^R(x_0 + \eta h(l))|^2 \, dl \right)^{1/2} \right].$$

The proposition is proved once we prove that the right-hand side of the above inequality converges to 0 as $|h|_V \to 0$.

Fix $x \in V$, $h \in H$ and write $\theta = \theta_x^R$, $v = v^R$, $u = u^R$, $D \theta = D_h \theta$ for simplicity. The term $D \theta$ solves the following equation:

$$\frac{d}{dt} D \theta + \kappa \Lambda^{2s} D \theta = - \chi_R(\theta^2_W) [D u \cdot \nabla \theta + u \cdot \nabla D \theta] + 2 \chi'_R(\theta^2_W)(\theta, D \theta) W_u \cdot \nabla \theta,$$

with initial value $D \theta(0) = h$ and $D u$ satisfying (1.3) with $\theta$ replaced by $D \theta$. Multiplying the above equation with $\Lambda^{2s} D \theta$ and taking the inner product in $L^2$, we have

$$\frac{1}{2} \frac{d}{dt} |\Lambda^{s} D \theta|^2 + \kappa |\Lambda^{s+\alpha}(D \theta)|^2 = - \left[ \chi_R(\theta^2_W) [D u \cdot \nabla \theta + u \cdot \nabla D \theta] + 2 \chi'_R(\theta^2_W)(\theta, D \theta) W_u \cdot \nabla \theta \right].$$

For the first term on the right-hand side, we have for $|\theta^2_W| \leq R$

$$\langle D u \cdot \nabla \theta, \Lambda^{2s} D \theta \rangle = \langle |\Lambda^{s-\alpha} \nabla \cdot (D u \theta), \Lambda^{s+\alpha} D \theta \rangle \leq C |\Lambda^{s-\alpha+1+\sigma_1} \theta| \cdot |\Lambda^{\sigma_2} D \theta| \cdot |\Lambda^{s+\alpha} D \theta|$$

$$+ C |\Lambda^{s-\alpha+1+\sigma_1} D \theta| \cdot |\Lambda^{\sigma_2} \theta| \cdot |\Lambda^{s+\alpha} D \theta| \leq \epsilon |\Lambda^{s+\alpha} D \theta|^2$$

$$+ C (C(R) + |\Lambda^{s+\alpha} v|^2 + |\Lambda^{s-\alpha+1+\sigma_1} z|^2)|\Lambda^{s} D \theta|^2$$

for $\sigma_1, \sigma_2$ as (6.6), where we used div $D u = 0$ in the first equality and Lemmas 2.1 and 2.2 in the first inequality as well as the interpolation inequality (2.3) and Young’s inequality in the second inequality.

The second term can be estimated similarly. For the third term, by Lemmas 2.1 and 2.2, we have

$$\langle u \cdot \nabla \theta, \Lambda^{2s} D \theta \rangle = \langle |\Lambda^{s-\alpha} \nabla \cdot (u \theta), \Lambda^{s+\alpha} D \theta \rangle \leq C |\Lambda^{s-\alpha+1+\sigma_1} \theta||\Lambda^{\sigma_2} \theta| \cdot |\Lambda^{s+\alpha} D \theta|$$

$$\leq C (|\Lambda^{s+\alpha} v| + |\Lambda^{s-\alpha+1+\sigma_1} z|)|\Lambda^{s} \theta| \cdot |\Lambda^{s+\alpha} D \theta|$$

The proposition is proved once we prove that the right-hand side of the above inequality converges to 0 as $|h|_V \to 0$.
where in the first equality we used \( \text{div} u = 0 \). Then we obtain

\[
\frac{1}{2} \frac{d}{dt} |\Lambda^s D\theta|^2 + \kappa |\Lambda^{s+\alpha}(D\theta)|^2 \\
\leq \frac{\kappa}{2} |\Lambda^{s+\alpha}(D\theta)|^2 + C(C(R) + |\Lambda^{s+\alpha} v|^2 + |\Lambda^{s-\alpha+1+\sigma_1} z|^2) |\Lambda^s D\theta|^2.
\]

From Gronwall’s inequality and (6.6), we finally get

\[
\int_0^t |\Lambda^{s+\alpha}(D\theta(l))|^2 dl \\
\leq C |\Lambda^s h|^2 + \exp\left( C \int_0^t (C(R) + |\Lambda^{s+\alpha} v|^2 + |\Lambda^{s-\alpha+1+\sigma_1} z|^2) dl \right) |\Lambda^s h|^2 \\
\leq C |\Lambda^s h|^2 + \exp\left( C \left( |\Lambda^s x|^2 + \int_0^t (C(R) + |\Lambda^{s-\alpha+1+\sigma_1} z|^2) dl \right) \right) |\Lambda^s h|^2.
\]

Since by \( s - \alpha + 1 + \sigma_1 < s + 2\alpha - 1 \), \( z \) is a Gaussian random variable in \( C([0, \infty); H^{s-\alpha+1+\sigma_1}) \) (cf. [10], Proposition 2.15), by Fernique’s theorem we could choose \( t_0 \) small enough and obtain

\[
E \int_0^{t_0} |\Lambda^{s+\alpha}(D\theta(l))|^2 dl \leq c(t_0, R) |\Lambda^s h|^2,
\]

which, as \( G^{-1} = Q_0^{-1/2} \Lambda^{s+\alpha} \), implies the assertion for \( t_0 \). For general \( t \), by the semigroup property the assertion follows easily. \( \square \)

6.2. A support theorem for \( \alpha > 2/3 \). A Borel probability measure \( \mu \) on \( H \) is fully supported on \( \mathcal{W} \) if \( \mu(U) > 0 \) for every nonempty open set \( U \subset \mathcal{W} \). Set \( \mathcal{W}_1 := H^{s-\alpha+1+\sigma_1} \), where \( \sigma_1 \) is the same as (6.6) and we will use it below.

**Lemma 6.6 (Approximate controllability).** Let \( R > 0, T > 0 \). Let \( x \in \mathcal{W} \) and \( y \in \mathcal{W} \), with \( A_\alpha y \in \mathcal{W}_1 \), such that

\[
|x|_\mathcal{W}^2 \leq \frac{R}{2}, \quad |y|_\mathcal{W}^2 \leq \frac{R}{2}.
\]

Then there exist (a control function) \( \omega \in \text{Lip}([0, T]; \mathcal{W}_1) \) and \( \theta \in C([0, T]; \mathcal{W}) \cap L^2([0, T]; H^{s+\alpha}) \), such that \( \theta \) solves the equation

\[
\theta(t) - x + \int_0^t A_\alpha \theta(r) + \chi_R(|\theta|^2_\mathcal{W}) u(r) \cdot \nabla \theta(r) \, dr \\
= \omega(t) \quad dt \cdot \text{a.e. } t \in [0, T],
\]

(6.11) \( \theta(0) = x \) and \( \theta(T) = y \), and

\[
\sup_{t \in [0, T]} |\theta(t)|^2_\mathcal{W} \leq R.
\]

(6.12)
STOCHASTIC QUASI-GEOSTROPHIC EQUATION

PROOF. First consider \( \omega = 0 \). By similar arguments as in Theorems A.1 and A.2, there exist a unique solution \( \theta \in C([0, T], \mathcal{W}) \). Then by a similar calculation as (6.6), we get

\[
\frac{d}{dt} |\theta|^2_{\mathcal{W}} + \kappa |\Lambda^{\alpha+s}\theta|^2_{\mathcal{W}} \leq C(R).
\]

Hence, \( \theta(t) \in H^{s+\alpha} \) for almost every \( t \in [0, T] \) and, by solving again the equation with one of these regular points as initial condition and using Lemmas 2.1 and 2.2 we have

\[
\frac{d}{dt} |\Lambda^{\alpha+s}\theta|^2 + \kappa |\Lambda^{2\alpha+s}\theta|^2 = \chi_R(|\theta|^2_{\mathcal{W}})(\Lambda^{s+1}\cdot (u\theta), \Lambda^{2\alpha+s}\theta)
\]

\[
\leq C \chi_R(|\theta|^2_{\mathcal{W}}) |\Lambda^{2\alpha+s}\theta||\Lambda^{s+\alpha}\theta||\theta||_{L^p}
\]

\[
\leq C(R) |\Lambda^{s+\alpha}\theta|^2 + \frac{\kappa}{2} |\Lambda^{s+2\alpha}\theta|^2,
\]

where \( \sigma_3 = \frac{2}{3} < 2\alpha - 1 \) and we used div \( u = 0 \) in the first equality and \( H^s \subset L^p \) and the interpolation inequality (2.3), Young’s inequality in the last step. Then by a boot strapping argument, we find a small \( T_\ast \in (0, \frac{1}{\epsilon}) \) such that \( |\theta(t)|^2_{\mathcal{W}} \leq R \) and \( A_\alpha \theta(T_\ast) \in \mathcal{W}_1 \) for all \( t \leq T_\ast \). Define \( \theta \) to be the solution above for \( t \in [0, T_\ast] \) and extended by linear interpolation between \( y \) and \( \theta(T_\ast) \) in \([T_\ast, T]\). Then obviously (6.12) follows.

Next, if we set

\[
\eta := \partial_t \theta + A_\alpha \theta + \chi_R(|\theta|^2_{\mathcal{W}})u \cdot \nabla \theta, \quad T_\ast \leq t \leq T,
\]

\( \omega := 0 \) for \( t \leq T_\ast \) and \( \omega(t) = \int_{T_\ast}^t \eta_s \, ds \) for \( t \in [T_\ast, T] \), we also have (6.11). It remains to prove that \( \eta \in L^\infty(0, T; \mathcal{W}_1) \). For the first two terms of \( \eta \), this is obvious.

For the nonlinear term, we have that

\[
|u \cdot \nabla \theta|_{\mathcal{W}_1} = |\nabla \cdot (u\theta)|_{\mathcal{W}_1} \leq C |\Lambda^{2\alpha}\theta|^2_{\mathcal{W}_1}
\]

for any \( \theta \in \mathcal{W}_1 \), where in the first equality we used div \( u = 0 \) and in the last step we used Lemma 2.1. \( \square \)

Let \( l \in (0, \frac{1}{2}) \) and \( p > 1 \) such that \( l - \frac{1}{p} > 0 \). Under Hypothesis E.3, we see that for every \( \alpha_1 < \frac{s+\alpha-1}{2\alpha} \) the map

\[
\omega \mapsto z(\cdot, \omega) : W^{l, p}([0, T]; D(\Gamma^{g_i})) \rightarrow C([0, T]; D(\Gamma^{g_i+\frac{1}{p} - \epsilon}))
\]

is continuous, for all \( \epsilon > 0 \) (cf. [13]), where \( z \) is the solution to the following equation:

\[
z(t) + \int_0^t A_\alpha z(s) \, ds = \omega(t).
\]

In particular, it is possible to find \( \alpha_1 \in (0, \frac{s+\alpha-1}{2\alpha}) \) and \( p \) such that the above map is continuous from \( W^{l, p}([0, T]; D(\Gamma^{g_i}))) \) to \( C([0, T]; H^{s-\alpha+1+\sigma_1}) \) since \( \alpha > \frac{2}{3} \) and \( \sigma_1 < 3\alpha - 2 \).
LEMMA 6.7 (Continuity with respect to the control functions). Let \( l, p \) and \( \alpha_1 \) be chosen as above, and let \( \omega_n \to \omega \) in \( W^{l,p}([0,T]; D(A_n^{\alpha_1})) \). Let \( \theta \) be the solution to equation (6.11) corresponding to \( \omega \) and some initial condition \( x \in \mathcal{W} \) (the solution exists by the same arguments as the proof of Theorem 52 M. RÖCKNER, R. ZHU AND X. ZHU

\[
\tau = \inf \{ t \geq 0 : |\theta(t)|^2_{\mathcal{W}} \leq R \},
\]

where as usual we set \( \inf \mathcal{W} = \infty \). For each \( n \in \mathbb{N} \), define similarly \( \theta_n \) and \( \tau_n \) corresponding to \( \omega_n \) with the same initial condition \( x \). If \( \tau > T \), then \( \tau_n > T \) for \( n \) large enough and

\[
\theta_n \to \theta \quad \text{in} \quad C([0,T]; \mathcal{W}).
\]

PROOF. Set \( v_n := \theta_n - z_n \) for each \( n \in \mathbb{N} \), and \( v := \theta - z \), where \( z_n, z \) are the solutions to (6.13) corresponding to \( \omega_n, \omega \), respectively. Since \( \omega_n \to \omega \) in \( W^{l,p}([0,T]; D(A_n^{\alpha_1})) \), we can find a common lower bound for \( (\tau_n)_{n \in \mathbb{N}} \) and \( \tau \) by (6.6). For every time smaller than this lower bound \( t_0 \), by (6.6), we have

\[
\sup_{(0,t_0)} |\Delta^s \theta_n|^2 \leq R, \quad \sup_{(0,t_0)} |\Delta^s \theta|^2 \leq R, \quad \sup_{(0,t_0)} |\Delta^{s-\alpha+1+\alpha_1} z_n| \leq C,
\]

and

\[
\sup_{(0,t_0)} |\Delta^{s-\alpha+1+\alpha_1} z| \leq C,
\]

\[
\int_0^{t_0} |\Delta^{s+\alpha} v_n(l)|^2 \, dl \leq C(R), \quad \int_0^{t_0} |\Delta^{s+\alpha} v(l)|^2 \, dl \leq C(R),
\]

where \( C(R) \) is a constant depending only on \( R \). Moreover, we obtain for \( t \leq t_0 \)

\[
\frac{d}{dt} |v - v_n|_{\mathcal{W}}^2 + 2\kappa |\Delta^\alpha (v_n - v)|_{\mathcal{W}}^2
\]

\[
= (u_n \cdot \nabla \theta_n, \Delta^{2s} (v - v_n)) + |u \cdot \nabla \theta, \Delta^{2s} (v - v_n))
\]

\[
= |(u_{v_n} - u_v) \cdot \nabla \theta_n, \Delta^{2s} (v - v_n)) + |u \cdot \nabla (v_n - v), \Delta^{2s} (v - v_n))
\]

\[
+ |(u_{z_n} - u_z) \cdot \nabla \theta_n, \Delta^{2s} (v - v_n)) + |u \cdot \nabla (z_n - z), \Delta^{2s} (v - v_n))|
\]

where \( u_{v_n}, u_{z_n} \) satisfy (1.3) with \( \theta \) replaced by \( v_n, z_n \), respectively. For the first term on the right-hand side, we have

\[
|\langle (u_{v_n} - u_v) \cdot \nabla \theta_n, \Delta^{2s} (v - v_n) \rangle|
\]

\[
= |\langle \Delta^{s-\alpha} \nabla \cdot ((u_{v_n} - u_v)\theta_n), \Delta^{s+\alpha} (v - v_n) \rangle|
\]

\[
\leq C \|\Delta^{s+\alpha} (v - v_n)\| \|\Delta^{s-\alpha+1+\alpha_1} (v - v_n)\| \|\Delta^{\alpha} \theta_n\|
\]

\[
+ C \|\Delta^{s+\alpha} (v - v_n)\| \|\Delta^{s-\alpha+1+\alpha_1} \theta_n\| \|\Delta^{\alpha} (v - v_n)\|
\]

\[
\leq \kappa \frac{\|\Delta^{s+\alpha} (v - v_n)\|^2 + C(C(R) \|\Delta^{s+\alpha} v_n\|^2 \|\Delta^{s} (v - v_n)\|^2
\]

\[
+ C \|\Delta^{s-\alpha+1+\alpha_1} z_n\|^2 \|\Delta^{s} (v - v_n)\|^2.
\]
Here, $\sigma_1, \sigma_2$ are as (6.6) and we used $\text{div}(u_v - u_v) = 0$ in the first equality and Lemmas 2.1 and 2.2 in the first inequality and the interpolation inequality (2.3) and Young’s inequality in the last step. The other term can be estimated similarly. Then we obtain
\[
\frac{d}{dt} |v - v_n|^2_W + 2\kappa |\Lambda^\alpha (v_n - v)|^2_W
\leq \kappa |\Lambda^\alpha (v_n - v)|^2_W
+ C(C(R) + |\Lambda^\alpha v_n|^2_W + |\Lambda^\alpha ||^2_W)(|v - v_n|^2_W + |\Lambda^{s/2 + \sigma_1} (z - z_n)|^2).
\]

Then Gronwall’s lemma yields that
\[
|v - v_n|^2_W \leq \Theta_n \exp \left( C \int_0^t (C(R) + |\Lambda^\alpha v_n|^2_W + |\Lambda^\alpha ||^2_W) dt \right)
\times \int_0^t (C(R) + |\Lambda^\alpha v_n|^2_W + |\Lambda^\alpha ||^2_W) dt,
\]
where $\Theta_n = \sup_{[0, T]} |\Lambda^{s/2 + \sigma_1} (z - z_n)|$. We conclude $\theta_n \to \theta$ in $C([0, T]; W)$. Now, since $\tau > T$, if $S = \sup_{|t| \leq T} |\Lambda^\alpha \theta(t)|^2$, then $S < R$ and we find $\delta > 0$ (depending only on $R$ and $S$) and $n_0 \in \mathbb{N}$ such that $\Theta_n^2 < \delta$ and $|v_n - v|^2_W < \delta$ for all $n \geq n_0$, and so
\[
|\theta_n(t)|^2_W \leq |v_n(t) - v(t)|^2_W + \Theta_n + |\theta(t)|^2_W \leq 2\sqrt{\delta} + \sqrt{S} \leq \sqrt{R - \delta}.
\]
Then $\tau_n > T$ for all $n \geq n_0$. 

**Theorem 6.8.** Fix $\alpha > \frac{2}{3}$. Suppose Hypothesis E.3 holds and for $x \in W$ let $P_x$ be the distribution of the solution of (3.1) with initial value $\theta(0) = x$. Then for every $x \in W$ and every $T > 0$, the image measure of $P_x$ at time $T$ is fully supported on $\mathcal{W}$.

**Proof.** Fix $x \in W$ and $T > 0$. We need to show that for every $y \in W$ and $\varepsilon > 0$, $P_x[|\theta_T - y|^2_W < \varepsilon] > 0$. Let $\tilde{y} \in W \cap D(A_{\alpha})$ such that $A_{\alpha} \tilde{y} \in W_1$ and $|y - \tilde{y}|^2_W < \frac{\varepsilon}{2}$. Choose $R > 0$ such that $3|x|^2_W < R$ and $3|y|^2_W < R$. Then by Theorem 6.4,
\[
P_x[|\theta_T - y|^2_W < \varepsilon] \geq P_x\left[|\theta_T - \tilde{y}|^2_W < \frac{\varepsilon}{2}\right] \geq P_x\left[|\theta_T - \tilde{y}|^2_W < \frac{\varepsilon}{2}, \tau_R > T\right]
= P_x(R)\left[|\theta_T - \tilde{y}|^2_W < \frac{\varepsilon}{2}, \tau_R > T\right].
\]

By Lemma 6.6, there is a control $\bar{\omega} \in W^{l,p}([0, T]; D(A_{\alpha}^{l\alpha}))$, with $l, p$ and $\alpha_1$ chosen as in Lemma 6.7, such that the solution $\bar{\theta}$ to the control problem (6.11) corresponding to $\bar{\omega}$ satisfies $\bar{\theta}(0) = x, \bar{\theta}(T) = \tilde{y}$ and $|\bar{\theta}(t)|^2_W \leq \frac{\varepsilon}{2} R$. By
Lemma 6.7, there exists $\delta > 0$ such that for all $\omega \in W^{l,p}([0, T]; D(A_{\alpha}^{\dagger}))$ with $|\omega - \bar{\omega}|_{W^{l,p}([0, T]; D(A_{\alpha}^{\dagger}))} < \delta$, we have

$$|\theta(T, \omega) - \bar{y}|_{W} < \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{t \in [0, T]} |\theta(t, \omega)|_{W} < R,$$

where $\theta(\cdot, \omega)$ is the solution to the control problem (6.11) corresponding to $\omega$ and starting at $x$. Hence,

$$P_{x}^{(R)} \left[ \left| \theta_{T} - \bar{y} \right|_{W} < \frac{\varepsilon}{2}, \tau_{R} > T \right] \geq P_{x}^{(R)} \left[ \left| \eta - \bar{\omega} \right|_{W^{l,p}([0, T]; D(A_{\alpha}^{\dagger}))} < \delta \right].$$

where $\eta_{t} = \theta_{t} - x + f_{0}^{t}(A_{\alpha} \theta_{s} + \chi_{R}(\left|\theta_{s}\right|_{W})u \cdot \nabla \theta_{s}) ds$, hence $\theta_{T} = \theta(T, \eta)$, and the right-hand side of the inequality above is strictly positive since by Hypothesis E.3 $\eta$ is a Gaussian process in $D(A_{\alpha}^{\dagger})$. □

**Theorem 6.9.** Let $\alpha > \frac{3}{2}$ and suppose Hypothesis E.3 holds. Then there exists a unique invariant measure $\nu$ on $W$ for the transition semigroup $(P_{t})_{t \geq 0}$. Moreover:

(i) The invariant measure $\nu$ is ergodic.

(ii) The transition semigroup $(P_{t})_{t \geq 0}$ is $W$-strong Feller, irreducible and, therefore, strongly mixing. Furthermore, $(P_{t}(x, dy), t > 0, x \in W)$, are mutually equivalent.

(iii) There exist $0 < \delta_{1} < \frac{s + \alpha - 1}{2\alpha}$ and $0 < \gamma_{0} < 1$ such that

$$\int |A_{\alpha}^{\dagger} x|^{2\gamma_{0}} d\nu < \infty.$$

**Proof.** By similar methods as the proof of Theorem 5.12, we obtain the existence of the invariant measures. In fact, under Hypothesis E.3, we could choose the following approximation:

$$d\theta_{n}(t) + A_{\alpha} \theta_{n}(t) dt + u_{n}(t) \cdot \nabla \theta_{n}(t) dt = k_{\delta_{n}} \ast G dW(t),$$

with initial data $\theta_{n}(0) = x \in H^{s}, u_{n}$ satisfying (1.3) with $\theta$ replaced by $\theta_{n}$ and $k_{\delta_{n}}$ is the periodic Poisson kernel as in the proof of Theorem 3.3. By the same arguments as Theorems A.1 and A.2, we obtain that there exist a unique solution to the above equation with $\theta_{n} \in C([0, \infty), H^{s}) \cap L_{loc}^{2}([0, \infty), H^{s+\alpha}) P$-a.s. Then do the same calculations for $\theta_{n}$ as in Lemma 5.11, we obtain that there exists $0 < \gamma_{0} < 1, 0 < \delta_{1} < s + \alpha - 1$ such that

$$E \left[ \int_{0}^{t} \left| A_{\alpha}^{\dagger} \right|^{2\gamma_{0}} \right] \leq C(1 + t)(\|x\|_{L^{q}}^{q} + 1).$$

Choose $x_{0} \in H^{1}$ and define

$$\mu_{t} = \frac{1}{t} \int_{0}^{t} P_{r} \ast \delta_{x_{0}} dr.$$
Since by similar arguments as in the proof of Theorem A.1, we have $P$-a.s. $\theta_n \to \theta$ in $L^2([0, T], H)$ and for $2\alpha \delta_1 \leq \delta_1$, $0 < \gamma_0 < 1$

$$\int |A^\delta_1 x|^{2\gamma_0}_H \mu_t(dx) = \frac{1}{t} E_{x_0} \left[ \int_0^t |A^\delta_1 \theta|^{2\gamma_0} H^r \, dr \right],$$

by the above estimates we have for $t > 1$

$$\int |A^\delta_1 x|^{2\gamma_0}_H \mu_t(dx) \leq C.$$

This implies that $\mu_t$ is tight on $H^4$. Hence, any limit point of $\mu_t$ is an invariant measure for $(P_t)_{t \geq 0}$. Therefore, by Doob’s theorem, the strongly mixing property is a consequence of Theorem 6.3 and Theorem 6.8. □

**Remark 6.10 (Mildly degenerate noise).** We can also consider the ergodicity of the equation driven by a mildly degenerate noise as in [18]. For this, we have to use an extension of the Bismut–Elworthy–Li formula. We have the same problem as explained in Remark 6.2. So, we can just get the result for $\alpha > 2/3$.

6.3. **Exponential convergence for $\alpha > 2/3$**. In this subsection, we assume that $\alpha > \frac{2}{3}$ and $s > 3 - 2\alpha$. Then under Hypothesis E.3 the associated O–U process $z \in C((0, \infty), H^{2+\delta_0})$ for some $0 < \delta_0 < s + 2\alpha - 3$.

**Lemma 6.11.** Fix $\alpha > 2/3$. Let $\theta$ denote the solution of (3.1) and take $p > \frac{2}{3\alpha - 2}$, then for every $R_0 \geq 1$, there exist values $T_1 = T_1(R_0)$ and $\tilde{C}_1 = \tilde{C}_1(R_0)$ such that if $\sup_{t \in [0, T_1]} \|\theta(t)\|^p_{L^p} \leq R_0$, and $\sup_{t \in [0, T_1]} |\Lambda^{s+2\alpha-1-\varepsilon} z(t)|^2 \leq R_0$ for some $0 < \varepsilon < 3\alpha - 2 - \frac{2}{p}$, then $|\Lambda^{s+\delta} \theta(T_1)|^2 \leq \tilde{C}_1$ for some $\delta > 0$.

**Proof.** For $v = \theta - z$, we have the following estimate:

$$\frac{1}{2} \frac{d}{dt} \|v\|^2 + \kappa |\Lambda^\alpha v|^2 = \langle -u \cdot \nabla(v + z), v \rangle = \langle -u \cdot \nabla z, v \rangle$$

$$\leq C \|\nabla z\|_L^\infty \left[ \|v\|^2 + |v| \cdot |z| \right],$$

which implies that there exists $\tilde{C}_0 = \tilde{C}_0(R_0) > 0$ and for $P$-a.s. $\omega$, $\exists \theta(t_0(\omega) < 1$ such that

$$|\Lambda^\alpha \theta(t_0)|^2 \leq \tilde{C}_0.$$

For any $\tilde{r} > 0$ with $\tilde{r} - \alpha + 1 + \sigma_3 < s + 2\alpha - 1 - \varepsilon$ for $\sigma_3 = \frac{2}{p}$, we have the following a priori estimate for $v$, $r = \frac{\alpha}{\alpha - \frac{1}{2} - \frac{2}{p}}$:

$$\frac{d}{dt} |\Lambda^\tilde{r} v|^2 + 2\kappa |\Lambda^{\tilde{r} + \alpha} v|^2 \leq 2 \left[ |\Lambda^{\tilde{r} - \alpha} \nabla \cdot (u \theta), \Lambda^{\tilde{r} + \alpha} v| \right]$$

(6.14)
\[ \leq C \left| \Lambda^{\frac{\alpha}{2}+\frac{\alpha}{2}} v \right| \cdot \left| \Lambda^{\frac{\alpha}{2}-\frac{\alpha}{2}+\frac{\alpha}{2}} \theta \right| \cdot \| \theta \|_{L^p} \]
\[ \leq \frac{\kappa}{4} \left| \Lambda^{\frac{\alpha}{2}+\frac{\alpha}{2}} v \right|^2 + C \| \theta \|_{L^p} \left| \Lambda^{\frac{\alpha}{2}} v \right|^2 + C \left| \Lambda^{\frac{\alpha}{2}-\frac{\alpha}{2}+\frac{\alpha}{2}} \right|^2 \cdot \| \theta \|_{L^p}^2, \]

where we used \( \text{div } u = 0 \) in the first inequality and Lemmas 2.1, 2.2 in the second inequality and the interpolation inequality (2.3) and Young’s inequality in the last inequality. We choose the approximation \( v_n \) as in the proof of Theorem A.1 with initial time \( t = 0 \) replaced by initial time \( t = t_0(\omega) \). Then by a similar argument as in the proof of Theorem A.1 we have the following \( L^p \)-norm estimate of \( v_n \),
\[ \frac{d}{dt} \| v_n \|_{L^p}^p \leq C p \| \nabla v \|_\infty (\| v_n \|_{L^p}^p + \| z \|_{L^p} \| v_n \|_{L^p}^{p-1}). \]
Thus, we have
\[ \frac{d}{dt} \| v_n \|_{L^p} \leq C \| \nabla z \|_\infty (\| v_n \|_{L^p} + \| z \|_{L^p}). \]

Then by Gronwall’s lemma and \( s > 3 - 2\alpha \), we obtain the uniform \( L^p \)-norm estimates as (A.6) for \( v_n \). Moreover, by (6.14) and Gronwall’s lemma, we obtain the uniform \( H^s \)-norm estimates as (A.7) for \( v_n \). By a similar argument as in the proof of Theorem A.1, we have \( v_n \) converges to some process \( \tilde{v} \) in \( L^2([t_0, T], H) \) such that \( \tilde{v} + z \) is the solution of (3.1) in \([t_0, T]\). Then by the uniqueness proof in Theorem 4.2, we have \( \tilde{v} = v \), which implies for \( P\text{-a.s. } \omega \),
\[ v \in L^p_{\text{loc}}([t_0, \infty), H^\alpha) \land L^2_{\text{loc}}([t_0, \infty), H^{2\alpha}). \]
Therefore, (6.14) also holds for \( v \) with \( \tilde{r} = \alpha \), which implies that
\[ |\Lambda^\alpha v(t)|^2 + \kappa \int_{t_0}^t |\Lambda^{2\alpha} v(l)|^2 \, dl \leq \left( |\Lambda^\alpha v(t_0)|^2 + C(R_0) \right) \exp[C(R_0) t] + 1, \]
which implies that there exist \( \tilde{C}_1 = \tilde{C}_1(R_0) > 0 \) and \( \tilde{T}_0(R_0) \) such that \( |\Lambda^\alpha v(\tilde{T}_0)| \leq \tilde{C}_1(R_0). \) Moreover, there exists \( t_1 = t_1(\omega) > T_0(\omega) \) such that \( |\Lambda^{2\alpha} v(t_1)| \leq \tilde{C}_1. \) Using (6.14) for \( \tilde{r} = 2\alpha \) and by similar arguments as above, we obtain that there exists \( T_0 = T_0(R_0) \) independent of \( \omega \) such that \( |\Lambda^{2\alpha} v(T_0)| \leq \tilde{C}_1 \). Then we proceed analogously and obtain that there exists \( T_1 = T_1(R_0) > T_0(R_0) \) such that \( |\Lambda^{s+\delta} v(T_1)| \leq \tilde{C}_1 \) for some \( 0 < \delta < 3\alpha - 2 - \sigma_3 - \epsilon. \]

**Lemma 6.12.** Let \( \alpha > 2/3 \). Suppose Hypothesis E.3 holds with \( s > 3 - 2\alpha \). Then for each \( R \geq 1 \) there exist \( T_1 > 0 \) and a compact subset \( W \subset W \) such that
\[ \inf_{\| x \|_{L^p} \leq R} P_{T_1}(x, K) > 0 \]
for \( p \) in Lemma 6.11.

**Proof.** Define \( K := \{ x : |\Lambda^{s+\delta} x|^2 \leq \tilde{C}_1(R_0) \} \), where \( \tilde{C}_1(R_0), \delta \) comes from the previous lemma. By Lemma 6.11, for \( R \leq R_0 \), we have
\[ \inf_{\| x \|_{L^p} \leq R} P_{T_1}(x, K) \geq \inf_{\| x \|_{L^p} \leq R} \left( 1 - P_\delta \left( \sup_{t \in [0, T_1]} |\Lambda^{s+2\alpha-1-\delta} z(t)|^2 > R_0 \right) \right. \]
\[ - P_\delta \left( \sup_{t \in [0, T_1]} \| \theta(t) \|_{L^p}^p > R_0 \right) \].
Under Hypothesis E.3, since $\varepsilon$ is a Gaussian process, one deduces that there exist $\eta, C > 0$ such that

$$P_x\left[ \sup_{t \in [0, T_1]} \left| A^{s + 2\alpha - 1 - \varepsilon} \varepsilon(t) \right|^2 > R_0 \right] \leq C e^{-\eta(R_0^2/T_1)}$$

(see, e.g., [21], Proposition 15). Also by Theorem 3.3, we obtain

$$\sup_{\|x\| \leq R} \sup_{t \in [0, T_1]} \| \theta(t) \|_{L^p} \leq R_0 \sup_{\|x\| \leq R} \| \Theta(s) \|_{L^p} = C(R).$$

Choosing $R_0$ big enough, we prove the assertion. \(\Box\)

The exponential convergence now follows from Lemma 6.12 and an abstract result of [23], Theorem 3.1. For $p > \frac{2}{3\alpha - 2}$ let $V : L^p \to \mathbb{R}$ be a measurable function and define $\phi v := \sup_{x \in L^p} \frac{|\phi(x)|}{V(x)}$ and $\|v\| = \sup_{\|\phi\| \leq 1} \langle v, \phi \rangle$ for a signed measure $v$.

**Theorem 6.13.** Let $\alpha > 2/3$. Assume that Hypothesis E.3 holds with $s > 3 - 2\alpha$ and let $V(x) := 1 + \|x\|_p^p$ for $p > \frac{2}{3\alpha - 2}$. Then there exist $C_{\exp} > 0$ and $\alpha > 0$ such that

$$\|P_t^s \delta_{x_0} - \mu\|_{\var} \leq \|P_t^s \delta_{x_0} - \mu\|_V \leq C_{\exp} (1 + \|x_0\|_P^p) e^{-\alpha t}$$

for all $t > 0$ and $x_0 \in L^p$, where $\|\cdot\|_{\var}$ is the total variation distance on measures.

**Proof.** By [23], Theorem 3.1, we need to verify the following four conditions:

1. the measures $(P_t(x, \cdot))_{t > 0, x \in L^p}$ are equivalent,
2. $x \to P_t(x, \Gamma)$ is continuous in $\mathcal{W}$ for all $t > 0$ and all Borel sets $\Gamma \subset H$,
3. for each $R \geq 1$ there exist $T_1 > 0$ and a compact subset $K \subset \mathcal{W}$ such that

$$\inf_{\|x\|_{L^P} \leq R} P_{T_1}(x, K) > 0,$$

4. there exist $k, b, c > 0$ such that for all $t \geq 0$,

$$E^{P_t} \left[ \| \theta(t) \|_{L^p}^p \right] \leq k \|x\|_{L^P}^p e^{-bt} + c.$$

Condition 1 can be verified by [23], Lemma 3.2, and $P_t(x, \mathcal{W}) = 1$ for $x \in L^p$ since for fixed $t > 0$ the solution $\Theta$ will go into $H^s$ space if the initial value $x \in L^p$.

Other conditions can be verified by Theorem 6.9, Lemma 6.12 and Proposition 5.6. \(\Box\)

**Remark 6.14.** For $\alpha > \frac{3}{4}$, we could get a better result following a similar argument as in [50]. Namely, there exist $C_{\exp} > 0$ and $\alpha > 0$ such that

$$\|P_t^s \delta_{x_0} - \mu\|_{TV} \leq \|P_t^s \delta_{x_0} - \mu\|_V \leq C_{\exp} (1 + |x_0|^2) e^{-\alpha t}.$$
for all $t > 0$ and $x_0 \in H$. Here, $P_t$ could be every Markov selection obtained in Theorem B.5 associated to the solution of equation (3.1). The reason why $\alpha > \frac{3}{4}$ is needed is as follows.

As in Theorem 6.3, we can prove $P_t$ is $H^s$-strong Feller with $s > 3 - 3\alpha$. And for a solution $\theta$ of equation (3.1) starting from $x \in H$, we can only prove that it will enter $H^s$ under Hypothesis E.3. If the process $\theta$ enters $H^s$, we can prove that it satisfies the above four conditions. Hence, to obtain exponential convergence for every $x \in H$, we need the process starting from $x \in H$ to enter $H^s$. Hence, we need $3 - 3\alpha < s \leq \alpha$, that is, $\alpha > \frac{3}{4}$.

APPENDIX A

In this appendix, we construct a measurable map associated with the stochastic quasi-geostrophic equation, which will be used in the proof of Section 6. This proof is similar as done in [54], Section 3. Here, we give it for the reader’s convenience.

Assume that for any $m < 2 + \sigma$, $z \in C((0, \infty), H^m)$ with $\sigma$ in Hypothesis E.1. Then consider the following equation:

\[
\frac{dv}{dt} + A_{\alpha}v + (u_v + u_z) \cdot \nabla (v + z) = 0.
\]

(A.1)

For (A.1), we obtain the following existence and uniqueness result if the initial value starts from $H^1$.

**THEOREM A.1.** Fix $\alpha > 1/2$. Suppose that for any $m < 2 + \sigma$, $z \in C((0, \infty), H^m)$. For any $v_0 \in H^1$, there exists a unique solution $v \in L^\infty_{loc}([0, \infty); H^1) \cap L^2_{loc}([0, \infty); H^{1+\alpha})$ of equation (A.1) with $v(0) = v_0$, that is, for any $\varphi \in C^1(T^2)$

\[
\langle v(t), \varphi \rangle - \langle v_0, \varphi \rangle + \int_0^t \langle A_{\alpha/2}v(r), A_{\alpha/2}\varphi \rangle dr - \int_0^t \langle (u_v + u_z)(r) \cdot \nabla \varphi, (v + z)(r) \rangle dr = 0,
\]

where $u_v, u_z$ satisfy (1.3) with $\theta$ replaced by $v, z$, respectively.

**PROOF.** We construct an approximation of (A.1) by a similar construction as in the proof of Theorem 3.3.

We pick a smooth $\phi \geq 0$, with supp $\phi \subset [1, 2]$, $\int_0^\infty \phi = 1$, and for $\delta > 0$ let

\[
U_\delta(\theta)(t) := \int_0^\infty \phi(\tau)(k_\delta \ast R^{-\frac{1}{2}}\theta)(t - \delta \tau) d\tau,
\]

where $k_\delta$ is the periodic Poisson kernel in $T^2$ given by $k_\delta(\xi) = e^{-|\xi|}$, $\xi \in Z^2$, and we set $\theta(t) = 0$, $t < 0$. We take a zero sequence $\delta_n, n \in N$, and consider the equation:

\[
dv_n(t) + A_{\alpha}v_n(t) dt + u_n(t) \cdot \nabla (v_n(t) + z) dt = 0,
\]

(A.2)
with initial data \( v_n(0) = v_0 \) and \( u_n = U_\delta |v_n + z| \). For a fixed \( n \), this is a linear equation in \( v_n \) on each subinterval \([k, t_{k+1}]\) with \( t_k = k\delta \), since \( u_n \) is determined by the values of \( v_n \) on the two previous subintervals. By a similar argument as in the proof of Theorem 3.3, we obtain the existence and uniqueness of a solution \( v_n \in C([0, T], H^1) \cap L^2([0, T], H^{1+\alpha}) \) to (A.2) (for more details, we refer to [54]).

Now we take any \( p \) satisfying \( \frac{2}{\alpha - 1} < p < \infty \). From now on, we fix such \( p \) and we have \( H^1 \subset L^p \) by Lemma 2.2. Since the periodic Riesz transform is bounded on \( L^p \), we have for \( t > 0 \)

\[
\sup_{[0, t]} \| U_\delta[\theta] \|_{L^p} \leq C \sup_{[0, t]} \| \theta \|_{L^p},
\]

and also

\[
\int_0^t \| U_\delta[\theta] \|_{L^p}^p d\tau \leq C \int_0^t \| \theta \|_{L^p}^p d\tau.
\]

By Lemma 5.5, we obtain for \( v_n \) the following inequality by taking inner product with \( |v_n|^{p-2}v_n \) in \( L^2 \):

\[
\frac{d}{dt} \| v_n \|_{L^p}^p + 2\lambda_1 \| v_n \|_{L^p}^p \leq p |\langle u_n \cdot \nabla(v_n + z), |v_n|^{p-2}v_n \rangle| \leq p \| \nabla z \|_{L^\infty} \| v_n \|_{L^p} \| v_n \|_{L^p}^{p-1},
\]

where we used \( \text{div} u_n = 0 \) and Hölder’s inequality in the last inequality. Therefore,

\[
\| v_n(t) \|_{L^p}^p - \| v_n(0) \|_{L^p}^p + \int_0^t 2\lambda_1 \| v_n(\tau) \|_{L^p}^p d\tau \\
\leq \varepsilon \int_0^t (\| u_n \|_{L^p}^p + \| v_n \|_{L^p}^p) d\tau + pC(\varepsilon) \int_0^t \| \nabla z \|_{L^\infty}^{p/(p-1)} \| v_n \|_{L^p}^p d\tau \\
\leq \varepsilon \int_0^t \| v_n \|_{L^p}^p d\tau + pC(\varepsilon) \int_0^t \| \nabla z \|_{L^\infty}^{p/(p-1)} \| v_n \|_{L^p}^p d\tau + C \int_0^t \| z \|_{L^p}^p d\tau,
\]

where we used (A.4) in the last inequality. Then Gronwall’s lemma and \( H^1 \subset L^p \) yield that for any \( T > 0 \)

\[
\sup_{t \in [0, T]} \| v_n(t) \|_{L^p} \leq C,
\]

where \( C \) is a constant independent of \( n \).

Moreover, we get the following estimate by taking the inner product in \( L^2 \) with \( \Lambda e_k \) for (A.2), multiplying both sides by \( \langle \nu, \Lambda e_k \rangle \) and summing up over \( k \):

\[
\frac{1}{2} \frac{d}{dt} \| \Lambda v_n \|^2 + \kappa \| \Lambda^{1+\alpha} v_n \|^2 \\
\leq \| \Lambda^{1-\alpha} (u_n \cdot \nabla(v_n + z)) \| \| \Lambda^{1+\alpha} v_n \| \\
\leq C \| \Lambda^{1+\alpha} v_n \| \left( \| \Lambda^{2-\alpha+\sigma_1} (v_n + z) \|_{L^p} + \| \Lambda^{2-\alpha+\sigma_1} u_n \|_{L^p} + \| v_n + z \|_{L^p} \right),
\]
where \( \sigma_1 = 2/p < (2\alpha - 1) \) and we used Lemma 2.1 in the last inequality. Hence, we obtain that for \( r = \frac{2\alpha}{\alpha - 1 - \sigma_1} \),

\[
\frac{1}{2} (|\Lambda v_n(t)|^2 - |\Lambda v_n(0)|^2) + \kappa \int_0^t |\Lambda^{1+\alpha} v_n|^2 \, d\tau
\]

\[
\leq C \int_0^t |\Lambda^{1+\alpha} v_n| \left[ |\Lambda^{2-\alpha+\sigma_1}(v_n + z)| + \|u_n\|_{L^p} + |\Lambda^{2-\alpha+\sigma_1} u_n| \right] \|v_n + z\|_{L^p} \, d\tau
\]

\[
\leq \frac{\kappa}{2} \int_0^t |\Lambda^{1+\alpha} v_n|^2 \, d\tau
\]

\[
+ \frac{C}{2} \left[ \sup_{t \in [0, T]} \left( \|v_n(t) + z(t)\|_{L^p} + \|v_n(t) + z(t)\|_{L^p}^2 \right) + 1 \right]
\]

\[
\times \int_0^t |\Lambda v_n|^2 + |\Lambda^{2-\alpha+\sigma_1} z|^2 \, d\tau,
\]

where we used (A.3), (A.4), the interpolation inequality (2.3) and Young’s inequality in the last inequality. By Gronwall’s lemma and (A.6), we get that for \( v_0 \in H^1 \)

\[
(A.7) \quad \sup_{0 \leq t \leq T} |\Lambda v_n(t)|^2 + \kappa \int_0^T |\Lambda^{1+\alpha} v_n|^2 \, d\tau \leq C,
\]

where \( C \) is a constant independent of \( n \). Now decompose \( v_n \) as

\[
v_n(t) = v_0 - \int_0^t A_{\alpha} v_n(s) \, ds - \int_0^t (u_n(s) \cdot \nabla (v_n(s) + z(s))) \, ds.
\]

By (A.7), we obtain

\[
\left\| \int_0^t A_{\alpha} v_n(s) \, ds \right\|_{W^{1,2}(0, T; H^{-\alpha})} \leq C
\]

and

\[
\left\| \int_0^t (u_n(s) \cdot \nabla (v_n(s) + z(s))) \, ds \right\|_{W^{1,2}(0, T; H^{-3})} \leq C.
\]

So, we have proved

\[
\|v_n\|_{W^{1,2}([0, T], H^{-3})} \leq C,
\]

where \( C \) is a constant independent of \( n \). By the compactness embedding \( W^{1,2}([0, T], H^{-3}) \cap L^2([0, T], H^{1+\alpha}) \subset L^2([0, T], H^1) \) we have that there exists a subsequence of \( v_n \) converging in \( L^2([0, T], H^1) \) to a solution \( v \in L^\infty([0, \infty); H^1) \cap L^2([0, \infty); H^{1+\alpha}) \) of equation (A.1). Thus, (A.7) is also satisfied for \( v \). Uniqueness can be deduced from a similar argument as in the proof of Theorem 4.2. QED

**Theorem A.2.** Fix \( \alpha > 1/2 \). Suppose that for any \( m < 2 + \alpha, \sigma \in C([0, \infty), H^m) \). The solution \( v \) obtained in Theorem A.1 is in \( C([0, \infty); H^1) \).
STOCHASTIC QUASI-GEOSTROPHIC EQUATION

PROOF. It is sufficient to show that

$$\Lambda \frac{dv}{dt} \in L^2_{\text{loc}}([0, \infty); H^{-\alpha}).$$

For $\varphi$ smooth enough, we have

$$\left| \left( \frac{dv}{dt}, \varphi \right) \right| = |\kappa| \left| -\Lambda^\alpha v, \Lambda^{1+\alpha} \varphi \right| - \left| (u \cdot \nabla (\Lambda \varphi)), v + z \right|$$

$$\leq |\kappa| |\Lambda^{1+\alpha} v| + C |\Lambda^{2-\alpha} (u \cdot (v + z))| |\Lambda^\alpha \varphi|$$

$$\leq C \left[ |\Lambda^{1+\alpha} v| + |\Lambda^{2-\alpha+\sigma_1} (v + z)\|L_p\| |\Lambda^\alpha \varphi|, \right.$$

where $0 < \sigma_1 < 2\alpha - 1$, $p = \frac{2}{\sigma_1}$ and we used Lemma 2.1 in the last inequality.

Then

$$\left\| \Lambda \frac{dv}{dt} \right\|_{H^{-\alpha}} \leq C \left( \|v + z\|_{L^p} + 1 \right) |\Lambda^{1+\alpha} v| + C \|v + z\|_{L^p} |\Lambda^{2-\alpha+\sigma_1} z|.$$  

By (A.6) and (A.7), we obtain for $0 < T < \infty$

$$\int_0^T \left\| \Lambda \frac{dv}{dt} (\tau) \right\|_{H^{-\alpha}}^2 d\tau < \infty,$$

which implies that $v \in C([0, \infty); H^1)$. □

THEOREM A.3. Fix $\alpha > 1/2$. Suppose that for any $m < 2 + \sigma, z \in C([0, \infty), H^m)$. For any fixed $t > 0$, the map $v_0 \mapsto v(t, v_0)$ is a continuous map from $H^1$ into itself, where $v(t, v_0)$ is the solution of equation (A.1) with $v(0) = v_0$.

PROOF. Let $v_1, v_2$ be two solutions of (A.1) and $\xi = v_1 - v_2, \theta_1 = v_1 + z, \theta_2 = v_2 + z$. Then $\xi$ satisfies the following equation:

$$\left( \frac{d}{dt} \xi, \varphi \right) + \kappa (\Lambda^{\alpha} \xi, \Lambda^\alpha \varphi) = -(u_1 \cdot \nabla \xi, \varphi) - (u_2 \cdot \nabla \theta_2, \varphi),$$

where $u_1, u_2$ satisfy (1.3) with $\theta$ replaced by $\theta_1, \xi$, respectively.

Taking $\varphi = \Lambda e_k$, multiplying both sides by $(\xi, \Lambda e_k)$ and summing up over $k$ we have the following estimate since $v_i \in C([0, \infty); H^1) \cap L^2_{\text{loc}}([0, \infty); H^{1+\alpha}), i = 1, 2$, by Theorems A.1 and A.2:

$$\frac{1}{2} \frac{d}{dt} |\Lambda \xi|^2 + \kappa |\Lambda^{1+\alpha} \xi|^2$$

$$= -\langle \Lambda (u_1 \cdot \nabla \xi), \Lambda \xi \rangle - \langle u_\xi \cdot \nabla \theta_2, \Lambda^2 \xi \rangle$$

$$\leq C \left| \Lambda^{1+\alpha} \xi \right| \left[ |\Lambda^{2-\alpha} (u_1 \theta_2)| + |\Lambda^{2-\alpha} (u_1 \xi)| \right]$$

$$\leq C \left| \Lambda^{1+\alpha} \xi \right| \left[ \left| \Lambda^{2-\alpha+\sigma_1} \theta_2 \right| |\Lambda^{\sigma_2} \xi| + \left| \Lambda^{2-\alpha+\sigma_1} \theta_2 \right| |\Lambda^{\sigma_2} \xi| \right.$$
\[
\frac{\kappa}{2} |\Lambda^{1+\alpha} \zeta|^2
\]
\[
+ C(|\Lambda \theta_2|^r + |\Lambda \theta_1|^r + |\Lambda^{1+\alpha} v_2|^2 + |\Lambda^{2-\alpha+\sigma_1} z|^2 + |\Lambda^{s+\alpha} v_1|^2) |\Lambda \zeta|^2,
\]
where \( r = \frac{2\alpha - 1}{2(\alpha - 1)} \), \( \sigma_2 = 1 - \sigma_1 \) for some \( 0 < \sigma_1 < (2\alpha - 1) \) and we used Lemma 2.1 in the second inequality and Lemma 2.2, the interpolation inequality (2.3), \( H^1 \subset H^{\sigma_2} \) and Young's inequality in the last inequality. Then Gronwall's lemma yields that

\[
|\Lambda \zeta|^2 \leq C|\Lambda \zeta(0)|^2 \exp\left\{ \int_0^T |\Lambda \theta_2(\tau)|^r + |\Lambda \theta_1(\tau)|^r + |\Lambda^{2-\alpha+\sigma} z|^2 + |\Lambda^{1+\alpha} v_1(\tau)|^2 + |\Lambda^{1+\alpha} v_2(\tau)|^2 d\tau \right\}.
\]

Thus, the result follows. \( \square \)

Now for \( v_0 \in H^1, \overline{W} \in C(\mathbb{R}^+, H^{-1-\varepsilon_0}) \) we define

\[
v(t, W, v_0) := \begin{cases} v(t, v_0, z(\overline{W})), & \text{if } z(\overline{W}) \in C(\mathbb{R}^+, H^m) \text{ for } m < 2 + \sigma, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( v(t, v_0, z(\overline{W})) \) is the solution to (A.1) we obtained in Theorem A.1.

Combining Theorems A.1–A.3 we obtain the following results.

**Theorem A.4.** Fix \( \alpha > 1/2 \). \( v : \mathbb{R}^+ \times C(\mathbb{R}^+, H^{-1-\varepsilon_0}) \times H^1 \mapsto H^1, (t, \overline{W}, v_0) \mapsto v(t, \overline{W}, v_0) \) is a measurable map.

**Proof.** By Theorems A.1–A.3 \( t \mapsto v(t, t, v_0) \) and \( v_0 \mapsto v(t, \overline{W}, v_0) \) is continuous. Then it is sufficient to prove that if \( z_n \to z \) in \( C(\mathbb{R}^+, H^m) \), \( m < 2 + \sigma \), \( v_n \to v \) in \( C([0, T], H^1) \), where \( v_n = v(\cdot, v_0, z_n) \), \( v = v(\cdot, v_0, z) \). By the same arguments as in the proof of Theorem A.1, we have the following estimates:

\[
\sup_{[0,T]} |\Lambda v_n|^2 \leq C(T), \quad \sup_{[0,T]} |\Lambda v|^2 \leq C(T),
\]

and

\[
\int_0^T |\Lambda^{1+\alpha} v_n(l)|^2 dl \leq C(T), \quad \int_0^T |\Lambda^{1+\alpha} v(l)|^2 dl \leq C(T).
\]

Since \( v, v_n \in C([0, +\infty), H^1) \cap L^2_{\text{loc}}((0, +\infty), H^{1+\alpha}) \), we obtain

\[
\frac{d}{dt} |\Lambda (v - v_n)|^2 + 2\kappa |\Lambda^{1+\alpha} (v_n - v)|^2 \\
= [(u_{v_n} + u_{z_n}) \cdot \nabla (v_n + z_n), \Lambda^2 (v - v_n)] \\
- [(u_v + u_z) \cdot \nabla (v + z), \Lambda^2 (v - v_n)] \\
= [(u_{v_n} - u_v) \cdot \nabla (v_n + z_n), \Lambda^2 (v - v_n)]
\]
where \( u_{v_n}, u_{z_n} \) satisfy (1.3) with \( \theta \) replaced by \( v_n, z_n \) respectively. For the first term on the right-hand side, we have

\[
\frac{d}{dt} |\Lambda (v - v_n)|^2 + 2\kappa |\Lambda^{1+\alpha} (v_n - v)|^2 \\
\leq \kappa |\Lambda^{1+\alpha} (v_n - v)|^2 + C (C(T) + |\Lambda^{1+\alpha} v_n|^2 + |\Lambda^{1+\alpha} v|^2) \\
\times (|\Lambda (v - v_n)|^2 + |\Lambda^{2-\alpha+\sigma_1} (z - z_n)|^2).
\]

Gronwall’s lemma yields that

\[
|\Lambda (v - v_n)(t)|^2 \leq \Theta_n \exp \left( C \int_0^t \left( C(T) + |\Lambda^{1+\alpha} v_n|^2 + |\Lambda^{1+\alpha} v|^2 \right) dt \right) \\
\times \int_0^t \left( C(T) + |\Lambda^{1+\alpha} v_n|^2 + |\Lambda^{1+\alpha} v|^2 \right) dt,
\]

where \( \Theta_n = \sup_{[0,T]} |\Lambda^{2-\alpha+\sigma_1} (z - z_n)|. \) Then the results follow. \( \square \)

APPENDIX B

In this appendix, we prove the following lemma to complete the proof of Theorem 3.3.

**Lemma B.1.** For any \( x_0 \in B_0 \) defined in the proof of Theorem 3.3, there exists \( Q_{x_0} \in Q_{x_0} \) such that the map \( x_0 \mapsto Q_{x_0} \) from \( B_0 \) to \( \mathcal{P} (\Omega^n_0) \) is measurable with respect to \( B_0^n \).
PROOF. Let $B^{1}_{t_{i}}$ be the Borel $\sigma$-algebra on $\tilde{B}_{0} := \{ x(\cdot)1_{[0,f]}(\cdot) + x(t_{1}^{n}) \times 1_{[r_{1},\infty)}(\cdot) : x \in B_{0} \}$ with the topology induced by $\sup_{0 \leq t \leq r_{1}^{n}} \| x(t) \|_{H^{3}}$. Since $\{ \sup_{0 \leq t \leq r_{1}^{n}} \| x(t) \|_{H^{3}} < a \} \in B^{1}_{t_{i}}$, we know $B^{1}_{t_{i}} \subset B^{1}_{t_{i}}$. It suffices to prove that if for $\{ x_{m}, m \in \mathbb{N} \} \subset \tilde{B}_{0}$, $\sup_{0 \leq t \leq r_{1}^{n}} \| x_{m}(t) - x_{0}(t) \|_{H^{3}} \to 0$ and $Q_{m} \in Q_{x_{m}}$, then for some subsequence $m_{k}$, $Q_{m_{k}}$ weakly converges to some $Q \in Q_{x_{0}}$, because then [52], Lemma 12.1.8, Theorem 12.1.10, implies the existence of $Q_{x_{0}} \in Q_{x_{0}(\cdot)1_{[0,r_{1}^{n}]}(\cdot) + x_{0}(t_{1}^{n})1_{[r_{1},\infty)}(\cdot)}$ such that the map $x_{0} \mapsto x_{0}(\cdot)1_{[0,r_{1}^{n}]}(\cdot) + x(t_{1}^{n})1_{[r_{1},\infty)}(\cdot)$ from $B_{0}$ to $\tilde{B}_{0}$ to $\mathcal{P}(\Omega_{1}^{m})$ is measurable with respect to $B^{1}_{t_{i}}$. Moreover, by $Q_{x_{0}} \in Q_{x_{0}}$, the result follows.

Step 1: We prove that $(Q_{m})_{m \in \mathbb{N}}$ is tight in $\mathbb{S} := C([t_{1}^{n}, +\infty), H^{1}) \cap L_{\text{loc}}^{q}([t_{1}^{n}, +\infty), H^{3})$ for some $q \in \mathbb{N}$. Define for each $m \in \mathbb{N}$,

$$M^{m}(t,x) := \sum_{i=1}^{\infty} M_{i}^{m}(t,x)e_{i},$$

where $M_{i}^{m}$ is given in the proof of Theorem 3.3 (Step 2) with $x_{m}$ replaced by $x_{m}$. Then $(M^{m}(t,x))_{t \leq t_{1}^{n}}$ is a continuous $H^{3}$-valued $B_{t}$-martingale with respect to $Q_{m}$ and the following equality holds in $H^{1}$:

$$x(t) = x_{m}(t_{1}^{n}) - \int_{t_{1}^{n}}^{t_{1}^{n}} (A_{a}x(s) + U_{b_{m}}[x_{m}](s) \cdot \nabla x(s)) ds + M^{m}(t), \quad Q_{m} \text{-a.s.}$$

(B.1)

By Hölder’s inequality and (M3), (M1) for $Q_{m}$, we have

$$E^{Q_{m}} \left[ \sup_{s \neq t \in [t_{1}^{n}, t_{1}^{n}]} \left( \left\| \int_{s}^{t} A_{a}x(r) + U_{b_{m}}[x_{m}](r) \cdot \nabla x(r) dr \right\|_{H^{1}}^{\gamma} / |t - s|^{\gamma - 1} \right) \right]$$

$$\leq CE^{Q_{m}} \left[ \int_{t_{1}^{n}}^{t_{1}^{n}} \left\| A_{a}x(r) + U_{b_{m}}[x_{m}](r) \cdot \nabla x(r) \right\|_{H^{1}}^{\gamma} dr \right]$$

$$\leq CE^{Q_{m}} \left[ \sup_{0 \leq r \leq t_{1}^{n}} \left\| x(r) \right\|_{H^{3}}^{\gamma} \right] \left( 1 + \sup_{0 \leq r \leq t_{1}^{n}} \left\| x_{m}(r) \right\|_{H^{1}}^{\gamma} \right)$$

$$\leq C \left( \left\| x_{m}(t_{1}^{n}) \right\|_{H^{1}}^{\gamma} + 1 \right) \left( 1 + \sup_{0 \leq r \leq t_{1}^{n}} \left\| x_{m}(r) \right\|_{H^{1}}^{\gamma} \right),$$

where $C$ is independent of $m$. For $t_{1}^{n} \leq s < t \leq t_{1}^{n}$ and $q \in \mathbb{N}$, we have

$$E^{Q_{m}} \| M^{m}(t,x) - M^{m}(s,x) \|_{H^{1}}^{2q} \leq C_q E^{Q_{m}} \left( \int_{s}^{t} \left\| \Lambda(k_{\delta_{n}} \ast G(x(r))) \right\|_{L_{2}(U;H)}^{2} dr \right)^{q}$$

$$\leq C_q |t - s|^{q - 1} \int_{s}^{t} E^{Q_{m}} \| G(x(r)) \|_{L_{2}(U;H)}^{2q} dr$$

$$\leq C_q |t - s|^{q - 1} \int_{s}^{t} E^{Q_{m}} \| G(x(r)) \|_{L_{2}(U;H)}^{2q} dr$$
\[
\leq C_q |t - s|^{q-1} \int_s^t E Q_\beta^m (|x(r)|^{2q} + 1) \, dr \\
\leq C_q |t - s|^{q} (|A^3 x_m(t_1^m)|^{2q} + 1),
\]
where we used Hypothesis G.1 in the third inequality and (M3) in the last inequality. By Kolmogorov's criterion for any \( \beta \in (0, \frac{q-1}{2q}) \), we get
\[
\begin{align*}
&\sup_{s \neq t \in [t_1^m, t_2^m]} \frac{|M^m(t, x) - M^m(s, x)|_{H^1}}{|t - s|^\beta} \\
&\leq C_\beta(|A^3 x_m(t_1^m)|^{2q} + 1).
\end{align*}
\]
Combining (B.1)-(B.3) and \( Q_m((x : x(s) = x(t_2^m), s \in [t_1^m, +\infty))) = 1 \), we obtain for \( \beta_1 = 1 - \frac{1}{p} \) and any \( T > 0 \)
\[
\sup_{m \in \mathbb{N}} E Q_m \left( \sup_{s \neq t \in [t_1^m, T]} \frac{|x(t) - x(s)|_{H^1}}{|t - s|^\beta_1} \right) < \infty.
\]
Thus, by (M3) for \( Q_m \) and [24], Lemma 4.3, \( (Q_m)_{m \in \mathbb{N}} \) is tight in \( \mathbb{S} \).

Without loss of generality, we assume that \( Q_m \) weakly converges to some probability measure \( Q \) in \( \mathbb{S} \). We need to prove \( Q \in Q_{x_0} \).

Step 2: By Skorohod’s representation theorem, there exist a probability space \((\tilde{\Omega}, \tilde{\mathbb{B}}, \tilde{P})\) and \( \mathbb{S} \)-valued random variable \( \tilde{x}_m \) and \( \tilde{x} \) such that:

(i) \( \tilde{x}_m \) has the law \( Q_m \) for each \( m \in \mathbb{N} \);

(ii) \( \tilde{x}_m \to \tilde{x} \) in \( \mathbb{S} \), \( P \)-a.e., and \( \tilde{x} \) has the law \( Q \).

First, we easily deduce that
\[
Q(x(t_1^m)) = x_0(t_1^m)) = \tilde{P} (\tilde{x} (t_1^m) = x_0(t_1^m)) = \lim_{m \to \infty} Q_m (x(t_1^m) = x_m(t_1^m)) = 1,
\]
\[
Q(x(t) = x(t_2^m), t \geq t_2^m) = \tilde{P} (\tilde{x} (t) = \tilde{x}_m(t_2^m), t \geq t_2^m) = \lim_{m \to \infty} Q_m (x(t) = x(t_2^m), t \geq t_2^m) = 1.
\]

For \( q \in \mathbb{N} \), set
\[
\xi_q(x) := \sup_{r \in [t_1^m, t_2^m]} \|x(r)\|_{H^1}^{2q} + \int_{t_1^m}^{t_2^m} \|x(r)\|_{H^1}^{2q-1} \|x(r)\|_{H^1}^{2} dr.
\]
Then
\[
E Q(\xi_q(x)) = E \tilde{P}(\xi_q(\tilde{x})) \leq \liminf_{m \to \infty} E Q_m(\xi_q(x)) \leq \liminf_{m \to \infty} C(\|x_m(t_1^m)\|_{H^1}^{2q}) = C(\|x_0(t_1^m)\|_{H^1}^{2q} + 1).
\]
Thus, (M1) and (M3) follow.
Now we want to show that $(M_i(t,x))_{t \geq t_1^n}$ in the proof of Theorem 3.2 (Step 2) is a continuous $B_t$-martingale with respect to $Q$, whose square variation process is given by

$$\langle M_i \rangle(t,x) = \int_{t_1^n}^{t \wedge t_2^n} \| (k_{\delta_n} * G)^*(x(s)) (e_i) \|_U^2 ds.$$  

Since $\sup_{0 \leq t \leq t_1^n} \| x_m(t) - x_0(t) \|_{H^3} \to 0$ and $\tilde{x}_m \to \tilde{x}$ in $S$, we have

$$\lim_{m \to \infty} E_{\tilde{P}} \int_{t_1^n}^{t \wedge t_2^n} \left| (U_{\delta_n} [x_m] (s) \cdot \nabla \tilde{x}_m (s) + A_\alpha \tilde{x}_m (s) \right| ds$$

$$\leq \lim_{m \to \infty} E_{\tilde{P}} \int_{t_1^n}^{t \wedge t_2^n} \left| (U_{\delta_n} [x_0] (s) \cdot \nabla \tilde{x}_m (s) - U_{\delta_n} [x_0] (s) \cdot \nabla \tilde{x}_m (s) \right| ds$$

$$+ U_{\delta_n} [x_0] (s) \cdot \nabla (\tilde{x}_m (s) - \tilde{x}(s))$$

$$+ A_\alpha (\tilde{x}_m (s) - \tilde{x}(s), e_i) \| ds$$

$$= 0,$$

which implies that for $t \geq t_1^n$

$$(B.4) \quad \lim_{m \to \infty} E_{\tilde{P}} | M_i^m (t, \tilde{x}_m) - M_i (t, \tilde{x}) | = 0.$$  

Then we obtain for $t_1^n \leq s < t$,

$$E_{\tilde{Q}} (M_i(t,x)| B_s) = M_i(s,x).$$  

On the other hand, by the BDG inequality, we have

$$\sup_m E_{\tilde{P}} \left| M_i^m (t, \tilde{x}_m) \right|^{2q} \leq C \sup_m \int_{t_1^n}^{t \wedge t_2^n} E_{\tilde{P}} \left( \| (k_{\delta_n} * G)^* (\tilde{x}_m (s)) (e_i) \|_U^2 \right)^{2q} ds < +\infty.$$  

By (B.4), we have

$$\lim_{m \to \infty} E_{\tilde{P}} | M_i(t, \tilde{x}_m) - M_i (t, \tilde{x}) |^2 = 0.$$  

Then we obtain

$$E_{\tilde{Q}} \left( M_i^2 (t,x) - \int_{t_1^n}^{t} \| (k_{\delta_n} * G)^* (x(r)) (e_i) \|_U^2 dr | B_s \right)$$

$$= M_i^2 (s,x) - \int_{t_1^n}^{s} \| (k_{\delta_n} * G)^* (x(r)) (e_i) \|_U^2 dr.$$  

Now the results follow. $\square$
APPENDIX C: MARKOV SELECTIONS IN THE GENERAL CASE

In this section, we will use [24], Theorem 4.7, to get an almost sure Markov family \((P_x)_{x \in L^2}\) for equation (3.1). Here, we will use the same notation as in [24]. Below we choose

\[
H = \mathbb{Y} = L^2(\mathbb{T}^2)
\]

and

\[
\mathbb{X} = (H^{2+2\alpha})^*, \quad \mathbb{X}^* = H^{2+2\alpha}.
\]

Then \(\mathbb{X}\) is a Hilbert space and \(\mathbb{X}^* \subset \mathbb{Y}\) compactly. Let \(E = \{e_i, i \in \mathbb{N}\}\) be the orthonormal basis of \(H\) introduced in Section 2. We define the operator \(A\) as follows:

\[
A(\theta) := -\kappa(\Delta)^\alpha \theta - u \cdot \nabla \theta,
\]

where \(u\) satisfies (1.3). Then by Lemma C.3 below, \(A\) can be extended to an operator \(A : H \rightarrow \mathbb{X}\). For \(\theta\) not in \(H\) define \(A(\theta) := \infty\).

Set

\[
\Omega := C([0, \infty); \mathbb{X}),
\]

and let \(B\) denote the \(\sigma\)-field of Borel sets of \(\Omega\) and let \(\mathcal{P}(\Omega)\) denote the set of all probability measures on \((\Omega, B)\). Define the canonical process \(x : \Omega \rightarrow \mathbb{X}\) as

\[
x_t(\omega) = \omega(t).
\]

For each \(t, B_t = \sigma(x_s : 0 \leq s \leq t)\). Given \(P \in \mathcal{P}(\Omega)\) and \(t > 0\), let \(P(\cdot | B_t)(\omega)\) denote a regular conditional probability distribution of \(P\) given \(B_t\). In particular, \(P(\cdot | B_t)(\omega) \in \mathcal{P}(\Omega)\) for every \(\omega \in \Omega\) and for any bounded \(B\)-measurable function \(f\) on \(\Omega\)

\[
EP[f | B_t] = \int_{\Omega} f(y) P(dy | B_t), \quad P\text{-a.s.},
\]

and there exists a \(P\)-null set \(N \in B_t\) such that for every \(\omega\) not in \(N\)

\[
P(\cdot | B_t)(\omega) | B_t = \delta_\omega \quad (= \text{Dirac measure at } \omega),
\]

hence,

\[
P\{y : y(s) = \omega(s), s \in [0, t]\} | B_t)(\omega) = 1.
\]

In particular, we can consider \(P(\cdot | B_t)(\omega)\) as a measure on \((\Omega', B')\), that is,

\[
P(\cdot | B_t)(\omega) \in \mathcal{P}(\Omega'),
\]

where \(\Omega' := C([t, \infty); \mathbb{X})\) and \(B' := \sigma(x_s : s \geq t)\).

We say \(P \in \mathcal{P}(\Omega)\) is concentrated on the paths with values in \(H\), if there exists \(A \in B\) with \(P(A) = 1\) such that \(A \subset \{ \omega \in \Omega : x_t(\omega) \in H, \forall t \geq 0\}\). The set of such measures is denoted by \(\mathcal{P}_H(\Omega)\). The shift operator \(\Phi_t : \Omega \rightarrow \Omega'\) is defined by

\[
\Phi_t(\omega)(s) = \omega(s-t), \quad s \geq t.
\]

Following [24], Definitions 2.5, we introduce the following notions.
DEFINITION C.1. A family \((P_x)_{x \in H}\) of probability measures in \(\mathcal{P}_H(\Omega)\), is called an almost sure Markov family if for any \(A \in \mathcal{B}\), \(x \mapsto P_x(A)\) is \(\mathcal{B}(H)/\mathcal{B}([0,1])\)-measurable, and for each \(x \in H\) there exists a Lebesgue null set \(T_{P_x} \subset (0, \infty)\) such that for all \(t\) not in \(T_{P_x}\) and \(P_x\)-almost all \(\omega \in \Omega\)

\[
P_x(\cdot | B_t)(\omega) = P_{\omega(t)} \circ \Phi_t^{-1}.
\]

We now introduce the following notion of a martingale solution to equation (3.1) and write \(x(t)\) instead of \(x_t\).

DEFINITION C.2. Let \(x_0 \in H\). A probability measure \(P \in \mathcal{P}(\Omega)\) is called a martingale solution of equation (3.1) with initial value \(x_0\) if:

1. \((M1)\) \(P(x(0) = x_0) = 1\) and for any \(n \in \mathbb{N}\)
   \[
P\left\{ x \in \Omega : \int_0^n \| A(x(s)) \|_X ds + \int_0^n \| G(x(s)) \|_{L^2(U; H)} ds < +\infty \right\} = 1;
   \]
2. \((M2)\) for every \(l \in E\), the process
   \[
   M_l(t, x) := X_{\langle x(t), l \rangle_X} - \int_0^t X_{\langle A(x(s)), l \rangle_X} ds
   \]
   is a continuous square-integrable \(\mathcal{B}_t\)-martingale under \(P\), whose quadratic variation process is given by
   \[
   (M_l)(t, x) := \int_0^t \| G^*(x(s))(l) \|_U^2 ds,
   \]
   where the asterisk denotes the adjoint operator of \(G(x(s))\);
3. \((M3)\) for any \(p \in \mathbb{N}\), there exist a continuous positive real function \(t \mapsto C_{t, p}\) (only depending on \(p\) and \(A, G\)), a lower semicontinuous positive real functional \(N_p : Y \to [0, \infty]\), and a Lebesgue null set \(T_P \subset (0, \infty)\) such that for all \(0 \leq s \in [0, \infty)\) \(T_P\) and for all \(t \geq s\)
   \[
   E_P \left[ \sup_{r \in [s, t]} |x(r)|^{2p} + \int_s^t N_p(x(r)) dr \right] \leq C_{t-s}(|x(s)|^{2p} + 1).
   \]

First, we prove the following lemma.

LEMMA C.3. For any \(\theta_1, \theta_2 \in C^\infty(\mathbb{T}^2)\),
\[
\|(-\Delta)^a \theta_1 - (-\Delta)^a \theta_2\|_X \leq C_1|\theta_1 - \theta_2|,
\]
\[
\|u_1 \cdot \nabla \theta_1 - u_2 \cdot \nabla \theta_2\|_X \leq C_2(|\theta_1| + |\theta_2|)|\theta_1 - \theta_2|
\]
for constants \(C_1, C_2\). In particular, the operator \(A : C^\infty(\mathbb{T}^2) \to X\) extends to an operator \(A : H \to X\) by continuity.
PROOF. We only prove the second assertion, the first can be proved analogously. By the Sobolev embedding theorem, we have

\[
\|u_1 \cdot \nabla \theta_1 - u_2 \cdot \nabla \theta_2\|_X = \sup_{w \in C^\infty(T^3):\|w\|_{H^{2+2\alpha}} \leq 1} |\langle u_1 \cdot \nabla \theta_1 - u_2 \cdot \nabla \theta_2, w \rangle|
\]

\[
= \sup_{w \in C^\infty(T^3):\|w\|_{H^{2+2\alpha}} \leq 1} |\langle u_1 \cdot \nabla w, \theta_1 \rangle - \langle u_2 \cdot \nabla w, \theta_2 \rangle|
\]

\[
= \sup_{w \in C^\infty(T^3):\|w\|_{H^{2+2\alpha}} \leq 1} |\langle (u_1 - u_2) \cdot \nabla w, \theta_1 \rangle + \langle u_2 \cdot \nabla w, \theta_1 - \theta_2 \rangle|
\]

\[
\leq C \left( \sup_{w \in C^\infty(T^3):\|w\|_{H^{2+2\alpha}} \leq 1} \| \nabla w \|_{C(T^3)} \right) \left( |u_1 - u_2| \cdot |\theta_1| + |\theta_1 - \theta_2| \cdot |u_2| \right)
\]

\[
\leq C \left( |\theta_1| + |\theta_2| \right) |\theta_1 - \theta_2|.
\]

In the last inequality, we use (2.1) and the constant \(C\) changes from line to line. \(\square\)

In order to use [24], Theorem 4.7, we define the functional \(N_1\) on \(Y\) as follows:

\[
N_1(\theta) := \begin{cases} 
|\Lambda^\alpha \theta|^2, & \text{if } \theta \in H^\alpha, \\
+\infty, & \text{otherwise.}
\end{cases}
\]

It is obvious that \(N_1 \in \Omega^2\), defined in [24], Section 4. We recall that a lower semi-continuous function \(N: Y \to [0, \infty]\) belongs to \(\Omega^2\) if \(N(x) = 0\) implies \(x = 0\), \(N(cy) \leq c^2 N(y), \forall c \geq 0, y \in Y\) and \(\{y \in Y: N(y) \leq 1\}\) is relatively compact in \(Y\).

THEOREM C.4. Let \(0 < \alpha < 1\) and assume \(G\) satisfies Hypothesis G.1 with \(\rho_1 = 0\). Then for each \(x_0 \in H\), there exists a martingale solution \(P \in P(\Omega)\) starting from \(x_0\) to equation (3.1) in the sense of Definition C.2.

PROOF. We only need to check (C1)–(C3) in [24], Section 4, for the above \(A\) and \(G\).

The demi-continuity condition (C1) holds since Lemma C.3 and Hypothesis G.1 imply demi-continuity of \(A\) and \(G\).

The coercivity condition (C2) follows, because noting that for \(\theta \in H^\ast\)

\[
\langle u \cdot \nabla \theta, \theta \rangle = 0,
\]

we have

\[
\langle A(\theta), \theta \rangle = -N_1(\theta).
\]
Also the growth condition (C3) is clear since by Lemma C.3
\[ \|A(\theta)\|_\infty \leq C|\theta|^2 \]
and
\[ \|G(\theta)\|_{L^2(K;H)} \leq C(|\theta| + 1). \]
□

The set of all such martingale solutions with initial value \(x_0\) is denoted by \(C(x_0)\).

Using [24], Theorem 4.7, we now obtain the following.

**THEOREM C.5.** Let \(\alpha \in (0, 1)\). Assume \(G\) satisfies Hypothesis G.1 with \(\rho_1 = 0\). Then there exists an almost sure Markov family \((P_{x_0})_{x_0 \in H}\) for equation (3.1) and \(P_{x_0} \in C(x_0)\) for each \(x_0 \in H\).

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STOCHASTIC QUASI-GEOSTROPHIC EQUATION 71


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