The degree resistance distance of cacti

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Graph invariants, based on the distances between the vertices of a graph, are widely used in theoretical chemistry. The degree resistance distance of a graph $G$ is defined as $D_{R}(G) = \sum_{\{u,v\}\subseteq V(G)} [d(u) + d(v)]R(u, v)$, where $d(u)$ is the degree of the vertex $u$, and $R(u, v)$ the resistance distance between the vertices $u$ and $v$. Let $\text{Cact}(n; t)$ be the set of all cacti possessing $n$ vertices and $t$ cycles. The elements of $\text{Cact}(n; t)$ with minimum degree resistance distance are characterized.

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1. Introduction

The graphs considered in this paper are finite, loopless, and contain no multiple edges. Given a graph $G$, let $V(G)$ and $E(G)$ be, respectively, its vertex and edge sets. The ordinary distance $d(u, v) = d_{G}(u, v)$ between the vertices $u$ and $v$ of the graph $G$ is the length of the shortest path between $u$ and $v$.

The Wiener index $W(G)$ is the sum of ordinary distances between all pairs of vertices, that is, $W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u, v)$. It is the oldest and one of the most thoroughly studied distance-based graph invariants [8, 9, 25].

A modified version of the Wiener index is the degree distance defined as $D(G) = \sum_{\{u,v\}\subseteq V(G)} \{d(u) + d(v)\}d(u, v)$, where $d(u) = d_{C}(u)$ is the degree of the vertex $u$ of the graph $G$. The degree distance was also widely studied [24, 1, 15, 6, 7, 18–21, 31]. Tomescu [19] determined the unicyclic and bicyclic graphs with minimum degree distance. Yuan and An [31] determined the unicyclic graphs with maximum degree distance.

In 1993, Klein and Randić [16] introduced a new distance function named resistance distance, based on the theory of electrical networks. They viewed $G$ as an electric network $N$ by replacing each edge of $G$ with a unit resistor. The resistance distance between the vertices $u$ and $v$ of the graph $G$, denoted by $R(u, v)$, is then defined to be the effective resistance between the nodes $u$ and $v$ in $N$. This new kind of distance between vertices of a graph was eventually studied in detail [16, 3, 4, 11, 12, 23, 30, 28].

If the ordinary distance is replaced by resistance distance in the expression for the Wiener index, one arrives at the Kirchhoff index

$$Kf(G) = \sum_{\{u,v\}\subseteq V(G)} R(u, v)$$

which also has been widely studied [2, 10, 13, 22, 27, 26, 32].

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Similarly, if the ordinary distance is replaced by resistance distance in the expression for the degree distance, then one arrives at the degree resistance distance [14]:

$$D_R(G) = \sum_{(u,v) \in V(G)} [(d(u) + d(v))R(u,v)].$$

Palacios [17] named the same graph invariant “additive degree–Kirchhoff index”.


A cactus is a connected graph in which any two simple cycles have at most one vertex in common. Equivalently, every edge in such a graph belongs to at most one simple cycle. Denote by $Cact(n; t)$ the set of cacti possessing $n$ vertices and $t$ cycles. If $G \in Cact(n; t)$, then $|E(G)| = n + t - 1$. In this paper, we determine the minimum degree resistance distance among graphs in $Cact(n; t)$ and characterize the corresponding extremal graphs.

2. Preliminaries

Let $R_C(u, v)$ denote the resistance distance between $u$ and $v$ in the graph $G$. Recall that [16] $R_C(u, v) = R_C(v, u)$ and $R_C(u, v) \geq 0$ with equality if and only if $u = v$.

For a vertex $v$ in $G$, we define

$$Kf_v(G) = \sum_{u \in G} R_C(u, v) \quad \text{and} \quad D_v(G) = \sum_{u \in G} d_C(u)R_C(u, v).$$

In the above formulas and in what follows, for the sake of conciseness, instead of $u \in V(G)$ we write $u \in G$.

By the definition of $D_R(G)$, we also have

$$D_R(G) = \sum_{v \in G} d_C(v) \sum_{u \in G} R_C(u, v).$$

**Lemma 1** ([16]). Let $G$ be a graph, $x$ be a cut vertex of $G$ and let $u$, $v$ be vertices belonging to different components which arise upon deletion of $x$. Then $R_C(u, v) = R_C(u, x) + R_C(x, v)$.

**Lemma 2** ([14]). Let $G$ be a connected graph with a cut-vertex $v$ such that $G_1$ and $G_2$ are two connected subgraphs of $G$ having $v$ as the only common vertex and $V(G_1) \cup V(G_2) = V(G)$. Let $n_1 = |V(G_1)|$, $n_2 = |V(G_2)|$, $m_1 = |E(G_1)|$, $m_2 = |E(G_2)|$. Then

$$D_R(G) = D_R(G_1) + D_R(G_2) + 2m_2Kf_v(G_1) + 2m_1Kf_v(G_2) + (n_2 - 1)D_v(G_1) + (n_1 - 1)D_v(G_2).$$

Let $v$ be a vertex of degree $p + 1$ in a graph $G$, such that $vv_1, vv_2, \ldots, vv_p$ are pendent edges incident with $v$, and $u$ is the neighbor of $v$ distinct from $v_1, v_2, \ldots, v_p$. We form a graph $G' = \sigma(G, v)$ by deleting the edges $vv_1, vv_2, \ldots, vv_p$ and adding new edges $uv_1, uv_2, \ldots, uv_p$. We say that $G'$ is a $\sigma$-transform of $G$ (see Fig. 1).

**Lemma 3** ([14]). Let $G' = \sigma(G, v)$ be a $\sigma$-transform of the graph $G$, $d_C(u) \geq 1$. Then $D_R(G) \geq D_R(G')$. Equality holds if and only if $G$ is a star with $v$ as its center.

Let $G - v$ be the graph obtained from the graph $G$ by deleting its vertex $v$ and all edges incident to $v$.

**Lemma 4.** Let $u$ be a vertex of $G$ such that there are $p$ pendent vertices $u_1, u_2, \ldots, u_p$ attached to $u$. Let $v$ be another vertex of $G$ such that there are $q$ pendent vertices $v_1, v_2, \ldots, v_q$ attached to $v$. Let

$$G_1 = G - \{uv_1, uv_2, \ldots, uv_p\} + \{u_1, u_2, \ldots, u_p\}$$

and

$$G_2 = G - \{wu_1, wu_2, \ldots, wu_p\} + \{v_1, v_2, \ldots, v_p\}.$$

Then either $D_R(G) > D_R(G_1)$ or $D_R(G) > D_R(G_2)$.
**Proof.** Let \( A = \{u_1, u_2, \ldots, u_p\}, B = \{v_1, v_2, \ldots, v_q\} \) and \( H = V(G) \setminus (A \cup B \cup \{u, v\}) \). Let \( R_G(u, v) = r \).

In the transformation \( G \rightarrow G_1 \) for any pair of vertices \( x, y \) satisfying either \( x, y \in H \), or \( x \in A \), or \( x \in B \), or \( x \in A \), \( y \in H \), then the term \( \sum_{x,y} [d_G(x) + d_G(y)] R_G(x, y) \) does not change. Then

\[
D_R(G) = \left[ \sum_{x,y \in H} + \sum_{x \in A} + \sum_{x \in B} + \sum_{y \in H} \right] [d_G(x) + d_G(y)] R_G(x, y) + \sum_{x \in B, y \in H} [d_G(x) + d_G(y)] R_G(x, y)
+ \sum_{x \in A, y \in B} [d_G(x) + d_G(y)] R_G(x, y) + \sum_{x \in A, y = u} [d_G(x) + d_G(y)] R_G(x, y)
+ \sum_{x \in B, y = v} [d_G(x) + d_G(y)] R_G(x, y) + \sum_{x \in H, y = u} [d_G(x) + d_G(y)] R_G(x, y)
+ \sum_{x \in H, y = v} [d_G(x) + d_G(y)] R_G(x, y) + \sum_{x = u, y = v} [d_G(x) + d_G(y)] R_G(x, y)
\]

and analogously,

\[
D_R(G_1) = \left[ \sum_{x,y \in H} + \sum_{x \in A} + \sum_{x \in B} + \sum_{y \in H} \right] [d_{G_1}(x) + d_{G_1}(y)] R_{G_1}(x, y) + \sum_{x \in B, y \in H} [d_{G_1}(x) + d_{G_1}(y)] R_{G_1}(x, y)
+ \sum_{x \in A, y \in B} [d_{G_1}(x) + d_{G_1}(y)] R_{G_1}(x, y) + \sum_{x \in A, y = u} [d_{G_1}(x) + d_{G_1}(y)] R_{G_1}(x, y)
+ \sum_{x \in B, y = v} [d_{G_1}(x) + d_{G_1}(y)] R_{G_1}(x, y) + \sum_{x \in H, y = u} [d_{G_1}(x) + d_{G_1}(y)] R_{G_1}(x, y)
+ \sum_{x \in H, y = v} [d_{G_1}(x) + d_{G_1}(y)] R_{G_1}(x, y) + \sum_{x = u, y = v} [d_{G_1}(x) + d_{G_1}(y)] R_{G_1}(x, y)
\]

So we get

\[
D_R(G) - D_R(G_1) = q \left[ \sum_{y \in H} [2 + d_G(y)] [R_C(v, y) - R_C(u, y)] + r(3p + q) + r[d_G(u) - d_G(v)] \right]
\]
and by a similar reasoning,
\[ D_R(G) - D_R(G_2) = p \left[ \sum_{y \in H} [2 + d_C(y)][R_C(u, y) - R_C(v, y)] + r(3q + p) + r[d_C(v) - d_C(u)] \right]. \]

If \( D_R(G) - D_R(G_1) \leq 0 \), then
\[ \sum_{y \in H} [2 + d_C(y)][R_C(v, y) - R_C(u, y)] + r(d_C(u) - d_C(v)) \leq -r(3p + q). \]

Now,
\[ D_R(G) - D_R(G_2) = p \left[ \sum_{y \in H} [2 + d_C(y)][R_C(u, y) - R_C(v, y)] + r(3q + p) + r[d_C(v) - d_C(u)] \right] \]
\[ = p \left[ - \sum_{y \in H} [2 + d_C(y)][R_C(v, y) - R_C(u, y)] - r[d_C(u) - d_C(v)] \right] + rp(3q + p) \]
\[ \geq p \cdot r(3p + q) + rp(3q + p) > 0. \]

Thus, either \( D_R(G) > D_R(G_1) \) or \( D_R(G) > D_R(G_2) \).

**Lemma 5 ([14])**. Let \( C_k \) be the cycle of size \( k \), and \( v \in C_k \). Then, \( Kf(C_k) = \frac{k^3 - k}{12} \), \( D_R(C_k) = \frac{k^3 - k}{3} \), \( Kf_v(C_k) = \frac{k^2 - 1}{6} \), and \( D_v(C_k) = \frac{k^2 - 1}{3} \).

### 3. Main results

In this section, we characterize the graph with minimum degree resistance distance among the elements of \( \text{Cact}(n; t) \).

**Lemma 6.** Let \( G \) be the graph with minimum degree resistance distance among graphs in \( \text{Cact}(n; t) \). Then \( G \) must satisfy the following three conditions:

(i) \( G \) contains no pendent path with length greater than 1.

(ii) All pendent edges of \( G \) (if any) are incident to a common vertex.

(iii) If \( e \) is a cut edge of \( G \), then \( e \) is a pendent edge.

**Proof.** By Lemma 3, (i) holds. By Lemma 4, (ii) holds.

(iii) Assume that \( G \) has a cut edge \( uv \) which is not a pendent edge. Let \( G_1 \) and \( G_2 \) be the connected components of \( G - uv \), so that \( u \in G_1 \) and \( v \in G_2 \). Construct a new graph \( G' \) as indicated in Fig. 2.

In what follows, we denote by \( H \) the graph obtained by attaching to the vertex \( u \) of \( G \) the pendent vertex \( v \).

It is easy to see that \( G' \) also belongs to \( \text{Cact}(n; t) \). Let \( n_1 = |V(H)| \), \( n_2 = |V(G_2)| \), \( m_1 = |E(H)| \) and \( m_2 = |E(G_2)| \). By Lemma 2,

\[ D_R(G) = D_R(H) + D_R(G_2) + 2m_2Kf_v(G_2) + 2m_1Kf_v(G_2) + (n_2 - 1)D_v(H) + (n_1 - 1)D_v(G_2) \]

and

\[ D_R(G') = D_R(H) + D_R(G_2) + 2m_2Kf_v(H) + 2m_1Kf_v(G_2) + (n_2 - 1)D_u(H) + (n_1 - 1)D_v(G_2). \]
Then
\[ D_K(G) - D_K(G^*) = 2m_2[Kf_u(H) - Kf_u(H)] + (n_2 - 1)[D_h(H) - D_u(H)] \]
\[ = 2m_2 \sum_{x \in H} [R_{h}(x, v) - R_{h}(x, u)] + (n_2 - 1) \sum_{x \in H} d_h(x)[R_{h}(x, v) - R_{h}(x, u)] \]
\[ = 2m_2(n_1 - 2) + (n_2 - 1) \left( \sum_{x \in H} d_h(x) - 2 \right). \]

Since \( uv \) is not a pendant edge, \( n_1 \geq 3, m_1 \geq 2, n_2 \geq 2, \) and \( m_2 \geq 1. \) Thus \( D_K(G) - D_K(G^*) > 0, \) i.e., \( G^* \) is a graph with smaller \( D_K \)-value than \( G, \) a contradiction. \( \blacksquare \)

**Lemma 7.** Let \( G = (V, E) \) be a graph belonging to \( \text{Cact}(n; t), t \geq 3. \) Let \( C_h \) be a cycle with \( h \geq 4 \) vertices, contained in \( G. \) Let there be a unique vertex \( u \) in \( C_h \) which is adjacent to a vertex in \( V(G) \setminus V(C). \) Assuming that \( uv, vw \in E(C), \) construct a new graph \( G^* = G - uw + uv \) as shown in Fig. 3. Then, \( D_K(G) > D_K(G^*). \)

**Proof.** Let \( S \) be the graph obtained by attaching to the vertex \( u \) of \( C_{h-1} \) the pendant vertex \( v. \) From **Lemma 2** it then follows,

\[ D_K(G) = D_K(C_h) + D_K(H) + 2|E(H)|Kf_u(C_h) + 2hKf_u(H) + (|V(H)| - 1)D_u(C_h) + (h - 1)D_u(H) \]

and

\[ D_K(G^*) = D_K(S) + D_K(H) + 2|E(H)|Kf_u(S) + 2hKf_u(H) + (|V(H)| - 1)D_u(S) + (h - 1)D_u(H). \]

Both \( G \) and \( G^* \) belong to \( \text{Cact}(n; t), \) which implies \( |E(G)| = |E(G^*)| = n + t - 1. \) From \( t \geq 3 \) and \( h \geq 4 \) it follows that \( n \geq 8. \) Therefore

\[ D_K(G) - D_K(G^*) = D_K(C_h) - D_K(S) + 2(n + t - 1 - h)[Kf_u(C_h) - Kf_u(S)] + (n - h - 1)[D_u(C_h) - D_u(S)] \]
\[ = \frac{h^2 - 8h + 3}{3} + 2(n + t - 1 - h) + (n - h - 1) \frac{2h - 7}{6} + (n - h - 1) \frac{2h - 4}{3} \] (by **Lemma 5**)
\[ = \frac{h^2 - 8h + 3}{3} + (n - 1 - h) \frac{4h - 11}{3} + (n - h - 1) \frac{2h - 7}{3} \]
\[ \geq \frac{h^2 - 2h - 18}{3} + (n - 1 - h) \frac{4h - 11}{3} \] (by \( t \geq 3). \)

If \( h = 4, \) then \( D_K(G) - D_K(G^*) = \frac{5}{3}n - 12 \geq \frac{4}{3} > 0. \) If \( h = 5, \) then \( D_K(G) - D_K(G^*) = 3n - 19 > 0. \) If \( h \geq 6, \) then \( D_K(G) - D_K(G^*) \geq h^2 - 2h - 18 > 0. \)

This completes the proof. \( \blacksquare \)

**Lemma 8.** Let \( G \) be a cactus graph. Let \( C^*_i = u_a, b, u \) (\( i = 1, 2, \ldots, s \)) be cycles of size 3, attached at a common vertex \( u. \) In addition, let \( C^*_v = v_f, g, v \) (\( j = 1, 2, \ldots, h \)) be cycles of size 3, attached at a common vertex \( v. \) Suppose \( C^*_i \) and \( C^*_v \) are vertex-disjoint for all \( i = 1, 2, \ldots, s, j = 1, 2, \ldots, h. \) Construct two new graphs

\[ G_1 = G - \bigcup_{j=1}^{h} \{vf_j, f_jg, g_v\} + \bigcup_{j=1}^{h} \{uf_j, f_jg, g_u\} \]

and

\[ G_2 = G - \bigcup_{i=1}^{s} \{ua_i, a_i, b_i, u\} + \bigcup_{i=1}^{s} \{va_i, a_i, b_i, v\}. \]

Then either \( D_K(G) > D_K(G_1) \) or \( D_K(G) > D_K(G_2). \)
Proof. Let \( A = \{a_1, b_1, \ldots, a_i, b_i\}, B = \{f_1, g_1, \ldots, f_j, g_k\} \) and \( H = V(G) - A - B - \{u, v\} \). Assume, \( R_c(u, v) = r \).

In the transformation \( G \rightarrow G_1 \) for any pair of vertices \( x, y \) satisfying either \( x, y \in H \), or \( x \in A, y \in B \), or \( x \in A, y \in H \), then the term \( \sum_{x, y} [d_c(x) + d_c(y)]R_c(x, y) \) remains unchanged. Thus,

\[
D_b(G) = \left[ \sum_{x, y \in H} + \sum_{x \in A, y \in B} + \sum_{x \in A, y \in H} [d_c(x) + d_c(y)]R_c(x, y) \right] R_c(x, y) \\
+ \sum_{x \in B, y \in H} [d_c(x) + d_c(y)]R_c(x, y) + \sum_{x \in A, y \in B} [d_c(x) + d_c(y)]R_c(x, y) \\
+ \sum_{x \in A, y = u} [d_c(x) + d_c(y)]R_c(x, y) + \sum_{x \in A, y = v} [d_c(x) + d_c(y)]R_c(x, y) \\
+ \sum_{x \in B, y = u} [d_c(x) + d_c(y)]R_c(x, y) + \sum_{x \in B, y = v} [d_c(x) + d_c(y)]R_c(x, y) \\
+ \sum_{x \in H, y = v} [d_c(x) + d_c(y)]R_c(x, y) + \sum_{x \in H, y = v} [d_c(x) + d_c(y)]R_c(x, y) \\
+ \sum_{x = u, y = v} [d_c(x) + d_c(y)]R_c(x, y)
\]

and analogously,

\[
D_b(G_1) = \left[ \sum_{x, y \in H} + \sum_{x \in A, y \in B} + \sum_{x \in A, y \in H} [d_c(x) + d_c(y)]R_c(x, y) \right] R_c(x, y) \\
+ \sum_{x \in B, y \in H} [d_c(x) + d_c(y)]R_c(x, y) + \sum_{x \in A, y \in B} [d_c(x) + d_c(y)]R_c(x, y) \\
+ \sum_{x \in A, y = u} [d_c(x) + d_c(y)]R_c(x, y) + \sum_{x \in A, y = v} [d_c(x) + d_c(y)]R_c(x, y) \\
+ \sum_{x \in B, y = u} [d_c(x) + d_c(y)]R_c(x, y) + \sum_{x \in B, y = v} [d_c(x) + d_c(y)]R_c(x, y) \\
+ \sum_{x \in H, y = v} [d_c(x) + d_c(y)]R_c(x, y) + \sum_{x \in H, y = v} [d_c(x) + d_c(y)]R_c(x, y) \\
+ \sum_{x = u, y = v} [d_c(x) + d_c(y)]R_c(x, y)
\]
which combines yields (see Fig. 4.Lemma 9.) adjacent to a vertex in \( V \) different from \( v \).

Let \( D \) be the spanning subgraph of \( G \) with vertex set \( V \). Let \( H_1 \) and \( H_2 \) be, respectively, the spanning subgraphs of \( G_1 \) and \( G_2 \) with vertex sets \( V(G_1) \setminus V(C) \) and \( V(G_2) \setminus V(G) \cup \{ v \} \). It is easy to see that \( H_1 \cong H_2 \cong K_{1,r} \). Then by Lemma 2,

\[
D(G_1) = D(G) + D(H_1) + 2rK_f(v) + 2(n + t - 1)K_f(H_1) + rD_u(G) + (n - 1)D_v(H_1)
\]

and

\[
D(G_2) = D(G) + D(H_2) + 2rK_f(v) + 2(n + t - 1)K_f(H_2) + rD_u(G) + (n - 1)D_v(H_2)
\]

which combined yields

\[
D(G_1) - D(G_2) = 2rK_f(v)G - K_f(G_2) + rD_u(G) - D_v(G)
\]

\[
= 2r \left[ \sum_{x \in H} R_C(x, u) + \sum_{x \in C} R_C(x, u) - \sum_{x \in H} R_C(x, v) - \sum_{x \in C} R_C(x, v) \right]
\]

So we get

\[
D(G) - D(G_1) = 2h \left[ \sum_{y \in H} [3 + d_C(y)][R_C(y, y) - R_C(u, y)] + r[10s + 2h] \right]
\]

and in a fully analogous manner:

\[
D(G) - D(G_2) = 2s \left[ \sum_{y \in H} [3 + d_C(y)][R_C(u, y) - R_C(y, y)] + r[10h + 2s] \right].
\]

If \( D(G) - D(G_1) \leq 0 \), then

\[
\sum_{y \in H} [3 + d_C(y)][R_C(y, y) - R_C(u, y)] + r[d_C(u) - d_C(v)] \leq 0.
\]

Now,

\[
D(G) - D(G_2) = 2s \left[ - \sum_{y \in H} [3 + d_C(y)][R_C(y, y) - R_C(u, y)] - r[d_C(u) - d_C(v)] \right]
\]

\[
+ 2s \cdot r[10h + 2s] = 2s \cdot r[10h + 2s] > 0.
\]

Thus, either \( D(G) > D(G_1) \) or \( D(G) > D(G_2) \), which completes the proof.  

**Definition 1.** Let \( G \in \text{Cact}(n; t) \), \( t \geq 2 \). A cycle \( C \) of \( G \) is said to be an *end cycle* if there is a unique vertex \( v \) in \( C \) which is adjacent to a vertex in \( V(G) \setminus V(C) \). This unique vertex \( v \) in \( C \) is called the *anchor* of \( C \).

**Lemma 9.** Let \( G \in \text{Cact}(n; t) \), \( t \geq 2 \), be a cactus without cut edges. Let \( C \) be an end cycle of \( G \) and \( v \) be its anchor. Let \( u \) be a vertex of \( C \) different from \( v \). The graphs \( G_1 \) and \( G_2 \) are constructed by adding \( r \) pendant edges to the vertices \( u \) and \( v \), respectively (see Fig. 4). Then \( D(G_1) > D(G_2) \).

**Proof.** We first note that \( |V(G_1)| = |V(G_2)| = n + r \) and \( |E(G_1)| = |E(G_2)| = |E(G)| + r \).

Let \( H \) be the spanning subgraph of \( G \) with vertex set \( V(G) \setminus V(C) \). Let \( H_1 \) and \( H_2 \) be, respectively, the spanning subgraphs of \( G_1 \) and \( G_2 \) with vertex sets \( V(G_1) \setminus V(G) \cup \{ u \} \) and \( V(G_2) \setminus V(G) \cup \{ v \} \). It is easy to see that \( H_1 \cong H_2 \cong K_{1,r} \). Then by Lemma 2,

\[
D(G_1) = D(G) + D(H_1) + 2rK_f(u) + 2(n + t - 1)K_f(H_1) + rD_u(G) + (n - 1)D_v(H_1)
\]

and

\[
D(G_2) = D(G) + D(H_2) + 2rK_f(v) + 2(n + t - 1)K_f(H_2) + rD_u(G) + (n - 1)D_v(H_2)
\]

which combined yields

\[
D(G_1) - D(G_2) = 2r[K_f(u)G - K_f(G_2) + rD_u(G) - D_v(G)]
\]

\[
= 2r \left[ \sum_{x \in H} R_C(x, u) + \sum_{x \in C} R_C(x, u) - \sum_{x \in H} R_C(x, v) - \sum_{x \in C} R_C(x, v) \right]
\]

\[
+ r \left[ \sum_{x \in H} d_C(x)R_C(x, u) + \sum_{x \in L} d_C(x)R_C(x, u) - \sum_{x \in H} d_C(x)R_C(x, v) - \sum_{x \in C} d_C(x)R_C(x, v) \right]
\]
This completes the proof. ■

**Theorem 1.** Let $G^0(n; t)$ be the graph depicted in Fig. 5. Then $G^0(n; t)$ is the unique element of Cact$(n; t)$, $t \geq 1$, having minimum degree resistance distance.

**Proof.** Cact$(n; 1)$ consists of unicyclic graphs. Thus, based on results from [14], Theorem 1 holds for $t = 1$.

If $t = 2$, then Cact$(n; 2)$ consists of bicyclic graphs. Assume that $G$ is the unique graph having the minimum degree resistance distance in Cact$(n; 2)$. By Lemmas 6 and 9, we conclude that $G$ contains two cycles attached to a common vertex $u$, and that all pendant edges (if any) are also attached to $u$.

By a straightforward calculation we get

$$D_R(G) = -\frac{h_1^3}{3} + \frac{(2n + 1)h_1^2}{3} - \frac{(9n - 1)h_1}{3} - \frac{h_2^2}{3} + \frac{(2n + 1)h_2^2}{3} - \frac{(9n - 1)h_2}{3} + \frac{3n^2 + 5n}{3} - \frac{2}{3}$$

where $h_1$, $h_2$ are the sizes of the cycles of $G$. Then

$$D_R(G) - D_R(G^0(n; 2)) = \left( -\frac{h_1^3}{3} + \frac{(2n + 1)h_1^2}{3} - \frac{(9n - 1)h_1}{3} + 3n + 5 \right)$$

$$+ \left( -\frac{h_2^3}{3} + \frac{(2n + 1)h_2^2}{3} - \frac{(9n - 1)h_2}{3} + 3n + 5 \right).$$

Let

$$f(h) = -\frac{h_1^3}{3} + \frac{(2n + 1)h_1^2}{3} - \frac{(9n - 1)h_1}{3} + 3n + 5.$$

It is elementary to verify that $f(3) = 0$ and $f(h) > 0$ for $h > 3$. Therefore $D_R(G) \geq D_R(G^0(n; 2))$ with equality holding if and only if $G \cong G^0(n; 2)$. Thus the claim of Theorem 1 holds for $t = 2$.

If $t \geq 3$, then due to Lemma 9, if $G$ is a graph with minimum degree resistance distance among graphs in Cact$(n; t)$, then $G$ has at least two end cycles. By consecutive application of Lemmas 7–9, we arrive at the conclusion that $G$ contains $t$ cycles of size 3 attached to a common vertex and $n - 2t - 1$ pendant edges attached at the same vertex, i.e., $G \cong G^0(n; t)$. ■

By a straightforward calculation, we obtain

$$D_R(G^0(n; t)) = -\frac{4}{3}t^2 + \left( \frac{8}{3}n - 6 \right)t + 3n^2 - 7n + 4.$$

It can be shown that the value of $D_R(G^0(n; t))$ increases with $t$, for $1 \leq t \leq (n - 1)/2$. Thus $D_R(G^0(n; 1))$ is the graph with minimum degree resistance distance among all cacti.

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References