A stochastic multiscale computational model for predicting the mechanical properties of fiber reinforced concrete

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ABSTRACT

A stochastic multiscale computational model for predicting the mechanical properties of fiber reinforced concrete (FRC), subjected to tensile loading, is proposed. It involves the microscale, the mesoscale and the macroscale. On the mesoscale, the heterogeneity of the material is taken into account by a periodic layout of unit cells of matrix–fiber materials, consisting of short fibers and mortar. Material modeling on the microscale is characterized by a periodic layout of unit cells of matrix-aggregate composite materials, consisting of randomly distributed fine aggregate grains and cement matrix. A new unified micro–meso–macro homogenization procedure, based on two-scale asymptotic expressions, has been established. It is used for deriving formulae for multiscale analysis of FRC. The numerical results for the elastic modulus of FRC are compared with experimental results. The comparison shows that the proposed stochastic multiscale computational method is useful for determination of this mechanical property. The developed model is also applied to investigating the influence of different fiber materials on the elastic modulus, and Poisson’s ratio of FRC.

1. Introduction

Fiber reinforced concrete (FRC) is a relatively novel cementitious composite with fine grained concrete as the matrix and fibers as the reinforcement (Hannant, 1978; Bentur and Mindess, 1990; Brandt, 2008). For an assessment of the failure risk of civil engineering structures, it is important to predict the mechanical behavior and durability performance of FRC. Most existing macro models are phenomenological. They are essentially insufficient for assessing the mechanical performance of FRC on the basis of the underlying mechanical properties at different observation levels, because these properties are closely related to the compositions of the matrix and the reinforcement. Thus, micro–meso–macro coupled analysis is necessary to adequately describe the mechanical behavior of FRC.

Because of the excellent mechanical properties of FRC, numerous physical experimental studies on modifications of the material structure of traditional concrete were reported in the literature.

Hannant (1978) has provided a comprehensive treatment, covering the basic theory, the properties, production, and application of a wide range of fiber cements and concretes, such as steel fibers, polypropylene fibers, and glass fibers in cement and in concrete. Recently, FRC with hybrid combinations of steel and non-metallic (polyester, polypropylene, and glass) fibers were investigated by Sivakumar and Santhanam (2007) in order to quantify the influence of these combinations on plastic shrinkage cracking. Further reviews and descriptions of scientific progress can be found in Swamy and Barr (1989), Zollo (1997) and Harle (2014). Notwithstanding the wide use of FRC, adequate multiscale models of mechanical properties of FRC are still lacking.

Progress in computer technology has rendered the analysis of the microscopic mechanical behavior numerically feasible. Direct numerical modeling of large-scale structures by means of micro-mechanical analysis would, however, be prohibitive because of extensive computational costs. Because of the great difference in scale of macroscopic structures and inhomogeneous microstructures of materials, homogenization techniques for treatment of micro–macro coupling are becoming indispensable for understanding the influence of micromechanics on the mechanical behavior of composite materials.
The underlying theory of such techniques is documented in papers published in the 1970s and in the early 1980s, for instance, in works by Bensoussan et al. (1978), Oleinik et al. (1992), Sanchez-Palencia (1980), Bakhvalov and Panasenko (1984), and in several papers written by applied mathematicians. Practical applications of such techniques to the analysis of composite materials have been reported in the literature. Multiscale analysis of linear elastic reinforced composites by this method have been conducted by Guedes and Kikuchi (1990), Fish and Wagiman (1993), Fish and Belsky (1995), Ghosh et al. (1995). For nonlinear materials, homogenization methods have been extended by Suquet (1985), Fish et al. (1997), Guedes and Kikuchi (1990) and Chung et al. (2004).

Such methods have also been applied to the solution of thermal problems (Kamiński, 2003; Zhang et al., 2005; Yu et al., 2009), thermo-mechanical problems (Yu and Fish, 2002; Han et al., 2010; Terada et al., 2010), and damage problems (Devries et al., 1989; Lee et al., 1999; Fish et al., 1999; Ghosh et al., 2001). Relatively few attempts concerning the development of multiscale models for concrete have been reported in the literature. Hain and Wriggers (2008) have applied a homogenization technique to the elastic response of concrete, Cusatis and Cedolin (2007) have derived an equivalent macroscopic cohesive crack law from the meso-level response, using such a technique. Fracture at the mesoscale has been modeled by a lattice model. Following the same line, Grassl and Jirásek (2010) have analyzed the fracture process of mortar as the matrix. On the microscale, the material consists of mortar phase on the mesoscale. In this research, the fine aggregate and the paste. In a highly disordered arrangement, fine aggregate grains and of the cement paste as the matrix for composite materials with randomly distributed fibers, Li and Cui (2005) developed a statistical second-order two-scale analysis method by introducing a random sample model to predict physical and mechanical properties of random composites. The algorithm for generation of the random unit cell was developed by Yu et al. (2008). It is based on a model for the probability distribution of the aggregates. Furthermore, this analysis method can capture the mechanical characteristics inside the material (He et al., 2012). A substantial loss of data is avoided if the sample of the unit cell can be generated such that the statistical fluctuations on the level of the unit cell are reflected.

In this paper, a novel multiscale computational model for evaluation of the mechanical properties of fiber-reinforced concrete is proposed. The aim of this work is to provide higher accuracy with less effort and computational cost than with conventional methods. A two-step homogenization technique is the key point of this research. This technique comprises a mortar composite and a matrix–fiber composite where the matrix properties follow from the first homogenization step. The multiscale problem is solved by micro–meso–macro modeling and by homogenization procedure proposed in this paper.

The paper is organized as follows. In Section 2, the mesoscale and the microscale representation of FRC are described. Section 3 is devoted to the multiscale formulation for prediction of the mechanical properties of FRC. In Section 4, the algorithm for multiscale computation of FRC is presented. In Sections 5 and 6, numerical results for the mechanical parameters of FRC, obtained by the proposed multiscale analysis, are given.

2. Multiscale representation of FRC

A multiscale representation of FRC, as shown in Fig. 1, is proposed. Since characterization of FRC requires scale transitions between more than two scales, different unit cells must be defined. FRC can be modeled by means of three different length scales $l_j$ ($j = 0, 1, 2$). Herein, two unit cells are introduced: the microscale unit cell $Z$ with a characteristic length $l_0$ of 1–5 cm, which represents the mortar. It consists of two material phases, with $l_2$ ranging from 0.5 to 5 mm: the fine aggregate grains and the cement paste. In a highly disordered arrangement, fine aggregate grains are embedded in the cement paste, building up the isotropic mortar phase on the mesoscale. In this research, the fine aggregate...
grains are represented by spherical inclusions. The FRC, defined by
the mesoscale unit cell \( Y \) with \( l_1 \) of 10–30 cm, can be regarded as a
matrix built up by the mortar with inclusions in form of fibers,
with \( d_1 \) typically ranging from 1 to 3 cm. Similar to the microscale
unit cell, the fibers are represented by ellipsoidal inclusions. The
shape of the inclusions reflects the approximate isotropy of FRC.
Because of \( d_2 \ll l_1 \ll l_0 \), where \( l_0 \) characterizes the mac-
roscale of the investigated domain \( X \), the requirement of separation
of scales is satisfied.

For the FRC, the unit cell configuration is determined by means
of statistical fitting of data. The equivalent unit cell is then found
such that it approximates the target unit cell as closely as possible
in terms of real concrete. Then, two unit cells of FRC are repre-
sented as follows:

1. There exists a constant \( \epsilon_1 \), satisfying the following condition:
   \( 0 < \epsilon_1 \ll l_0 \). Thus, FRC can be regarded as a set of periodic
   unit cells \( Y \). That is, \( X = \bigcup Y \), as shown in Fig. 1, where it is
   assumed that all unit cells \( Y \) have the same probability dis-
   tribution \( x_1 \) of the fibers with the same size \( \epsilon_1 \).

2. In the mesoscale unit cell \( Y \), the fibers are treated as ellip-
soids, approximating their geometrical shapes, and the mor-
tar fills the remainder of the cubic cell. Considering the
shape and the orientation of the ellipsoids, they can be con-
structed with eight random parameters: the location param-
eters, i.e., the central points \( (y^1_0, y^2_0, y^3_0) \) of the ellipsoid, and
the size parameters, i.e., the radius \( r_L \) and the length \( L \) of the
ellipsoid, and the orientation parameters, i.e., the Euler
angles \( (\theta, \phi, \psi) \) of the rotations Fig. 2(a).

3. The mortar can then be regarded as a set of periodic unit
cells \( Z \), as shown in Fig. 1, where it is assumed that all unit
cells have the same probability distribution \( x_2 \) of fine aggre-
gates with the same size \( \epsilon_2 \).

4. In the microscale unit cells \( Z \), the fine aggregates are repre-
sented as spheres and the cement fills the remainder of
the cubic cells. Thus, the aggregates can be constructed with
four random parameters: the central points \( (z^1_0, z^2_0, z^3_0) \) and
the radius \( r_z \) of the sphere Fig. 2(b).

In the following, a detailed description of the proposed method
for multiscale analysis of FRC will be given. The purpose of this
method is to investigate the influence of the microscopic charac-
teristics on the macroscopic behavior. Moreover, the behavior of
each component can be studied by means of homogenization anal-
ysis. This enhances the understanding of the characteristics of the
compositions of the material and of its mechanical behavior at the
macrocontinuum level.

3. Multiscale analysis of FRC

3.1. Governing equations

The mechanical behavior of FRC is controlled by the fiber struc-
ture at the mesoscale and by the fine aggregates at the microscale.
Following the multiscale representation of FRC, the scale factor \( \epsilon_1 \)
at the mesoscale is defined as \( \epsilon_1 = \epsilon \), where \( \epsilon \) denotes the basic
scale factor. At the microscale, the scale factor \( \epsilon_2 \) is defined as

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**Fig. 1.** Multiscale representation of FRC: (a) macroscale FRC structure \( X \); (b) mesoscale unit cell \( Y \); (c) microscale unit cell \( Z \).

**Fig. 2.** Representation of (a) ellipsoids, (b) spheres.
\( \varepsilon_2 = \varepsilon^2 \). For the sake of simplicity, the probability distributions \( \omega_1 \) and \( \omega_2 \) at different scales are denoted as \( \omega \) with \( \omega = \{ \omega_1 : y \in Y; \omega_2 : z \in Z \} \). The governing equation for description of the mechanical behavior of RFR is then given by

\[
- \frac{\partial}{\partial X} \left[ a_{ijkl}(x, \omega) \frac{1}{2} \left( \frac{\partial u_i}{\partial X} + \frac{\partial u_j}{\partial X} \right) \right] = f_i, \quad \text{in } X, \tag{1}
\]

where \( a_{ijkl}(x, \omega) \) (i, j, k, l = 1, 2, 3) are the elastic coefficients with \( \varepsilon \)-periodicity, \( u_i \) is a displacement component, with the superscript \( \varepsilon \) denoting a scale factor, and \( f_i \) is a component of the body force vector.

The boundary conditions are specified as follows:

\[
u_j \frac{\partial u_i}{\partial X} + \frac{\partial u_i}{\partial X} \right) = p_i(x), \quad \text{on } \Gamma_p, \tag{2}
\]

where the boundary of \( X \) consists of the surfaces \( \Gamma_u \) and \( \Gamma_p \) with \( \Gamma_u \cap \Gamma_p = \emptyset \); \( u_i(x) \) is a component of the displacement vector, \( p_i \) is a component of the surface load vector, and \( \nu_j \) (j = 1, 2, 3) are the direction cosines of the outward unit normal to \( \Gamma_p \).

3.2. Equations for the microscale, the mesoscale, and the macroscale

Let \( x, y, z \) denote the coordinate system of the macroscale, the mesoscale, and the microscale, respectively. They are related to one another as follows:

\[
y = \frac{x}{\varepsilon}, \quad z = \frac{y}{\varepsilon}. \tag{3}
\]

A perturbation of the displacement vector \( u^0(x) \) is performed with respect to the meso coordinate system \( y \) and the micro coordinate system \( z \) (Oleinik et al., 1992; Sanchez-Palencia, 1980):

\[
u^0(x) = u^0(x) + \varepsilon u^1(x, y, z) + \varepsilon^2 u^2(x, y, z) + \ldots, \tag{4}
\]

where \( u^0(x) = x - 1,2, \ldots \) are \( Y \)-periodic functions, i.e.,

\[
u^0(x, y, z) = u^0(x, y + Y, z), \tag{5}
\]

and\( Z \)-periodic functions, i.e.,

\[
u^0(x, y, z) = u^0(x, y, z + Z), \tag{6}
\]

respectively. Herein, the asymptotic Eq. (4) is truncated after the quadratic term, which corresponds with the number of scales below the macroscale.

Differentiation with respect to \( x \) is defined as

\[
\frac{\partial}{\partial X} = \frac{\partial}{\partial x} + \varepsilon^{-1} \frac{\partial}{\partial y} + \varepsilon^{-2} \frac{\partial}{\partial z}, \quad (i = 1,2,3). \tag{7}
\]

Since Eq. (1) contains first partial derivatives with respect to \( x_i \) and \( x_l \) within the inner brackets and a first partial derivative of the expression within the outer brackets, substitution of (4) into (1), considering (7), gives

\[
e^{-4} \left\{ - \frac{\partial}{\partial X} \left[ a_{ijkl}(y, z, \omega) \frac{\partial u_i}{\partial X} \right] \right\} + e^{-3} \left\{ - \frac{\partial}{\partial Y} \left[ a_{ijkl}(y, z, \omega) \frac{\partial u_i}{\partial Y} \right] - a_{ijkl}(y, z, \omega) \frac{\partial u_i}{\partial Y} \right\} + e^{-2} \left\{ - \frac{\partial}{\partial Z} \left[ a_{ijkl}(y, z, \omega) \frac{\partial u_i}{\partial Z} \right] - a_{ijkl}(y, z, \omega) \frac{\partial u_i}{\partial Z} \right\} - \frac{\partial}{\partial Z} \left[ a_{ijkl}(y, z, \omega) \left( \frac{\partial u_i}{\partial X} + \frac{\partial u_i}{\partial Y} + \frac{\partial u_i}{\partial Z} \right) \right] \right\} \tag{8}
\]

\[
- e^{-1} \left\{ - \frac{\partial}{\partial X} \left[ a_{ijkl}(y, z, \omega) \left( \frac{\partial u_i}{\partial Y} + \frac{\partial u_i}{\partial Z} \right) \right] \right\} - \frac{\partial}{\partial Y} \left[ a_{ijkl}(y, z, \omega) \left( \frac{\partial u_i}{\partial Y} + \frac{\partial u_i}{\partial Z} \right) \right] \right\} - \frac{\partial}{\partial Z} \left[ a_{ijkl}(y, z, \omega) \left( \frac{\partial u_i}{\partial Y} + \frac{\partial u_i}{\partial Z} \right) \right] \right\} \right\} \right\}
\]

The differential equation which represents the vanishing \( O(\varepsilon^{-4}) \)-term in (8) is obtained as

\[
- \frac{\partial}{\partial Z} \left[ a_{ijkl}(y, z, \omega) \frac{\partial u_i}{\partial Z} \right] = 0. \tag{9}
\]

Since \( u^0 \) is only a function of \( x \) (Oleinik et al., 1992; Sanchez-Palencia, 1980), the above relation is automatically satisfied.

By analogy, the following partial differential equation is determined from the \( O(\varepsilon^{-3}) \)-term:

\[
\frac{\partial}{\partial Y} \left[ a_{ijkl}(y, z, \omega) \frac{\partial u_i}{\partial Y} \right] - \frac{\partial}{\partial Z} \left[ a_{ijkl}(y, z, \omega) \frac{\partial u_i}{\partial Z} \right] = 0. \tag{10}
\]

Since the terms \( \frac{\partial u_i}{\partial X} \) and \( \frac{\partial u_i}{\partial Y} \) vanish, \( u^0 \) is independent of \( z \), which allows writing \( u^0(x, y) \) as follows:

\[
u^0(x, y) = N^0(x, y) \frac{\partial u^1}{\partial X} (x) + \tilde{u}^1(x), \tag{11}
\]

where \( \tilde{u}^1 \) is a function which is independent of \( y \), and \( N^0(x, y) \) is a component of the first characteristic term \( N(y) \).

The following partial differential equation is determined from the \( O(\varepsilon^{-2}) \)-term:

\[
\frac{\partial}{\partial X} \left[ a_{ijkl}(y, z, \omega) \frac{\partial u_i}{\partial X} \right] + \frac{\partial}{\partial Y} \left[ a_{ijkl}(y, z, \omega) \frac{\partial u_i}{\partial Y} + \frac{\partial u_i}{\partial Z} \right] = 0. \tag{12}
\]

Since the first two terms vanish, the following equation is obtained:

\[
\frac{\partial}{\partial X} \left[ a_{ijkl}(y, z, \omega) \left( \frac{\partial u_i}{\partial X} + \frac{\partial u_i}{\partial Y} + \frac{\partial u_i}{\partial Z} \right) \right] = 0. \tag{13}
\]

The terms \( \frac{\partial u_i}{\partial X} \) and \( \frac{\partial u_i}{\partial Y} \) are only functions of \( x \) and \( y \). Hence, \( u^0 \) can be written as follows:

\[
u^0(x, y, z) = M^0(z) \left[ \delta_{xy} \partial_{xy} + \delta_{y} \partial_{y} + \delta_{x} \partial_{x} \right] (x) + \tilde{u}^1(x, y), \tag{14}
\]

where \( \delta_{xy} \) is the Kronecker delta, \( M^0(z) \) is a component of the second characteristic term \( M(z) \) and \( \tilde{u}^1 \) is a function which is independent of \( z \). With the help of (11),

\[
M(x, y, \omega) \frac{\partial u^1}{\partial X} + \frac{\partial u^1}{\partial Y} + \frac{\partial u^1}{\partial Z} \right] = a_{ijkl}(y, z, \omega) \left( \delta_{xy} \partial_{xy} + \delta_{y} \partial_{y} + \delta_{x} \partial_{x} \right) \left( \delta_{xy} \partial_{xy} + \delta_{y} \partial_{y} + \delta_{x} \partial_{x} \right) \tag{15}
\]

is obtained. Substitution of (15) into (13) yields the following partial differential equation, termed microscale equation, because of referring to the microscale domain \( Z \):
of the homogenized FRC, can be evaluated as
\[ \sigma_y = \frac{1}{\text{V}} \int_Y \sigma^{\text{M}}_y(y, \omega) \left( \frac{\partial u_y}{\partial x_i} + \frac{\partial u_y}{\partial x_j} \right) dy. \]  
(26)

From (11) and (14), approximate values of the stresses are obtained as
\[ \sigma_{ij}^{\text{M}} = a_{ijkl}(x, \omega) c_{ij}^{\text{M}} \]
\[ = a_{ijkl}(y, \omega) \left( \frac{\partial u_y}{\partial x_i} + \frac{\partial u_y}{\partial x_j} \right) \]
\[ + \sum_{k=1}^P \frac{1}{\text{V}_k} \int_{Y_k} a_{ijkl}(y, \omega) \left( \frac{\partial u_y}{\partial x_i} + \frac{\partial u_y}{\partial x_j} \right) dy. \]  
(27)

4. Analysis procedure for FRC

4.1. Computation of the elasticity tensor of the homogenized material

According to the unit cell model of FRC at the micro- and the mesoscale, different inclusions, such as fibers and fine aggregates, have different shapes and positions in different samples. From the multiscale formulae (21) and (26), it can be seen that this has an influence on the stiffness of FRC. Thus, as was mentioned in the context of the description of the microscale and mesoscale properties of FRC, the mechanical characteristics of this material, such as the elasticity tensor \( a_{ijkl}(\omega) \) of the homogenized FRC, can be evaluated by means of the proposed multiscale model.

Suppose that the elasticity tensor at the microscale for a sample \( \omega^m_k \) is given as \( a_{ijkl}(y, \omega^m_k) \) and that the elasticity tensor at the mesoscale for a sample \( \omega^c_i \) is given as \( a_{ijkl}^c(y, \omega^c_i) \). Then, the procedure for determination of the stiffness properties of the homogenized FRC with randomly distributed inclusions can be summarized as follows:

(1) Based on the statistical characteristics of FRC at the microscale, for a random distribution \( \omega^m_k \), a sample is generated. Then, Eq. (16) is solved in \( Z \) to obtain \( \mathbf{M}(\mathbf{Z}, \omega^c_i) \). Thereafter, the tensor \( a_{ijkl}^c(y, \omega^c_i) \) corresponding to sample \( \omega^c_i \) is computed by means of (21).

(2) Following the generation of \( P \) samples with random distributions \( \omega^m_k, k = 1, \ldots, P \), and based on Kolmogorov’s strong law of large numbers, the elasticity tensor for the homogenized mortar can be computed with the help of the formula
\[ a_{ijkl}^c(\omega^c_i) = \lim_{P \to \infty} \frac{\sum_{k=1}^P a_{ijkl}^c(y, \omega^c_i)}{P}. \]  
(28)

(3) Analogous to (1), for a random distribution \( \omega^c_i \) at the mesoscale, a sample is generated. Then, Eq. (20) is solved in \( Y \) to obtain \( \mathbf{N}(\mathbf{Y}, \omega^c_i) \). Thereafter, the tensor \( a_{ijkl}(\omega^c_i) \), corresponding to the sample \( \omega^c_i \), is computed by means of (26).
Following the generation of $Q$ samples with random distributions of $a_i^1$, $s_1 = 1, \ldots, Q$, and based on Kolmogorov’s strong law of large numbers, the elasticity tensor for the homogenized FRC can be computed with the help of the formula

$$a_{ijkl}^H = \lim_{Q \to \infty} \frac{\sum_{s=1}^{Q} a_{ijkl}^H(s)}{Q}$$

\hspace{0.5cm} (29)

4.2. Flowchart of the multiscale algorithm

The flowchart of the algorithm for the proposed multiscale method for prediction of the mechanical properties of FRC is given as follows (Table 1):

**Remark 1.** Concerning the microscale and the mesoscale unit cell, it is emphasized that the advocated approach is purely geometry-based. It is independent of the constitutive assumptions and of the physical theory used to describe the behavior of the material system. The basic idea is to replace a set of cells with different randomly distributed inclusions by a statistical, idealized distribution of the inclusions, such that the original set of cells is properly represented, see Fig. 1. An arbitrary cell may not be a useful representative of the mechanical behavior of FRC. Thus, the modified cells are treated by statistical analysis with random parameters. Statistical analysis of the investigated cells provides a good approximation of the solution.

5. Validation of the stochastic multiscale model

5.1. Strategy

So far, the homogenized mechanical properties of FRC were evaluated on the basis of the following main assumptions: (a) the FRC is considered as a three-scale composite medium, composed of cement and fine aggregates on the microscale and of mortar and fibers on the mesoscale, homogenized to an isotropic continuum on the macroscale; and (b) the heterogeneous microscale and mesoscale structures are modeled by periodic cubic cells at two different scales, respectively. Then, the multiscale problem is solved by micro–meso–macro modeling and by a two-step homogenization procedure. This technique comprises a mortar composite at the mortar level, and a matrix–fiber composite, with the matrix properties following from the results of the first homogenization, at the mortar level (see Fig. 1).

At the mortar level, the mortar is represented by a periodic layout of cubic cell units, composed of randomly distributed fine aggregate grains and cement paste as the matrix. The fine aggregates are represented as spheres, and the cement paste fills the remainder of the cubic cell at the micro level. The positions of spheres are assumed to be uniformly distributed inside the microscale cells with sizes $10 \times 10 \times 10$ (mm$^3$), and the radii of the spheres are uniformly distributed in the interval $[0.25, 1]$ mm. The homogenization stiffness properties (Young’s modulus $E$ and Poisson’s ratio $\nu$) at this level can be predicted with the help of Eq. (21).

At the FRC level, FRC is considered as a matrix–fiber composite, where the fibers are treated as long-thin ellipsoids, whereas the mortar, representing the matrix, fills the remainder of the mesoscale cubic cells. The positions of the ellipsoids are assumed to be uniformly distributed inside the mesoscale cells with sizes $25 \times 25 \times 25$ (cm$^3$). The long axes of the ellipsoids are uniformly distributed in the interval $[2, 2.5]$ cm, whereas the lengths of the middle and short axes are uniformly distributed in the interval $[0.3, 0.5]$ cm. The orientations (i.e., the angles between the axes of the ellipsoids and of the coordinates) are uniformly distributed in the interval $[0, \pi]$. The homogenized stiffness predictions for $E$ and $\nu$, computed from Eqs. (26), and the results of Eq. (21) are used as input parameters.

The main purpose of this section is to verify the capability of the proposed multiscale technique to determine the material stiffness of FRC. The developed model will hereafter be validated independently at two different levels: the mortar level and the FRC level, considering experimental results from the open literature. The predictive capacity of the model will then be demonstrated by calculating the material stiffness at different load levels and comparing the results with corresponding experimentally determined stiffness values. Finally, the stiffness of the FRC will be determined as a function of the type and the contents of the fibers. The influence of the number of samples and the one of different samples on the results will be quantified.

Table 1

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{ijkl}$, Volume fraction, elastic properties of fibers, fine aggregates, and cement paste</td>
<td>$\bar{a}_{ijkl}$, and the stresses</td>
</tr>
</tbody>
</table>

| (1) Generate a sample $a_i^2$ in the normalized cell $Z$ according to its random distribution of mechanical properties, and then generate the FE meshes (Fig. 6) | (8) Compute the stresses according to Eq. (27) and obtain the homogenized displacements $u^{\alpha}$ |
| (2) Compute $M(z, a_i^2)$ by solving Eq. (16) and evaluate $\bar{a}_{ijkl}^H(a_i^2)$ according to Eq. (21) from intrinsic material constants $a_{ijkl}$ | |
| (3) Repeat (1)–(2) for different samples $a_i^2$ and compute $\bar{a}_{ijkl}^H$ according to Eq. (28) | |
| (4) Generate a sample $a_i^3$ in the normalized cell $Y$ according to its random distribution of mechanical properties, and then generate the FE meshes | |
| (5) Compute $N(y, a_i^3)$ according to Eq. (20), then evaluate $\bar{a}_{ijkl}^H(a_i^3)$ by formula (26) | |
| (6) Repeat (4)–(5) for different samples $a_i^3$ and compute $\bar{a}_{ijkl}^H$ according to Eq. (29) | |
| (7) Solve the macroscale problem according to Eq. (25) and obtain the homogenized displacements $u^{\alpha}$ | |

5.2. Stiffness validation at the mortar level

As most of the available experimental results refer to the evaluation of Young’s modulus as a function of the hydration degree, herein, the aging Young’s modulus is predicted by means of the developed stochastic multiscale model. The result is compared with experimental results. Using the developed model, the influence of the degree of hydration of the cement paste on the effective elastic properties at the mortar scale is investigated for an overall water-cement ratio of 0.30. The input parameters for the validation sets are given in Table 2.

Fig. 3 contains the result of a comparison of the proposed method with experimental results. As shown in this figure, the model accurately captures the early-age stiffness development at the mortar level. The correlation coefficient between the model predictions and the corresponding experimental values amounts to 95%. The distribution of the fine aggregates has a relatively small influence on Young’s modulus, where the correlation coefficient changes from 95% to 93% for aggregates with uniform distribution.
Fig. 3 also contains a comparison of the elastic modulus, obtained by the proposed method, with results, based on the Hashin method (Hashin and Shtrikman, 1963), at the mortar level. Young’s modulus of the cement paste is the same as the one reported in Wyrzykowski and Lura (2013). It is seen that the proposed method gives a slightly better approximation of the experimental results than the Rosen–Hashin lower bound and a significantly better approximation than the Rosen–Hashin upper bound.

5.3. Stiffness validation at the FRC level

At this level, the homogenized stiffness predictions for $E$ and $\nu$ are compared to corresponding experimental measurements conducted on steel-fiber reinforced concrete (Williamson, 1974). The input parameters for the validation sets are given in Table 3.

Table 3

<table>
<thead>
<tr>
<th>Phase</th>
<th>Young’s modulus (GPa)</th>
<th>Poisson’s ratio</th>
<th>Volume ratio</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel fiber</td>
<td>200</td>
<td>0.300</td>
<td></td>
<td>Williamson (1974)</td>
</tr>
<tr>
<td>Concrete</td>
<td>20.8</td>
<td>0.208</td>
<td></td>
<td>Williamson (1974)</td>
</tr>
</tbody>
</table>

Fig. 3. Comparison of computed and measured stiffness values.

The input parameters for the validation sets at the mortar level (Wyrzykowski and Lura, 2013; Boumiz et al., 1996) are given in Table 2.

Table 2

<table>
<thead>
<tr>
<th>Phase</th>
<th>Young’s modulus (GPa)</th>
<th>Poisson’s ratio</th>
<th>Volume ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cement paste</td>
<td>22.8</td>
<td>0.2</td>
<td>0.6</td>
</tr>
<tr>
<td>Fine aggregate</td>
<td>60.0</td>
<td>0.2</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Table 4

<table>
<thead>
<tr>
<th>Phase</th>
<th>Young’s modulus (GPa)</th>
<th>Poisson’s ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel fibers</td>
<td>200</td>
<td>0.30</td>
</tr>
<tr>
<td>Polypropylene fibers</td>
<td>4</td>
<td>0.38</td>
</tr>
<tr>
<td>Glass fibers</td>
<td>73.1</td>
<td>0.22</td>
</tr>
<tr>
<td>Carbon fibers</td>
<td>380</td>
<td>0.20</td>
</tr>
<tr>
<td>Mortar</td>
<td>30</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Fig. 4. Comparison of values of the material stiffness obtained from experiments and from different numerical models.

Table 4

<table>
<thead>
<tr>
<th>Phase</th>
<th>Young’s modulus (GPa)</th>
<th>Poisson’s ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel fibers</td>
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</tr>
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<td>Mortar</td>
<td>30</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Fig. 5. Model-predicted dependence of the stiffness of FRC on the content of the fibers for four different types of fibers.

and Lura (2013). It is seen that the proposed method gives a slightly better approximation of the experimental results than the Rosen–Hashin lower bound and a significantly better approximation than the Rosen–Hashin upper bound.

5.3. Stiffness validation at the FRC level

At this level, the homogenized stiffness predictions for $E$ and $\nu$ are compared to corresponding experimental measurements conducted on steel-fiber reinforced concrete (Williamson, 1974). The input parameters for the validation sets are given in Table 3.

Fig. 5. Model-predicted dependence of the stiffness of FRC on the content of the fibers for four different types of fibers.
Fig. 4 displays experimentally obtained values of the stiffness as well as computed values. The green line shows the computed values and the black line indicates the experimentally obtained results. The results based on the proposed method agree reasonably well with the experimental results. Moreover, the numerical results based on this method represent better approximations of the experimental results than results obtained by other numerical models. Halpin’s model (Affdl and Kardos, 1976) gives slightly lower results, whereas Ahmad’s model (Ahmad and Lagoudas, 1991) yields slightly higher results. Fig. 4 shows that the mixture model proposed by Ezeldin and Balaguru (1992) results in the largest deviations from the experimental results, whereas the proposed stochastic multiscale model yields the smallest ones. Moreover, as the fiber volume fraction increases, the deviations of the results from Ezeldin’s model from the experimental results increase. This is not the case with the stochastic multiscale model. Hence, it is concluded that the proposed model is sufficiently accurate for computation of the material stiffness of FRC.

5.4. Stiffness of FRC as a function of the type and the content of the fibers

The mechanical properties of FRC do not only depend on the mechanical properties of the individual phases but also on the volume fractions of the fibers (Lo and Chim, 1992). Thus, it is necessary to study their influence on the properties of FRC. In this Section, four types of widely used fibers, i.e., steel fibers (Williamson, 1974; Yang and Liu, 2010; Kuang et al., 2009), polypropylene fibers (Dutra et al., 2010; Chen, 2005), glass fibers (Piggott and Harris, 1980), and carbon fibers (Dutra et al., 2010; Piggott and Harris, 1980), are considered in order to investigate their influence on the mechanical properties of FRC. The mechanical parameters of the fibers are given in Table 4. In the calculations, the volume fractions are limited to 3%, noting that a higher fiber content may result in fiber agglomeration during the mixing procedure.

Fig. 5 shows results concerning the stiffness of FRC for different volume fractions of fibers, obtained by the stochastic multiscale model. As far as the steel fibers, the carbon fibers, and the glass fibers are concerned, the elastic modulus of FRC increases with increasing fiber volume fraction (Fig. 5(a)), which obeys the mixture law, as their elastic moduli are much higher than the modulus of the matrix. However, for polypropylene fibers, the elastic modulus decreases with increasing fiber volume fraction (see Fig. 5(b)), because the elastic modulus of the polypropylene fibers is lower than that of the matrix.

Fig. 5(b) shows that the change of Poisson’s ratio with a change of the volume fraction of fibers made of different materials is less than the one of Young’s modulus. The relative change is in the range of 0.95–1.02. This indicates that the dependence of Poisson’s on the volume fractions of different types of fibers is insignificant. Poisson’s ratio is mainly determined by the mortar.

5.5. Influence of the number of samples and different samples

Because of the random characteristics of FRC on the microscale and the mesoscale, the results obtained from physical experiments exhibit a scatter. In order to simulate this phenomenon and validate the Eq. (29), a Monte-Carlo-like technique is used. Meanwhile, a statistical analysis method is used to study the influence of the number of samples on the investigated mechanical properties, i.e., the influence of the number of samples on the modulus of elasticity.

Fig. 6 displays two different microscale cells with the same statistical characteristic, used for the computation of mortar composite at the mortar level. In other word, they were generated by the same uniform distribution as described in Section 5.1. Fig. 6(a) is used for the computation of red circle in Figs. 7(a), and 6(b) used for the computation of red star in Fig. 7(a).

Fig. 7(a) displays experimentally obtained values of the stiffness on day 8 at the mortar level, whereas Fig. 7(b) shows experimentally obtained values of the stiffness for a fiber volume fraction of 1.5% at the FRC level. Statistically, the different samples should have different results, as shown in Fig. 7(a). But accompanied by the increasing number of samples with the same statistical characteristic, the mathematical expectation of the computation results should converge. Obviously, as shown in Fig. 7, the scatter of data decreases with increasing number of samples. The value of the error value decreases from 10% to 1%. As can be seen, for 30 or more samples a good prediction is obtained. Therefore, thirty samples were taken in this study to avoid an unacceptable scatter of the numerical results.
model is effective in predicting the elastic properties of short-fiber reinforced concrete. In order to investigate the influence of the type and the volume fraction of the fibers on the equivalent elastic modulus of short-fiber reinforced concrete, four different types of this material were chosen for the purpose of a comparison. These types are steel fibers, polypropylene fibers, glass fibers, and carbon fibers. They are widely used in engineering practice.

Despite a number of simplifying hypothesizes adopted in this study to model the geometric and mechanical characteristics of FRC, the numerical results agree very well with experimental results taken as reference. Future work will focus on the influence of the geometric distribution of the inclusions, adopted to represent the aggregates in the periodic cells. The high quality of the results encourages the application of the proposed method to investigate more complex problems such as determination of crack initiation and of the ultimate load of FRC.

6. Summary and conclusions

As a relatively new structural material, short-fiber reinforced concrete seems to have a bright future. As a composite material, its mechanical properties are related to both the mesoscopic characteristics, such as the ones of mortar and short-fibers, and the microscopic characteristics, such as those of cement and fine aggregates. Hence, determination of the mechanical properties of short-fiber reinforced concrete is not trivial. So far, research in this area has been mainly based on experimental tests, whereas in practical applications of this material, the determination of mechanical properties is primarily based on empirical formulae. Therefore, the development of reliable and efficient theoretical methods is of great importance for engineering practice.

In this paper a stochastic multiscale model for short-fiber reinforced concrete was presented. It combines microscale and mesoscale structural characteristics with macroscopic mechanical analysis. This combination represents an efficient link between three different scales. By comparing the obtained results with empirical formulae and experimental results, the proposed stochastic multiscale model was validated. It was shown that this method is of great importance for engineering practice.

Despite a number of simplifying hypothesizes adopted in this study to model the geometric and mechanical characteristics of FRC, the numerical results agree very well with experimental results taken as reference. Future work will focus on the influence of the geometric distribution of the inclusions, adopted to represent the aggregates in the periodic cells. The high quality of the results encourages the application of the proposed method to investigate more complex problems such as determination of crack initiation and of the ultimate load of FRC.

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References


Fig. 7. The expected stiffness value with different number of samples: (a) Day 8 at the mortar level; (b) Fiber volume fraction 1.5% at the FRC level.