Ray chaos in an architectural acoustic semi-stadium system
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Stability and chaos in an inertial two neuron system
Ray acoustics plays an important role in the acoustic designs of enclosed spaces. It provides useful techniques to design architectural shapes in order to avoid acoustic faults. A semi-stadium (SS) system consisting of a semicircular cap and a rectilinear platform represents the typical plan profile of some theaters or auditoriums, such as the Ancient Theatre of Orange, Teatro Olimpico in Vicenza, and Wuhan Qintai Arts Center Theatre. Although the SS system has been widely used in architectural acoustic design, its ray dynamics has not been previously studied. In this paper, we apply a theoretical model to describe the ray behaviours of the SS system. The model can be reduced to the semi-circular and rectilinear platform systems when the rectilinear length is sufficiently small and large. The Lyapunov exponents are employed to quantitatively describe ray dynamics. Two kinds of acoustic faults including flutter-echo in the rectilinear platform system and sound focusing in the semicircular system are discussed. The SS system, however, can effectively eliminate these two acoustic faults by producing ray chaos with positive Lyapunov exponent. Furthermore, the maximal Lyapunov exponent of the SS system shows the scaling law and two scaling exponents behaviors, such as ray chaos.4–8 Chaotic systems can show strong local instability of this system, and thus slight deviations of initial conditions to be exponentially amplified to Ray chaos in an architectural acoustic semi-stadium system

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The semi-stadium system is composed of a semicircular cap and a rectilinear platform. In this study, a dynamic model of the side, position, and angle variables is applied to investigate the acoustic ray chaos of the architectural semi-stadium system. The Lyapunov exponent is calculated in order to quantitatively describe ray instability. The model can be reduced to the semi-circular and rectilinear platform systems when the rectilinear length is sufficiently small and large. The quasi-rectilinear platform and the semicircular systems both produce regular trajectories with the maximal Lyapunov exponent approaching zero. Ray localizations, such as flutter-echo and sound focusing, are found in these two systems. However, the semi-stadium system produces chaotic ray behaviors with positive Lyapunov exponents and reduces ray localizations. Furthermore, as the rectilinear length increases, the scaling laws of the Lyapunov exponent of the semi-stadium system are revealed and compared with those of the stadium system. The results suggest the potential application of the proposed model to simulate chaotic dynamics of acoustic ray in architectural enclosed systems. © 2013 American Institute of Physics. [http://dx.doi.org/10.1063/1.4772969]
remarkably different outputs. Similar phenomena have also been observed in the Sinai billiard as well as in the room system with a convex and four straight walls. However, rectangular and circular rooms have been proven not to be ergodic. In a rectangular room, ray directions are well-defined, but ray positions are uniformly distributed in the room. In a circular room, a ray can take all directions, but does not take all positions.

Differing from the symmetric structures of rectangular, circular, and BS rooms, a semi-stadium (SS) acoustic system consists of a semicircular cap and a rectilinear platform. The SS system represents the typical plan profile of some theaters or auditoriums, such as the Ancient Theatre of Orange, Teatro Olimpico in Vicenza, and Wuhan Qintai Arts Center Theatre. Although the SS system has been widely observed in architectural acoustic design, its ray chaotic dynamics and the influence of room geometric parameters on ray movements have not been previously investigated. In this paper, we focus on the SS system and apply a theoretical model to describe its ray chaotic behaviours. The Lyapunov exponents are calculated to describe the ray instability of the SS system. Furthermore, the SS system can be reduced to the quasi-rectilinear (QR) platform and the SC system by setting \( e = 10^4 \) and 0 in order to investigate how the architectural shapes affect ray movements. Flutter-echo in the QR platform system and sound focusing in the SC system are discussed. Finally, the relationships between the maximal Lyapunov exponent and the rectilinear length are investigated for both SS and BS systems to reveal their differences.

II. RAY MODEL OF THE SS SYSTEM

Previous studies have shown that acoustic ray propagation in a closed room is strongly analogous to the particle trajectory of a billiard system in the limitation of high frequency. An acoustic ray propagates straight through the free space. It is reflected specularly at impact with the boundary according to the law of mirror reflection. The ray between two successive reflections can be determined by projecting the ray position and angle on the boundaries. Thus, we can apply the side, position, and angle variables in the high-frequency limit of geometrical acoustics to derive the acoustic ray model.

We consider the SS system with a semicircular cap of the radius \( R \) and a rectilinear platform of the length \( \varepsilon \), as illustrated in Fig. 1(a). Without loss of generality, a dimensionless variable \( \varepsilon = \varepsilon / R \). The SS system represents a simplified 2D theoretical model of the U-shaped architectural enclosed systems, such as Wuhan Qintai Arts Center Theatre, as shown in Fig. 1(b). The counter-clockwise direction is defined as the positive direction. For the reflection number \( n \), \( \eta_n \) is the side at which the ray is reflected. \( \eta = 0, 1, 2, 3 \) denote the four sides \( AB, BC, CD, DA \) of the SS system, respectively. \( s_n \) is the ratio of the length between the impact position and the left terminal vertex (corner) to the total length of the side \( \eta_n \), \( s_n = 0 \) and 1 corresponds to the left and right terminal vertex (corner) of the side \( \eta_n \), respectively. \( \phi_n \) is the angle between the reflected ray and the tangent direction of the side in the counter-clockwise fashion, as shown in Fig. 1(a).

According to these definitions and their geometrical relationship, we can obtain the following dynamic model:

\[
X_{n+1} = M_n X_n, \tag{1}
\]

where \( X_{n+1} = \{X_{n+1}^{(1)}, X_{n+1}^{(2)}, X_{n+1}^{(3)}\} = \{s_{n+1}, \phi_{n+1}, \eta_{n+1}\} \) is the three-dimensional vector, \( M \) represents the nonlinear map from \((s_n, \phi_n, \eta_n)\) at the n-time reflection to \((s_{n+1}, \phi_{n+1}, \eta_{n+1})\) at the \( n+1 \)-time reflection.

The mathematical expressions of \( M \) can be derived based on the following analysis:

(I) For the n-time reflection at \( \overline{AB} \) and the \( n+1 \)-time reflection at \( \overline{BC} \), the ray satisfies

\[
s_{n+1} = \pi(1 - s_n)\tan(\phi_n)/2, \tag{2}
\]

\[
\phi_{n+1} = \pi/2 - \phi_n, \quad \phi_n \in \left(0, \arctan\left(\frac{2}{\pi(1 - s_n)}\right)\right),
\]

\[
\eta_{n+1} = (\eta_n + 1) \mod 4.
\]

A singularity will appear when \( s_n = 1 \).

(II) For the n-time reflection at \( \overline{AB} \) and the \( n+1 \)-time reflection at \( \overline{CD} \), the ray satisfies
Similarly, the ray that has the n-time reflection at $\overline{CD}$ and the n+1-time reflection at $\overline{AB}$ satisfies Eq. (3) because of the symmetry. In both cases, singularities occur when $s_n = 0$ and 1.

(III) For the n-time reflection at $\overline{AB}$ and the n+1-time reflection at $\overline{DA}$, from Fig. 1(a), we have $\frac{l_{OA} - l_{AA'} \cos(\phi_{n+1})}{\cos(\phi_n)} = \frac{l_{OA} \cos(\phi_{n+1})}{\cos(\phi_n)}$, where $l_{OA} = 1$ and $l_{AA'} = -s_n \tan(\phi_n)$ represent the lengths of $\overline{OA}$ and $\overline{AA'}$, respectively. Then, the ray can be described as

$$s_{n+1} = \left(\phi_n + \arccos[\mp s_n \sin(\phi_n) + \cos(\phi_n)] - \pi\right) / \pi,$$

$$\phi_{n+1} = \arccos[\mp s_n \sin(\phi_n) + \cos(\phi_n)], \quad \phi_n \in \left(\pi - \arctan\left(\frac{2}{E(1-s_n)}\right), \pi\right),$$

$$\eta_{n+1} = (\eta_n + 2) \mod 4.$$

A singularity will appear when $s_n = 0$.

(IV) For the n-time reflection at $\overline{BC}$ and the n+1-time reflection at $\overline{CD}$, the ray satisfies

$$s_{n+1} = 2(1 - s_n)\tan(\phi_n) / \pi,$$

$$\phi_{n+1} = \pi / 2 - \phi_n, \quad \phi_n \in \left(0, \arctan\left(\frac{\pi}{2(1 - s_n)}\right)\right),$$

$$\eta_{n+1} = (\eta_n + 1) \mod 4.$$

A singularity will appear when $s_n = 1$.

(V) For the n-time reflection at $\overline{BC}$ and the n+1-time reflection at $\overline{DA}$, the ray satisfies

$$s_{n+1} = \left\{\phi_n + \arccos[\pi \cos(\phi_n) + (2s_n - 1)\sin(\phi_n)] - \pi / 2\right\} / \pi,$$

$$\phi_{n+1} = \arccos[\mp \cos(\phi_n) + (2s_n - 1)\sin(\phi_n)], \quad \phi_n \in \left(\pi - \arctan\left(\frac{\pi}{2(1 - s_n)}\right), \pi\right) - \arctan\left(\frac{\pi}{2s_n}\right),$$

$$\eta_{n+1} = (\eta_n + 2) \mod 4.$$

Singularities occur when $s_n = 0$ and 1.

(VI) For the n-time reflection at $\overline{BC}$ and the n+1-time reflection at $\overline{AB}$, the ray satisfies

$$s_{n+1} = 1 + 2s_n \tan(\phi_n) / \pi,$$

$$\phi_{n+1} = 1.5\pi - \phi_n, \quad \phi_n \in \left(\pi - \arctan\left(\frac{\pi}{2s_n}\right), \pi\right),$$

$$\eta_{n+1} = (\eta_n + 3) \mod 4.$$

A singularity will appear when $s_n = 0$.

(VII) For the n-time reflection at $\overline{CD}$ and the n+1-time reflection at $\overline{DA}$, the ray satisfies

$$s_{n+1} = \left\{\phi_n + \arccos[\pi (s_n - 1)\sin(\phi_n) + \cos(\phi_n)]\right\} / \pi,$$

$$\phi_{n+1} = \arccos[\mp (s_n - 1)\sin(\phi_n) + \cos(\phi_n)], \quad \phi_n \in \left(0, \arctan\left(\frac{2}{E(1-s_n)}\right)\right),$$

$$\eta_{n+1} = (\eta_n + 1) \mod 4.$$

A singularity will appear when $s_n = 1$.

(VIII) For the n-time reflection at $\overline{CD}$ and the n+1-time reflection at $\overline{BC}$, the ray satisfies

$$s_{n+1} = 1 + \pi s_n \tan(\phi_n) / 2,$$

$$\phi_{n+1} = 1.5\pi - \phi_n, \quad \phi_n \in \left(\pi - \arctan\left(\frac{2}{\pi s_n}\right), \pi\right),$$

$$\eta_{n+1} = (\eta_n + 3) \mod 4.$$

A singularity will appear when $s_n = 0$. 

$$s_{n+1} = 1 - s_n - 2\cot(\phi_n) / \pi,$$

$$\phi_{n+1} = \pi - \phi_n, \quad \phi_n \in \left(\arctan\left(\frac{2}{E(1-s_n)}\right), \pi - \arctan\left(\frac{2}{E s_n}\right)\right),$$

$$\eta_{n+1} = (\eta_n + 2) \mod 4.$$
(IX) For the n-time reflection at the semicircular DA and the n+1 reflection at the upper part of DA, the ray satisfies

\[ s_{n+1} = s_n + 2\phi_n / \pi, \]
\[ \phi_{n+1} = \phi_n, \quad \phi_n \in \left( 0, \frac{(1-s_n)\pi}{2} \right), \]
\[ \eta_{n+1} = \eta_n. \] (10)

(X) For the n-time reflection at DA and the n+1-time reflection at AB, the ray satisfies

\[ s_{n+1} = -\left[ 2\cos^2\left(\frac{s_n \pi}{2}\right) \cot(s_n \pi + \phi_n) + \sin(s_n \pi) \right] / \pi, \]
\[ \phi_{n+1} = \pi(1-s_n) - \phi_n, \quad \phi_n \in \left( \frac{(1-s_n)\pi}{2}, \arctan\left( \frac{\sin(s_n \pi) + \pi}{2 \cos^2\left(\frac{s_n \pi}{2}\right)} \right) - s_n \pi + \frac{\pi}{2} \right), \]
\[ \eta_{n+1} = (\eta_n + 1) \text{mod} 4. \] (11)

A singularity will appear when \( s_n = 1 \).

(XI) For the n-time reflection at DA and the n+1-time reflection at BC, the ray satisfies

\[ s_{n+1} = 0.5[\bar{z} + \sin(\pi s_n)]\tan(\pi s_n + \phi_n) + \cos^2(\pi s_n/2), \]
\[ \phi_{n+1} = 1.5\pi - \pi s_n - \phi_n, \quad \phi_n \in \left( \arctan\left( \frac{\sin(s_n \pi) + \bar{z}}{2 \cos^2\left(\frac{s_n \pi}{2}\right)} \right) - s_n \pi + \frac{\pi}{2} \right) - \arctan\left( \frac{\sin(s_n \pi) + \bar{z}}{2 \sin^2\left(\frac{s_n \pi}{2}\right)} \right) - s_n \pi \right), \]
\[ \eta_{n+1} = (\eta_n + 2) \text{mod} 4. \] (12)

Singularity occurs when \( s_n = 0 \) and 1.

(XII) For the n-time reflection at DA and the n+1-time reflection at CD, the ray satisfies

\[ s_{n+1} = 1 - \left[ 2\sin^2\left(\frac{s_n \pi}{2}\right) \cot(s_n \pi + \phi_n) - \sin(s_n \pi) \right] / \pi, \]
\[ \phi_{n+1} = \pi(2-s_n) - \phi_n, \quad \phi_n \in \left( \frac{3\pi}{2} - \arctan\left( \frac{\sin(s_n \pi) + \bar{z}}{2 \sin^2\left(\frac{s_n \pi}{2}\right)} \right) - s_n \pi, \frac{2-s_n\pi}{2} \right), \]
\[ \eta_{n+1} = (\eta_n + 3) \text{mod} 4. \] (13)

A singularity will appear when \( s_n = 0 \).

(XIII) For the n-time reflection at the semicircular DA and the n+1-time reflection at the lower part of DA, the ray satisfies

\[ s_{n+1} = s_n + 2\phi_n / \pi - 2, \]
\[ \phi_{n+1} = \phi_n, \quad \phi_n \in \left( \frac{(2-s_n)\pi}{2}, \pi \right), \]
\[ \eta_{n+1} = \eta_n. \] (14)

Equations (2)–(14) represent specific formula for the ray model Eq. (1) under 13 kinds of different geometric relationships. Ray movement can be theoretically derived by solving map equations (2)–(14). Thus, Eqs. (2)–(14) give the theoretical models or mathematical formula instead of numerical algorithm, which allow us to determine the acoustic ray motion accurately large whole region (0,1) of \( x \) and derive the variational equations to perform the Lyapunov exponent analysis. In addition, with \( \bar{z} = 0 \), this model is reduced to the SC system; while with \( \bar{z} \to \infty \), it is reduced to the rectilinear platform system. For the sufficiently large \( \bar{z} = 10^4 \), a QR platform system can be obtained. Furthermore, the variable \( s_n = 0 \) or 1 describe that the ray hits the corners A, B, C, and D. Mathematically, \( s_n = 0 \) and 1 causes the singularities in Eqs. (2)–(14); thus, these equation calculations will become errors and cannot be continuously performed beyond that point. However, with respect to the whole region (0,1) of \( s_n \), the measure of these two singularities (\( s_n = 0 \) and 1) is 0 and thus may not affect the ergodicity of the ray SS system. With the similar idea, we can derive the ray model of the 2D BS system that is given in the Appendix. More details about this stadium ray model has been reported in our previous study.\(^6\)
III. LYAPUNOV EXPONENT CALCULATION OF THE RAY SYSTEMS

A chaotic system has extreme sensitivity on initial conditions and a ray perturbation will exponentially diverge with time.\textsuperscript{11} In order to quantitatively measure the average rates of divergence or convergence of neighbouring ray trajectories, Lyapunov exponents of the SS system will be calculated. The classical algorithms\textsuperscript{17–19} for calculating the spectrum of Lyapunov exponents have been developed on the basis of Oseledec’s theorem.\textsuperscript{20} Briefly, for the sufficiently small perturbation of a volume element. For the conservative ray systems, all Lyapunov exponents represent the rate of expansion or divergence of neighbouring ray trajectories, their position difference, and the SS system becomes sufficiently chaotic and is always ergodic and diffuse.\textsuperscript{23,24} The SS system with positive $\lambda_1$ has ray instability and may need a short time to reach the diffuse sound field.

In numerical calculations, all vectors tend to fall along the path of most rapid growth,\textsuperscript{19} corresponding to the maximal Lyapunov exponent $\lambda_1$, that is,

$$\|\delta X_{n+1}\| \sim \|\delta X_1\| \exp(\lambda_1 n).$$

In order to obtain other Lyapunov exponents, the orthonormalization procedure will be used for $\delta X_1^{(i)}$ at each integration time step. An orthonormal set $\{\mathbf{e}_i\}, i = 1, 2, 3$ spanning the same subspace as $\{\delta X_1^{(i)}\}$ was generated using Gramm-Schmidt orthogonalization. Considering the convergent or divergent tendency of these three base vectors, for the sufficiently large time $n$, one can obtain Lyapunov exponents as

$$\hat{\lambda}_i \approx \frac{1}{n} \sum_{j=1}^{n} \log \|\mathbf{e}_j^{(i)}\|.$$ 

In this study, we use the method of Wolf \textit{et al.}\textsuperscript{19} to calculate all Lyapunov exponents of the theoretical ray models. The sum of the Lyapunov exponents represents the rate of expansion of a volume element. For the conservative ray systems, $\sum_{i=1}^{3} \hat{\lambda}_i = 0$ should be obtained.\textsuperscript{19,21,22}

IV. NUMERICAL CALCULATIONS AND DISCUSSIONS

To illustrate the ray instability of the SS system to initial perturbation, we plot the first 20 reflections of two rays in Fig. 2(a). The ray model equations (1)–(14) were numerically calculated with double-precision floating-point precision and $\pi = 1$. The round-off error is about the order of $10^{-16}$. The solid and dashed lines represent two rays with the initial launch angle difference $10^{-6}$ rad and they start at the same dark point. For these two rays, Fig. 2(b) gives the time series of the absolute value of their position difference $\Delta s_n$. In room acoustics, a room with positive Lyapunov exponents is chaotic, and in order of $10^{-16}$. Thus, the SS system with positive $\lambda_1$ has ray instability and may need a short time to reach the diffuse sound field.

For the quasi-rectilinear platform system with $\pi = 10^4$, Fig. 3(a) shows the ray movement. The ray is almost trapped between the two parallel sides $\overline{AB}$ ($\eta = 0$) and $\overline{CD}$ ($\eta = 2$). For an initial launch angle $\phi_1$, the ray trap time can be estimated as the order of $\frac{1}{2} \tan(\phi_1)$ from Eq. (3). The periodic

\begin{equation}
\delta X_{n+1} = DM_n\delta X_n,
\end{equation}

where $DM_n$ is the Jacobian matrix of $M$ with respect to $X_n$ which is initializing from the initial condition $X_1$. Using Eq. (15), we can obtain the relationship between $X_n$ and $X_1$ as

\begin{equation}
\delta X_{n+1} = DM_nDM_{n-1}\delta X_{n-1} = \cdots = \prod_{j=1}^{n} DM_j\delta X_1.
\end{equation}
time series of $s_n$, $\phi_n$, $\eta_n$ is given in Fig. 3(b). In Fig. 3(c), for sufficiently long time, all Lyapunov exponents $\lambda_1, \lambda_2, \lambda_3$ approach 0, indicating the regular dynamics of the quasi-rectilinear platform system. Such a sound ray trap happens more often and longer when $\tau$ is excessively higher. In room acoustics, the phenomenon of sound bouncing back and forth between two hard walls is known as flutter-echo. To minimize flutter echoes, making two facing walls out of parallel and installing high absorber to one or both walls have been practically applied.

For the semicircular system with $\tau = 0$, Fig. 4(a) shows the ray movement. The ray bounces back and forth between the straight side ($\eta = 1$) and the circular side ($\eta = 3$). The periodic time series of $s_n$, $\phi_n$, $\eta_n$ is given in Fig. 4(b). $s_n$ fluctuates around 0.5, showing that the ray moves densely around a small area. In Fig. 4(c), with the time $n$ increases, all Lyapunov exponents approach 0, suggesting that this semicircular system is also regular and is not ergodic. The ray moves densely around a small area. In architecture acoustics, such a ray localization may cause sound focusing in a room or auditorium where the sound intensity level is sufficiently higher than elsewhere. It is undesirable for a uniform diffuse sound field.

For the SS system with $\tau = 1$, Figs. 5(a) and 5(b) show the ray and the corresponding aperiodic time series of $s_n$, $\phi_n$, $\eta_n$, respectively. For sufficiently long time $n$, the Lyapunov exponents can be estimated as $\lambda_1 = 0.66, \lambda_2 \approx 0$, and $\lambda_3 = -0.66$, as shown in Fig. 5(c). In numerical calculations, the singularities of $s_n = 0$ and 1 have not been found even for a long time series (100000 ray collisions in Fig. 5) since their slight differences as low as $10^{-16}$ will significantly deviate from these singular values and allow further calculations due

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**FIG. 3.** The quasi-rectilinear platform system with $\tau = 10^4$. (a) Ray trajectory; (b) Time series of $s_n$, $\phi_n$, $\eta_n$; (c) Lyapunov exponents.

**FIG. 4.** The semicircular system with $\tau = 0$. (a) Ray trajectory; (b) Time series of $s_n$, $\phi_n$, $\eta_n$; (c) Lyapunov exponents.
to the high sensitivity of the SS system. The SS system is chaotic and thus is ergodic. The ray instability of the SS system could better reduce ray localization than both the QR platform and the SC system did, which is important to avoid sound focusing. After a sufficiently long time, ray trajectory will become long-term unpredictable and diffuse homogeneously throughout the whole space. Numerous rays launched from a single point will reach sufficient ergodization. The ergodicity of the chaotic SS ray system causes that the average over many initial conditions equals to the average over long time series. These may cause the uniform distribution characteristic of sound energy density in the diffuse sound field. In room acoustic application, designing a ray system with positive Lyapunov exponents may represent an effective approach to realize a diffuse sound field.

Figure 6 compares the maximal Lyapunov exponent $\lambda_1$ of the SS and BS systems with the increase of $\log(\bar{\tau})$, where the average value of $\lambda_1$ over 100 random initial positions $s_0$ was taken for a certain $\bar{\tau}$. For the SS system, there has the maximal value 0.69 of $\lambda_1$ at $\bar{\tau}_{\text{max}} \approx 1.35$, indicating the largest instability of the SS system. The maximal value $\lambda_1$ of the BS system is 0.94 at $\bar{\tau}_{\text{max}} / C_25^{1.76}$. When $0 < \bar{\tau} < \bar{\tau}_{\text{max}}$ and $\bar{\tau}_{\text{max}} / C_25^{1.76} \leq \bar{\tau} < \infty$, $\lambda_1$ of both systems satisfies the power laws with respect to $\bar{\tau}$ as

$$\lambda_1 \sim \bar{\tau}^\beta.$$  

(20)

Using the curve fitting, with the increase of $\bar{\tau}$, we can obtain two scale exponents of the SS system: $\beta = 0.503 \pm 0.0001$ for $0 < \bar{\tau} < \bar{\tau}_{\text{max}}$ and $\beta = -0.94 \pm 0.01$ for $\bar{\tau}_{\text{max}} < \bar{\tau} < \infty$, and the scale exponents of the BS system are $\beta = 0.502 \pm 0.0001$ for $0 < \bar{\tau} < \bar{\tau}_{\text{max}}$ and $\beta = -0.90 \pm 0.01$ for $\bar{\tau}_{\text{max}} < \bar{\tau} < \infty$. Such scale exponents can also be found in the minimal Lyapunov exponent $\lambda_3$ since $\lambda_3 \approx -\lambda_1$ and $\lambda_2$ approaches 0. Any enclosed space, such as auditoriums, theaters, and stadiums, can be modeled as the composition of arcs and straight lines. If the arc size is too small or too large (such as in the QR platform and SC system), a chaotic room may not be created. As shown in Fig. 6, when $\bar{\tau}$ is too small or too large, the Lyapunov exponents of both room systems approach zero, and thus rays are not chaotic. Chaotic field might be created when the sizes of the arcs and straight lines are comparable. Furthermore, with the increase of $\bar{\tau}$, two scale exponents can be found in both BS and SS systems. Such a power-law property of the BS system has also been numerically and theoretically found in previous studies. However, for the SS system with asymmetric geometric property, as far as we know, it is the first time to reveal such power-law properties, as shown in this study.

Compared with the BS system, the SS system has different characteristics in ray model and chaotic dynamics, which is not symmetry decomposition of the BS system. The
proposed SS ray model equations (2)–(14) may not be deduced from the stadium model equations (A1)–(A8) using linear transformations. The symmetry of the BS system can greatly simplify its ray model, and thus eight irreducible equations are needed to determine the ray motion. However, as an asymmetric ray model, the SS system includes more cases of ray motions. Ray motions from $\overline{AB}$, $\overline{DA}$, $\overline{CD}$ to $\overline{BC}$ should be considered, which results in thirteen irreducible equations governing the ray motion. In addition, as shown in Fig. 6, the maximal Lyapunov exponent of the SS model is lower than that of the BS system, which is more significant for $\varepsilon_{\text{max}} \ll \varepsilon \ll \infty$. It also suggests that there may not have a linear relationship between the geometric dynamics of these two systems; otherwise, their Lyapunov exponents should be equal. There is an auditoria acoustic relevance to be further investigated, but is out of the scope of this study.

V. CONCLUSION

In this paper, we applied a nonlinear model of ray side, position, and angle variables to study acoustic ray propagation with specular reflection in a U-shaped architectural SS system. The Lyapunov exponent analysis quantitatively described the ray instability. The QR platform ray system might cause flutter echoes and the SC system might cause sound focusing. Both produced regular rays with the maximal Lyapunov exponents approach zero. However, the SS system could produce strong ray instability with positive Lyapunov exponent and reduce ray localizations. The evidence of chaos in the SS system was provided. Furthermore, with increased rectilinear length $\varepsilon$, the maximal Lyapunov exponent showed the scaling laws $\lambda_1 \sim \varepsilon^p$ and two scaling exponents were derived and compared with those of the BS system. The maximal Lyapunov exponents of the SS system are lower than that of the BS system. The dynamic model may represent an effective method for analyzing acoustic ray instability in the SS system. This modeling method may also be generalized to investigate ray instability of other geometric acoustic systems, such as polygon, sector, and ellipse, etc. It may show valuable application in simulating chaotic dynamics of acoustic rays in architectural enclosed spaces.

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APPENDIX: RAY MODEL OF THE BS SYSTEM

The Bunimovich stadium consists of two semicircular cups of radius $r$ jointed by two rectilinear parts of length $\varepsilon$, as shown in Fig. 7. In the BS system, the trajectory can be reflected by $\overline{AB}$, $\overline{CD}$, $\overline{BC}$, and $\overline{DA}$. Consider different n-time reflection positions, the ray motions can be classified into two types:\(^8\)

(1) When the n-time reflection occurs at the lines ($\overline{AB}$ and $\overline{CD}$), the $n+1$-time reflection may occur at $\overline{AB}$, $\overline{CD}$, $\overline{BC}$, and $\overline{DA}$, respectively. Then, we have the following three cases:

(I) For the n-time reflection at the lines and the n+1-time reflection at the lines, the ray satisfies

$$s_{n+1} = 1 - s_n - 2 \cot(\phi_n)/\varepsilon,$$

$$\phi_{n+1} = \pi - \phi_n,$$

$$\phi_n \in \left( \arctan\left( \frac{2}{\varepsilon(1 - s_n)} \right), \pi - \arctan\left( \frac{2}{\varepsilon s_n} \right) \right).$$  (A1)

Singularity occurs when $s_n = 0$ and 1.

FIG. 7. The diagram of the BS system.
(II) For the $n$-time reflection at the lines and the $n+1$-time reflection at its left semi-circular, the ray satisfies
\[
\begin{align*}
s_{n+1} &= |\phi_n + \arccos(\sin(\phi_n) + \cos(\phi_n)) - \pi|/\pi, \\
\phi_{n+1} &= \arccos(\sin(\phi_n) + \cos(\phi_n)), \quad \phi_n \in \left[0, \frac{(1 - s_n)\pi}{2}\right).
\end{align*}
\] (A2)

A singularity will appear when $s_n = 0$.

(III) For the $n$-time reflection at the lines and the $n+1$-time reflection at its right semi-circular, the ray satisfies
\[
\begin{align*}
s_{n+1} &= |\phi_n + \arccos(\sin(\phi_n) + \cos(\phi_n))|/\pi, \\
\phi_{n+1} &= \arccos(\sin(\phi_n) + \cos(\phi_n)), \quad \phi_n \in \left[0, \arctan\left(\frac{2}{s_n}\right)\right).
\end{align*}
\] (A3)

A singularity will appear when $s_n = 0$.

(2) When the $n$-time reflection occurs at the semi-circulars ($\widehat{BC}$ and $\widehat{DA}$), the $n+1$-time reflection may occur at $\widehat{AB}, \widehat{CD}, \widehat{BC}$, and $\widehat{DA}$, respectively. Then, we have the following five different cases:

(IV) For the $n$-time reflection at the semi-circular and the $n+1$ reflection at its upper part, the ray satisfies
\[
\begin{align*}
s_{n+1} &= s_n + 2\phi_n/\pi, \\
\phi_{n+1} &= \phi_n, \quad \phi_n \in \left[0, \frac{(1 - s_n)\pi}{2}\right).
\end{align*}
\] (A4)

(V) For the $n$-time reflection at the semi-circular and the $n+1$ reflection at its lower part, the ray satisfies
\[
\begin{align*}
s_{n+1} &= s_n + 2\phi_n/\pi - 2, \\
\phi_{n+1} &= \phi_n, \quad \phi_n \in \left[0, \frac{(2 - s_n)\pi}{2}\right).
\end{align*}
\] (A5)

(VI) For the $n$-time reflection at the semi-circular and the $n+1$ reflection at its upper line, the ray satisfies
\[
\begin{align*}
s_{n+1} &= -\left[\cos^2\left(\frac{s_n\pi}{2}\right)\cot(s_n\pi + \phi_n) + \frac{\sin(s_n\pi)}{2}\right]/\pi, \\
\phi_{n+1} &= \pi(1 - s_n) - \phi_n, \quad \phi_n \in \left[0, \frac{(1 - s_n)\pi}{2}\right), \arctan\left(\frac{\sin(s_n\pi) + \pi}{2\cos^2\left(\frac{s_n\pi}{2}\right)}\right) - s_n\pi + \frac{\pi}{2}.
\end{align*}
\] (A6)

A singularity will appear when $s_n = 1$.

(VII) For the $n$-time reflection at the semi-circular and the $n+1$ reflection at its lower line, the ray satisfies:
\[
\begin{align*}
s_{n+1} &= 1 - \left[2\sin^2\left(\frac{s_n\pi}{2}\right)\cot(s_n\pi + \phi_n) - \frac{\sin(s_n\pi)}{2}\right]/\pi \\
\phi_{n+1} &= \pi(2 - s_n) - \phi_n, \quad \phi_n \in \left[0, \frac{3\pi}{2}\right) - \arctan\left(\frac{\sin(s_n\pi) + \pi}{2\sin^2\left(\frac{s_n\pi}{2}\right)}\right) - s_n\pi + \frac{(2 - s_n)\pi}{2}.
\end{align*}
\] (A7)

A singularity will appear when $s_n = 0$.

(VIII) For the $n$-time reflection at the semi-circular and the $n+1$-time reflection at its opposite semi-circular, the ray satisfies
\[
\begin{align*}
s_{n+1} &= \arccos[\cos(\phi_n) + \frac{\pi}{2}\sin(\phi_n - s_n\pi)], \\
\phi_{n+1} &= \arccos[\cos(\phi_n) + \frac{\pi}{2}\sin(\phi_n)], \\
\phi_n \in \arctan\left(\frac{\sin(s_n\pi) + \pi}{2\cos^2\left(\frac{s_n\pi}{2}\right)}\right) - s_n\pi + \frac{\pi}{2} - \arctan\left(\frac{\sin(s_n\pi) + \pi}{2\sin^2\left(\frac{s_n\pi}{2}\right)}\right) + s_n\pi.
\end{align*}
\] (A8)

Singularity will appear when $s_n = 0$ and 1. Equations (A1)–(A8) represent the ray model of the BS system.


