Nonlinear dynamics modeling and analysis of two rods connected by a joint with clearance

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Abstract

In this paper, the nonlinear dynamic behaviors of two elastic rods connected by a joint with clearance are investigated. The rods and joint system are modeled by two degrees of freedom of mass-spring system with a clearance. The equations of motion of the mass-spring system with clearance are established by means of d'Alembert's principle. Due to the nonlinearity caused by clearance, the dynamic properties of the system are studied using the averaging method and compared with numerical solutions. The frequency responses of the system subjected to cosinusoidal excitation are obtained as well as the effects on the vibration characteristics induced by different gap sizes are investigated. The stability condition for steady solutions is presented based on Lyapunov theory. The method for detecting the multi-value performances of the frequency response has been proposed. Based on this method, the effects of the clearance size on the multi-value response characteristics are investigated, and the critical value of the clearance is obtained.

1. Introduction

Mechanisms including joints with clearance are extensively applied in spatial structures especially in deployable trusses such as solar panels, satellite antennas and space stations. Some clearances are necessary due to the components assemblage or to allow the relative motion between the connecting bodies as well. While the manufacturing tolerances, wear and material deformations can also cause the clearance between the journal and the bearing. Clearances can be a source of impact forces resulting machining wear and tolerance between the components in the structure. In these cases, the impulsive force caused by the excessive joint clearances can lead to the degradation of the performance of the mechanical systems. Hence the effects of joint clearance are capturing the attention of a large number of researchers [1–3].

Many of these works are focus on how to model the joint with clearance more precisely. Generally, since the concept of the impact pair proposed by Dubowsky and Freudenstein [4,5], there are three main modeling methods for joint with clearance, i.e. the massless link approach [6], the spring-damping approach [7,8] and the momentum exchange approach [9]. These modeling strategies are applied extensively for different working conditions to simulate the motion of the system as closely as possible.

From the view of the nonlinear dynamics, the vibration of the structure with clearance shows typical nonlinear behaviors. And the system is a non-smooth one which might be called as Filippov system [10] with plenty of nonlinear dynamical
properties [11]. The number and the stability of the periodic solutions may change with the variation of the clearance, which can lead to various types of bifurcation such as pitchfork bifurcation [12], Hopf bifurcation [13] as well as discontinuous bifurcation [11,14]. Many methods are developed to solve this problem analytically and numerically [15–17], such as the harmonic balance method, Runge–Kutta algorithm and shooting method. Numerical methods may obtain the time domain response more conveniently and can solve systems with much more degrees of freedom than the approximately analytical methods, while the detailed parametric analysis for dynamic behaviors is often neglected. Thus, several analytical methods are developed to acquire the insights into the influences of structural parameters on the nonlinear dynamic characteristics. And the averaging method used in the present work is just an analytical method which can study these problems in phase space.

In this paper, two rods connected by a joint with clearance subjected to axial harmonic excitation are taken into account. The equations of motion of the rods and joint system for its axial vibration are established. Due to the nonlinearity caused by the clearance, the averaging method is applied to obtain the steady solutions directly instead of introducing smoothening functions which are frequently used in the harmonic balance technique. The influences of the parameters in the structure especially the size of clearance on the nonlinear dynamic behaviors of the structural system are investigated. The stability condition for steady motions is constructed based on Lyapunov theory.

2. Problem formulation for analytical study

Connecting rods system including joints with clearance can be seen extensively in the engineering mechanisms especially in the truss structures. Fig. 1(a) shows the schematic diagram of two rods connected by a joint with clearance subjected to axial excitation. This structure is an infinite degrees of freedom mechanical system. In order to investigate the essential nonlinear dynamic properties caused by the clearance and obtain the analytical solutions for the vibration of the structure by using of the averaging method, simplifications are obviously required. Lumped mass method has been used to simplify many types of continuous structures such as rods, beams and pipes. It is necessary to point out that this method has also been used to analyze the nonlinear vibration performances by many researchers [18,19] rather than the linear problems. Based on these facts, the dynamic model for the rods and joint system with clearance is modeled by the lumped mass method in this analysis and shown in Fig. 1(b).

In the simplified model, \(m_1\) and \(m_2\) denote the concentrated masses of the rods and joint, respectively. \(k_1\) is the equivalent stiffness of the rod 1. Unlike the linear model, there is one non-linear stiffness force in this model, i.e. \(k_2f(x_1,x_2)\), in which \(k_2\) is a constant representing the equivalent stiffness of the rod 2 and \(f(x_1,x_2)\) is referred as a clearance type non-linearity as shown in Fig. 2 where \(\delta\) denotes the size of clearance of the joint, and it is obvious that \(|\pm \delta|\) are stiffness break points.

The vibration region which can be divided into three subregions is changed instantly due to the switch of the structural parameters. When \(−\delta < x_2 − x_1 < \delta\) (case 1), it means that there is no contact between the two mass elements. For the case of \(x_2 − x_1 < −\delta\) (case 2), it can be viewed as the left mass element passing through the clearance from left to right and contacting the right spring during the vibration. And the state of motion for \(x_2 − x_1 > \delta\) is similar to the case 2, while the difference is that the left mass element passes through the gap in the opposite direction.

The parameters in mass-spring system are equivalent to the rods and joint system according to the principles in Mechanics of Materials. Since it is difficult to determine the structural damping coefficients of rods, an important and extensively applied approximation is adopted to obtain the equivalent linear viscous damping [20,21]. The equivalent principle is to deem that the energy dissipation of these two types of damping is equal to each other over one period of vibration. For rod 1, the work done by equivalent linear viscous damping is \(W_d = \frac{1}{2} c_1 x_{1n}^2\). Since the damping mainly takes effect in the resonance region. It is reasonable to take \(\omega = \omega_{1n}\), where \(\omega_{1n}\) is the natural frequency of rod 1. Hence,

![Fig. 1. Schematic diagram of two rods connected by a joint with clearance (a), and the two degrees of freedom mass-spring model for the rods and joint system with clearance (b).](image-url)
The work done by the structural damping can be expressed as \( W_s = -C_0 g A^2 \), where \( g \) is the structural damping coefficient. Based on the principle \( W_s = W_d \), which yields \( c_1 = \frac{W_s}{C_0} \), considering the damping ratio of the structure \( \eta = g/k_1 \), one can obtain the equivalent viscous damping coefficient \( c_1 = \frac{k_1 \eta}{k_1} \), in which \( k_1 \) is the efficient stiffness of rod 1. And \( c_2 \) for rod 2 can also be obtained in the same way. Then the equations of motion of the two degrees of freedom mass-spring system can be expressed as

\[
\begin{align*}
    m_1 \ddot{x}_1 + k_1 x_1 + c_1 x_1 - c_2 (\dot{x}_2 - \dot{x}_1) - k_2 f(x_1, x_2) &= 0 \\
    m_2 \ddot{x}_2 + c_2 (\dot{x}_2 - \dot{x}_1) + k_2 f(x_1, x_2) &= F,
\end{align*}
\]

where \( F = F_1 \cos(\omega t) \) denotes the external excitation, and \( f(x_1, x_2) \) represents the clearance type non-linearity as mentioned above and it can be written as

\[
f(x_1, x_2) = \begin{cases} 
    x_2 - x_1 - \delta, & x_2 - x_1 > \delta \\
    0, & -\delta \leq x_2 - x_1 \leq \delta \\
    x_2 - x_1 + \delta, & x_2 - x_1 < -\delta
\end{cases}
\]

The governing equation (1) can be rewritten in terms of the relative displacement for the subsequent analysis. Let \( x_3 = x_2 - x_1 \) denotes the relative displacement between the two mass elements. Then the simplified equations can be written as follows:

\[
\begin{align*}
    \ddot{x}_2 &= -\frac{k_2}{m_2} x_3 + F_1' \\
    \ddot{x}_3 &= \frac{k_1}{m_1} x_2 - \left( \frac{k_1}{m_1} + \frac{k_2}{m_1} + \frac{k_2}{m_2} \right) x_3 + F_2',
\end{align*}
\]

where

\[
F_1' = \frac{F_1 \cos(\omega t) + f_n - c_2 \dot{x}_3}{m_2}, \quad F_2' = \frac{F_1 \cos(\omega t) + f_n - c_2 \dot{x}_3 - c_2 \dot{x}_3 - f_n - c_1 (\dot{x}_2 - \dot{x}_1)}{m_1}
\]

here \( f_n \) is referred as residual force in [22], which can be expressed as

\[
f_n = \begin{cases} 
    k_2 \delta, & x_3 > \delta \\
    k_2 x_3, & -\delta \leq x_3 \leq \delta \\
    -k_2 \delta, & x_3 < -\delta
\end{cases}
\]

The linear and nonlinear characters of the structure can be clearly identified when the equations are written in this manner. And \( f_n \) can be considered to be a nonlinear term perturbing the linear system.

The primary interest of this study is the dynamic behaviors of the model and the influences of the clearance on the vibration performance of the system. The averaging method will be used to obtain the steady solutions and the stability will be studied by using the perturbation technique. The resonance case is taken into account in the following analysis. And the external excitation is composed of a single harmonic term with small amplitude.
3. Steady state solution for nonlinear system

3.1. Averaging method

There are plenty of methods to solve nonlinear ordinary differential equation (ODE) either analytically or numerically such as the harmonic balance method [23], multiple scales method, finite element method and Runge–Kutta algorithm. In this section, the averaging method is used to obtain the steady solutions of the nonlinear equations.

It is necessary to express the equations of motion in the state equations form [24]. Let $y_1 = x_1, y_2 = x_2, y_3 = x_2, y_4 = x_2$, and introduce parameter $e$ to indicate the tiny effect of damping, clearance and external excitation with respect to the linear parts of the system. Then the state equations can be expressed as

$$
\begin{align*}
    \dot{y}_1 &= y_2 \\
    \dot{y}_2 &= ay_3 - by_1 + eF_2 \\
    \dot{y}_3 &= y_4 \\
    \dot{y}_4 &= -cy_1 + eF_1
\end{align*}
$$

(6)

where $a = k_1/m_1, b = k_1/m_2 + k_2/m_2 + k_1/m_1$ and $c = k_2/m_2$.

The method of averaging is based on the assumed form of solutions. Since the system has two degrees of freedom, the solutions are sought of the form

$$
y_i = \sum_{k=1}^{2} A_k \varphi_{ik}(\theta_k) \quad s = 1, 2, 3, 4.
$$

(7)

where $A_k$ is the amplitude of the vibration and $\theta_k$ is the phase of the motion. Obviously, the difference of the solutions between the nonlinear and linear vibrations is that the amplitude $A_k$ and frequency $\dot{\theta}_k$ vary with respect to the time for nonlinear case while both of them are constants for linear case. Hence, the central task of this study is to analyze the nonlinear vibration characteristics through inspecting the variations of the amplitude and frequency. Thus the following equations should be necessary and can be derived by substituting Eq. (7) into Eq. (6). That is

$$
\begin{align*}
    \sum_{k=1}^{2} \frac{d A_k}{dt} \varphi_{sk}(\theta_k) - \sum_{k=1}^{2} A_k \varphi_{sk}^*(\theta_k) \left( \frac{d \theta_k}{dt} - \lambda_k \right) &= eF_s \quad s = 1, 2, 3, 4.
\end{align*}
$$

(8)

where

$$
F_1 = 0, \quad F_2 = F_2', \quad F_3 = 0, \quad F_4 = F_1',
$$

and $\lambda_k$ is the natural frequency of the derived system (linear system) of Eq. (6) from which $\varphi_{sk}$ and $\varphi_{sk}^*$ can also be obtained. Let $y_s = A_s e^{i\lambda t}, s = 1, 2, 3, 4$, and then substitute them into the derived system of Eq. (6) and acquire the fundamental solutions $\varphi_{sk}$ and $\varphi_{sk}^*$ for linear equations with the following forms

$$
\begin{align*}
    \varphi_{sk} &= \cos(\lambda_k t) & \varphi_{sk}^* &= \sin(\lambda_k t) \\
    \varphi_{sk} &= -i \sin(\lambda_k t) & \varphi_{sk}^* &= i \cos(\lambda_k t) \\
    \varphi_{sk} &= \frac{c \cos(\lambda_k t)}{\lambda_k} & \varphi_{sk}^* &= \frac{c \sin(\lambda_k t)}{\lambda_k} \\
    \varphi_{sk} &= -\frac{c \sin(\lambda_k t)}{\lambda_k} & \varphi_{sk}^* &= \frac{c \cos(\lambda_k t)}{\lambda_k}.
\end{align*}
$$

(9)

Actually, $\varphi_{sk}$ and $\varphi_{sk}^*$ are the real parts and imaginary parts of $y_s = A_s e^{i\lambda t}, s = 1, 2, 3, 4$, when $A_1 = 1$.

In order to determine $dA_k/dt$ and $d\dot{\theta}_k/dt$ from Eq. (8), we can introduce $\psi_{sk}$ and $\psi_{sk}^*$ which are the solutions to the conjugate equations with respect to the derived system of Eq. (6), and use the orthogonality among $\psi_{sk}, \psi_{sk}^*$ and $\varphi_{sk}, \varphi_{sk}^*$. The conjugate equations can be written as follows:

$$
\begin{align*}
    \dot{\psi}_1 &= cy_4 + by_2 \\
    \dot{\psi}_2 &= -y_1 \\
    \dot{\psi}_3 &= -ay_2 \\
    \dot{\psi}_4 &= -y_3
\end{align*}
$$

Similar to the acquisition of $\varphi_{sk}$ and $\varphi_{sk}^*$, solving the above equations can yield $\psi_{sk}$ and $\psi_{sk}^*$ which can be expressed as

$$
\begin{align*}
    \psi_{sk} &= \cos(\lambda_k t) & \psi_{sk}^* &= \sin(\lambda_k t) \\
    \psi_{sk} &= -\frac{\sin(\lambda_k t)}{\lambda_k} & \psi_{sk}^* &= \frac{\cos(\lambda_k t)}{\lambda_k}.
\end{align*}
$$
\[ \psi_{3k} = -\frac{a \cos(\lambda_2 t)}{\lambda_2^2} \quad \psi_{3k}^* = -\frac{a \sin(\lambda_2 t)}{\lambda_2^2} \]
\[ \psi_{4k} = \frac{a \sin(\lambda_2 t)}{\lambda_2^2} \quad \psi_{4k}^* = -\frac{a \cos(\lambda_2 t)}{\lambda_2^2}. \]

Multiplying Eq. (8) by \( \psi_{ik} \) and \( \psi_{ik}^* \), and summing them for \( s \) from 1 to 4, then the standard equations can be obtained as follows:
\[ \frac{dA_k}{dt} = \frac{\varepsilon}{\Delta_k} \sum_s F_s \psi_{sk}(\theta_k) = \varepsilon \phi_k \]
\[ \frac{d\theta_k}{dt} = \dot{\lambda}_k - \frac{\varepsilon}{\Delta_k A_k} \sum_s F_s \psi_{sk}^*(\theta_k) = \dot{\lambda}_k - \varepsilon \phi_k^*, \]

where
\[ \Delta_k = \sum_s \psi_{sk} \phi_{sk} = \sum_s \psi_{sk} \phi_{sk}^* = 1 - \frac{ac}{\lambda_2^2}. \]

According to Eq. (11), it should be noticed that the derivative of \( A_k \) with respect to the time is small and nonlinear frequency \( d\theta_k/dt \) can be considered to be a small modification over the natural frequency of the linear system. These facts are just in line with the assumption that the variation of amplitude and initial phase are small over a period during the procedure of averaging.

As mentioned before, the resonance case will be taken into account, namely choosing \( \omega \) such that \( \lambda_2 - \omega = O(\varepsilon) \) in which \( \lambda_2 \) is the second order natural frequency of the derived system. Then applying Krylov–Bogoliubov transformation on Eq. (11), one can obtain
\[ A_k = y_k + \varepsilon U_k \]
\[ \theta_k = \omega t + \dot{\theta}_k + \varepsilon V_k, \]

where \( U_k \) and \( V_k \) are periodic functions with \( 2\pi \) as their periods, and \( y_k, \theta_k \) satisfy the following condition
\[ \frac{\phi_k}{\varepsilon} = \varepsilon Y_k(y, \theta) \quad \text{and} \quad \frac{\phi_k^*}{\varepsilon} = \varepsilon Z_k(y, \theta), \]

where \( Y_k \) and \( Z_k \) are the implicit functions of \( t \), which can be determined by the average of \( \phi_k \) and \( \phi_k^* \) over a period \( T \) as
\[ Y_k = \frac{1}{T} \int_0^T \phi_k dt = \frac{1}{2\pi} \int_0^{2\pi} \phi_k d\psi \]
\[ Z_k = -\frac{1}{2\pi} \int_0^{2\pi} \phi_k^* d\psi \]

Without loss of generality, letting \( k = 2 \), then \( \phi_2 \) and \( \phi_2^* \) can be expressed as
\[ \phi_2 = \frac{1}{\Lambda_2} \left\{ \left[ \frac{a}{m_2} \frac{1}{\lambda_2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \right] f_n \sin \theta_2 + \frac{1}{2} \left( \frac{F_1 a}{m_2 \lambda_2^2} - \frac{F_1}{m_2 \lambda_2^2} \right) (\sin(2\omega t + \dot{\theta}_2) + \sin \theta_2) \right. \]
\[ + \left. \frac{c_1 A_1 c_1 \sin \theta_1 \sin \theta_2}{m_1 \lambda_2^3} + \frac{c_1 A_2 c_1 \sin^2 \theta_2}{m_1 \lambda_2^2} - \frac{c_1 A_1 \lambda_1 \sin \theta_1 \sin \theta_2}{m_1 \lambda_2} + \frac{c_1 A_2 \sin^2 \theta_2}{m_1 \lambda_2^2} - \frac{c_2 A_1 \lambda_1 \sin \theta_2 \sin \theta_1}{m_2 \lambda_2^2} \right\}, \]
\[ \phi_2^* = \frac{1}{\Lambda_2 \Lambda_3} \left\{ \left[ \frac{1}{\lambda_2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) - \frac{a}{m_2 \lambda_2^2} \right] f_n \cos \theta_2 + \frac{1}{2} \left( \frac{F_1}{m_2 \lambda_2^2} - \frac{F_1 a}{m_2 \lambda_2^2} \right) (\cos(2\omega t + \dot{\theta}_2) + \cos \theta_2) \right. \]
\[ - \frac{c_1 A_1 c_1 \sin \theta_1 \cos \theta_2}{m_1 \lambda_2^3} - \frac{c_1 A_2 c_1 \cos \theta_2}{m_1 \lambda_2^2} + \frac{c_1 A_1 \lambda_1 \sin \theta_1 \cos \theta_2}{m_1 \lambda_2} + \frac{c_1 A_2 \cos^2 \theta_2}{m_1 \lambda_2^2} + \frac{c_2 A_1 \lambda_1 \cos \theta_2 \sin \theta_1}{m_2 \lambda_2^2} \right\}, \]

Since the frequency of the external force is around \( \lambda_2 \), the following calculation is reasonable. Letting \( \delta = A_2 \cos \theta_0 \) and combining with the Fourier expansion of nonlinear function \( f_n \), then \( Y_2 \) and \( Z_2 \) can be drawn as follows:


\[
Y_2 = -\frac{1}{2} \int_1^2 \frac{F_1 \sin \vartheta_1 (a - \lambda_2)}{m_2 (ac - \lambda_2^2)} - \frac{1}{2} \frac{A_2 j_0^2 c_2 m_2 - \lambda_2^2 c_2 m_2 - \lambda_2^2 c_2 m_1 + ac_2 m_1}{(ac - \lambda_2^2) m_1 m_2}.
\]

\[
Z_2 = -\frac{1}{2} \int_1^2 \frac{F_1 \cos \vartheta_1 (a - \lambda_2)}{m_2 (ac - \lambda_2^2)} - \frac{k_2 A_2 j_2 (\pi - 2\theta_0 + \sin 2\theta_0)}{2m_2 (ac - \lambda_2^2)} \left( \frac{1}{m_1 A_2 \lambda_2} + \frac{1}{m_2 A_2 \lambda_2} - \frac{a}{m_1 A_2 \lambda_2^2} \right).
\]

By introducing equivalent damping coefficient

\[
\delta_k(A_2) = -\frac{1}{2} \frac{A_2 j_2^2 (c_2 m_2 - \lambda_2^2 c_2 m_2 - \lambda_2^2 c_2 m_1 + ac_2 m_1)}{m_1 m_2 (ac - \lambda_2^2)}.
\]

and equivalent linear natural frequency

\[
p_k(A_2) = \lambda_2 - \frac{k_2 A_2}{2 \pi (\lambda_2^2 - ac)} \left( \frac{1}{m_1 A_2 \lambda_2} + \frac{1}{m_2 A_2 \lambda_2} - \frac{a}{m_1 A_2 \lambda_2^2} \right),
\]

the standard equations can be rewritten as

\[
\frac{dA_2}{dt} = \delta_k(A_2) A_2 - \frac{1}{2} \frac{F_1 \lambda_2 \sin \vartheta_2 (a - \lambda_2)}{m_2 (ac - \lambda_2^2)}
\]

\[
\frac{d\theta_k}{dt} = p_k(A_2) - \omega - \frac{1}{2} \frac{F_1 \lambda_2 \cos \vartheta_2 (a - \lambda_2^2)}{m_2 (ac - \lambda_2^2) A_2}.
\]

and the steady solution can be obtained by letting the right hand side of Eq. (15) be equal to zero, that is

\[
A_\infty = \frac{1}{\sqrt{\delta_k^2(A_\infty) + (p_k(A_\infty) - \omega)^2}} \frac{F_1 \lambda_2 (a - \lambda_2^2)}{2m_2 (ac - \lambda_2^2)}
\]

\[
\theta_\infty = \arctan \left( \frac{\delta_k(A_\infty)}{p_k(A_\infty) - \omega} \right).
\]

3.2. Steady solutions and discussions

The methodology applied in the previous sections shall be validated by the comparisons with the results acquired from numerical simulation. Without loss of generality, the oscillator model chosen in the comparisons are pretty simple with unit stiffness, masses and the equivalent viscous coefficients \(c_1 = 0.01 \text{ N/(m/s)}\) and \(c_2 = 0.01 \text{ N/(m/s)}\), and \(\delta = 0.002 \text{ m}\) for convenience. Fig. 3 shows the steady frequency responses calculated by averaging method (solid line) and numerical simulation (asterisk), respectively. It can be seen that the results are consistent with each other quite well, which implies the validity of the averaging method for clearance type nonlinearity.

In the following analysis, the structural and material parameters based on Ref. [25] are chosen according to equivalent principles in Mechanics of Materials. In [25], connecting rods are alloy with elasticity modulus \(E = 7 \times 10^{10} \text{ N/m}^2\), density \(\rho = 4.73 \times 10^3 \text{ kg/m}^3\), length \(l = 450 \text{ mm}\), external diameter \(D = 7.5 \text{ mm}\), rod wall thickness \(d = 1.25 \text{ mm}\) and damping ratio is 0.01. It should be noticed that the size of the clearance is considered to be a variable in the vibration analysis so as to gain the insights into the nonlinear dynamic behaviors. Meanwhile, the influences of structural parameters other than clearance will be also taken into account.

The resonance curves for different clearances are presented in Fig. 4. It can be seen that the frequency corresponding to the resonance peak decreases with the increase of the clearance. And the natural frequency of the linear oscillator is highest (for \(\delta = 0 \text{ mm}\)). This might due to the fact that the existence of clearance can reduce the stiffness of the structure. And it is necessary to notice that the diagram pattern for small clearance (e.g. \(\delta = 0.1 \text{ mm}\)) is quite similar to the linear case in spite of
its nonlinearity, which indicates that there is no unstable area for this situation. Nonetheless, the remaining two curves show apparent nonlinear characteristics such as multi-value and jumping phenomenon just like the Duffing system. In this case, the vibration properties of connecting rods with clearance joint are similar to the hardening spring.

In order to acquire the insight into the nonlinear vibration characteristics of the structure, certain types of alloys with different elastic moduli are taken into account for rod 2. Thus the stiffness is the main equivalent parameter in the mass-spring system. And different stiffness $k_2$ is chosen to indicate different materials of rod 2. Fixing $\delta = 0.2$ mm, Fig. 5 shows the comparisons of response curves for different stiffness $k_2$. The relationship of the resonance frequency versus $k_2$ can be observed, namely the vibration frequency increases with $k_2$ increasing. And the hardening spring property also becomes stronger with the increase of $k_2$. The influences of the gap size on the frequency response characteristics will be discussed in the next section in detail.

4. Stability analysis of steady solutions

The local stability of the solutions can explain why for certain size of clearance there are no jumping phenomena, and the critical size of gap for this type of nonlinear dynamic characteristics can be determined. This can be done by perturbing the steady state solutions and studying the resulting motion. In this analysis, the criterion for the judgment is Lyapunov theory.

The solutions of the governing equations are perturbed such that:

$$A_2 = A_s + \delta A_2, \quad \dot{A}_2 = \dot{A}_s + \delta \dot{A}_2.$$  

As usual, the perturbed solutions are substituted into Eq. (15) and expanded for small perturbations by using Taylor series with only the linear terms in the perturbed variables retained. The following equations can be obtained

$$\frac{d\delta A_2}{dt} = \delta \epsilon(A_s) \delta A_2 - \frac{F_1 \dot{A}_2(a - \dot{A}_2^2)}{4m_2(ac - \dot{A}_2^2)} \cos(\dot{A}_s) \delta \dot{A}_2,$$

$$\frac{d\delta \dot{A}_2}{dt} = \frac{F_1 \dot{A}_2(a - \dot{A}_2^2)}{4m_2(ac - \dot{A}_2^2)} \frac{\cos(\dot{A}_s)}{A_s^2} - \rho \delta A_2 + \frac{F_1 \dot{A}_2(a - \dot{A}_2^2)}{4m_2(ac - \dot{A}_2^2)} \frac{\sin(\dot{A}_s)}{A_s^2} \delta \dot{A}_2,$$  

where

$$\rho = \frac{\dot{A}_2^4}{\dot{A}_s^2} \frac{k_2}{2\pi} \left( \frac{2\dot{\delta}^3}{A_s^2 \sqrt{A_s^2 - \dot{\delta}^2}} - \frac{2\dot{\delta}}{A_s \sqrt{A_s^2 - \dot{\delta}^2}} - \frac{2\delta}{A_s \sqrt{A_s^2 - \dot{\delta}^2}} \right).$$
Thus, the characteristic equation can be obtained as follows:

\[ \lambda^2 - 2\delta_e(A_i)\lambda + \delta_e^2(A_i) + (p_e(A_i) - \omega)^2 - \rho A_i(p_e(A_i) - \omega) = 0. \]

where \( \lambda \) is the eigenvalue for the Jacobian of Eq. (18). Since \( \delta_e(A_i) \) is always negative, according to the Lyapunov stability criteria [26], the condition for stability of steady solutions can be established as follows:

\[ \delta_e(A_i)^2 + (p_e(A_i) - \omega)^2 - \rho A_i(p_e(A_i) - \omega) \geq 0. \quad (19) \]

The following analysis is based on the result of \( \delta = 0.2 \) mm in Fig. 4. And Fig. 6 just displays the partial enlargement of the curve for \( \delta = 0.2 \) mm in Fig. 4 between the frequency of 13,000 and 14,000 Hz. Substituting the steady solutions in Fig. 6 into Eq. (19), it can be checked that the solutions on branch of ABC and DE satisfy stability condition while the steady motions on branch of CD are unstable. If the direction of sweeping frequency is forward, the amplitude will first vibrate along the curve ABC and jump vertically to a point on the DE branch from point C then decrease along the DE curve. While if the direction of sweeping frequency is backward, the vibration of amplitude will move along the DE branch first then jump at point D to a point on AB. From the view of bifurcation, there is a pitchfork bifurcation at point D (one periodic solution exists for \( \omega < \omega_D \) and three periodic solutions co-exist for \( \omega_D < \omega < \omega_C \), where \( \omega_C \) and \( \omega_D \) denote the frequencies corresponding to points C and D).

Furthermore, the time response history corresponding to the values chosen from the stable regions of Fig. 4 for \( \delta = 0.2 \) mm is calculated by Runge–Kutta algorithm and shown in Fig. 7, from which the stable vibration with equal amplitude can be observed. The stable limit cycle appears in the phase portrait in Fig. 8.

In order to identify for what size of the clearance the response diagram will present multi-value characteristic, the difference of the amplitudes between \( C \) and \( D \) in Fig. 6 is calculated. Here \( C \) and \( D \) may be called as a higher critical point and a lower critical point, respectively. Based on the first equation in Eq. (16), the curve in Fig. 6 can be just determined by the following function:

\[ W = A_i - \frac{1}{\sqrt{\delta_e^2(A_i) + (p_e(A_i) - \omega)^2}} \frac{F_1\lambda_2(a - \lambda_2^2)}{2m_2(ac - \lambda_2^2)}, \]

and the coordinates of \( C \) and \( D \) satisfy the condition \( \frac{\partial W}{\partial \lambda} = 0 \) according to the geometric relationship between these two points. Then combining Eq. (16), the differences of the amplitudes between them can be calculated finally and the results...
are displayed in Fig. 9. In Fig. 9 it can be observed when the clearance is smaller than that of point \( N \), the difference between \( C \) and \( D \) is always zero. It means that the response diagram is just similar to the linear case and the curve for \( d = 0.1 \text{ mm} \) in Fig. 4 just satisfies this situation. And the typical jumping phenomenon will appear when the size of clearance passes through that of point \( N \), which can also be verified in Fig. 4 (for the cases of \( d = 0.15 \text{ mm} \) and \( 0.2 \text{ mm} \)). Thus, the results of Figs. 4 and 9 accord with each other very well. To some extent, the value of the clearance at point \( N \) may be viewed as a critical parameter for the vibration of the model with multi-value behaviors.

### 5. Conclusions

The nonlinear dynamic characteristics of two elastic rods connected by a joint with clearance are investigated by the method of averaging. The nonlinear equations of motion are established. The nonlinear vibration properties are analyzed for different structural parameters based on the frequency responses of the model. Additionally, the stability condition for steady solutions is established by the averaging method combining with the Lyapunov approach. The jumping phenomenon in the frequency response curve is studied. Furthermore, the effects of the clearance size on the multi-value characteristics in the frequency response curve are also analyzed. Theoretical analysis and numerical stimulations are performed and the following conclusions can be drawn:

1. The approximate analytical solutions obtained by the averaging method match well with the results obtained from numerical simulation. Hence the averaging method is effective in studying the clearance type nonlinear problems.
2. Existence of clearance can reduce the resonance frequency of the structure. And the stiffness of the rods can also affect the structural dynamical characteristics.
3. According to the stability condition for steady solutions, the stable limit cycle appears when the parameters are chosen from the stable regions. Besides, the structural parameters for jumping phenomenon observed in the frequency response curve are also conform to the stability condition.
4. The critical clearance value for determining the multi-value characteristics of frequency response curve is obtained.

The results of this paper can be helpful for the nonlinear dynamical analysis and design of connecting rods and joint system with clearance.
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