The minimum numbers of nodal domains of spherical Grushin-harmonics

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\textbf{A R T I C L E I N F O}

\textbf{Article history:}
Received 8 December 2014
Accepted 25 April 2015
Communicated by S. Carl

\textbf{MSC:}
35H20
35B05
33C45

\textbf{Keywords:}
Grushin operator
Nodal domains
Grushin-harmonics

\textbf{ABSTRACT}

In this paper, nodal lines and nodal domains of homogeneous Grushin-harmonic polynomials of degree \(k\) in three variables are studied. It is proved that, for the cases \(k \neq 4m\), \(m \in \mathbb{N}\), there exist spherical Grushin-harmonics \(F\) of degree \(k\) such that the nodal curves of \(F\) divide the unit gauge sphere \(S_\rho\) into two parts; and for the case \(k = 4m\), \(m \in \mathbb{N}\), there is no a spherical Grushin harmonic \(F\) of degree \(k\) such that the nodal curve of \(F\) divides \(S_\rho\) into two parts. This provides a different picture comparing with that of classical spherical harmonics.

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1. Introduction

In this paper we investigate nodal domains of homogeneous Grushin-harmonic (or G-harmonic) polynomials of degree \(k\) in three variables, which are solutions to the so called Baouendi–Grushin operator. We recall that such an operator is generally defined on \(R^N = R^n \times R^1\), \(N = n + 1\) as follows:

\[ L_G u = \sum_{i=1}^{N} X_i X_i u, \]

where the vector fields are given by

\[ X_k = \frac{\partial}{\partial z_k}, \quad k = 1, \ldots, n, \quad X_{n+1} = |z| \frac{\partial}{\partial t}. \]

Here \(z = (z_1, \ldots, z_n) \in R^n, (z, t) \in R^N\). More precisely, we note that

\[ L_G = \Delta_z + |z|^2 \frac{\partial^2}{\partial t^2}, \]

\textsuperscript{*} This work is supported by National Natural Science Foundation of China (No.11401310, No.11401307) and Natural Science Foundation of Jiangsu Province (BK20140965).

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where $\Delta_z = \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2}$. Associated to the operator $L_G$, for $p = (z,t) \in R^N$, there is a gauge given by

$$\rho = \rho(p) = (|z|^4 + 4t^2)^{\frac{1}{2}}.$$ 

The gauge-balls with respect to $\rho$ centered at the origin with radius $r$ is $B^\rho_r = \{p = (z,t) : \rho(p) < r\}$. An important dilation with respect to the operator $L_G$ is as follows:

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t), \quad \lambda > 0, \quad (z,t) \in R^N.$$ 

The gauge $\rho$ is homogeneous of degree one with respect to the dilation. A polynomial $P(z,t)$ is called $G$-homogeneous (or homogeneous, for brief) of degree $k$ if $P(\delta_\lambda(z,t)) = \lambda^k(z,t)$ for every $\lambda > 0$.

$L_G$ is elliptic for $z \neq 0$ and degenerate on the characteristic submanifold $\{0\} \times R^N$. The hypoellipticity of this kind operator was established by V. Grushin [12,13]. N. Garofalo and D. Vassilev [9] obtained strong unique continuation for generalized Baouendi–Grushin operators. B. Franchi and E. Lanconelli [7] studied the Hölder regularity for them. For some other interesting properties related see [1,4,6].

The zero set of a function is called the nodal set. The nodal sets usually reveal some geometric structure, topological property and growth of their functions [14,16,17,20]. Let $n = 2$, $z = (x,y) \in R^2$, $u(z,t)$ be a spherical harmonic of degree $k$, i.e., a homogeneous polynomial solution of $\Delta u = 0$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2}$. When restricted to the unit sphere, these polynomials become eigenfunctions of the Laplace–Beltrami operator on the sphere, with eigenvalue $\lambda = n(n+1)$ of multiplicity $2n+1$, $n = 0, 1, \ldots$. The set \[ \{(x,y,t) : u(x,y,t) = 0, x^2 + y^2 + t^2 = 1\} \] is called the nodal lines of $u$. When $u$ is not constant, the nodal lines divide the unit sphere $S_1 = \{(x,y,t) : x^2 + y^2 + t^2 = 1\}$ into a certain number $N_k(u)$ of domains, called nodal domains. Courant’s nodal domain theorem [3] provides upper bounds of $N_k(u)$ for spherical harmonics. H. Lewy [15] proved that these lower bounds for $N_k(u)$ are actually assumed, i.e. to construct spherical harmonics of degree $k$ whose nodal lines divide the unit sphere in 2 resp. 3 domains for $k$ odd resp., even. See [2,5,10,18] for related results.

In this paper, we study the patterns of nodal lines and domains of homogeneous G-harmonic polynomials of degree $k$ in $R^3$. It is interesting to observe the fact that homogeneous polynomials of same degree may have different parities, this situation does not occur for ordinary polynomials. This causes complete different pictures between nodal domains of classical spherical harmonics and spherical G-harmonics. We obtain that, for the cases $k \neq 4m$, $m \in N$, there exist spherical G-harmonics $F$ of degree $k$ such that the nodal curves of $F$ divide $S_\rho$ into two parts. This indicates that the minimum number of nodal domains is 2 in these cases. We further prove that, for the case $k = 4m$, $m \in N$, there is no a spherical Grushin harmonic $F$ of degree $k$ such that the nodal curve of $F$ divides $S_\rho$ into two parts. This means that the minimum number of nodal domains is at least 3 in this case we also construct some spherical G-harmonic functions of degree $4m$ whose minimum number of nodal domains are exactly 3. It is interesting to point out that, although the minimum number of nodal domains of spherical G-harmonics of degree $k \neq 4m (m \in N)$ are same, the dividing styles of nodal curves of spherical G-harmonics of degree $k = 4m + 1, 4m + 2, 4m + 3 (m \in N)$ are fairly different. Thus we discuss these three cases respectively. More precisely, among other things, we obtain the following:

**Main Theorem.** For $k \neq 4m$, $m \in N$, there exists a spherical $G$-harmonic $F$ with degree $k$ such that the nodal curve of $F$ divides $S_\rho$ into two parts; for $k = 4m$, there are no G-harmonic functions $F$ with degree $k$ such that the nodal curve of $F$ divides $S_\rho$ into two parts.

The rest of this paper is written as follows. In Section 2, we describe the nodal lines of a basis of homogeneous G-harmonic polynomials of degree $k$ in $R^3$ and state a Courant’s type theorem which gives an upper bound of the nodal domains of the spherical G-harmonics. In Section 3, we derive the minimum number of nodal domains of the spherical G-harmonics by constructing the spherical G-harmonics whose number of nodal domains is the minimum. We actually divide the Main Theorem into four theorems: Theorems 3.1, 3.5, 3.6 and 3.7, and prove them, respectively in Section 3.
2. Nodal lines and nodal domains

In this section, we introduce a basis of all homogeneous G-harmonic polynomials of degree \( k \) in \( \mathbb{R}^3 \), and describe the nodal lines of these basis on the unit gauge sphere \( S_\rho = \{(x, y, t) | ((x^2 + y^2)^2 + 4t^2)^{1/4} = 1\} \). Moreover, we will prove an upper bound of numbers of the nodal domains of the spherical G-harmonics.

Firstly, we introduce some suitable polar coordinates for the Grushin operator in \( \mathbb{R}^3 \), see [8,11].

\[
L_G = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (x^2 + y^2) \frac{\partial^2}{\partial t^2}.
\]

(1)

We employ the gauge norm \( \rho \) associated to the operator \( L_G \). Let

\[
\begin{align*}
x &= \rho \sin \frac{\phi}{2} \cos \theta, \\
y &= \rho \sin \frac{\phi}{2} \sin \theta, \\
t &= \rho^2 \cos \phi.
\end{align*}
\]

(2)

Here, \( 0 < \phi < \pi \), and \( 0 < \theta < 2\pi \). We compute the Grushin operator \( L_G \) in the polar coordinates \((\rho, \phi, \theta)\).

A straightforward calculation gives

\[
L_G = \sin \phi \left( \frac{\partial^2}{\partial \rho^2} + \frac{Q - 1}{\rho} \frac{\partial}{\partial \rho} + 4 \frac{\partial^2}{\rho^2 L_G^*} \right),
\]

(3)

where

\[
L_G^* = \frac{\partial^2}{\partial \phi^2} + \cot \phi \frac{\partial}{\partial \phi} + \frac{1}{4} \sin^{-2} \phi \frac{\partial^2}{\partial \theta^2}.
\]

For \( k = 0, 1, \ldots \), let \( u_k = \rho^k g(\phi, \theta) \) be a homogeneous G-harmonic polynomial of degree \( k \) in \( \mathbb{R}^3 \). By using (3), we have

\[
L_G^* g(\phi, \theta) = -\frac{k(k + 2)}{4} g(\phi, \theta).
\]

Suppose that \( g(\phi, \theta) = h(\phi) Y_l(\theta) \), where \( Y_l(\theta) \) are spherical harmonics of degree \( l \) on \( \mathbb{R}^2 \), i.e. \( Y''(\theta) = -l^2 Y_l(\theta) \), that is

\[
Y_l(\theta) = A \cos l\theta + B \sin l\theta.
\]

Therefore, we have

\[
\frac{d^2 h}{d\phi^2} + \cot \phi \frac{dh}{d\phi} + \left( \frac{k(k + 2)}{4} - \frac{l^2}{4 \sin^2 \phi} \right) h = 0.
\]

(4)

Letting \( t = \cos \phi \) and \( w(t) = (1 - t^2)^{-\frac{1}{4}} h(\phi) \) in (4), one observes that \( w \) solves

\[
(1 - t^2) \frac{d^2 w}{dt^2} - (l + 2)t \frac{dw}{dt} + \frac{k - l}{2} \left( \frac{k - l}{2} + l + 1 \right) w = 0.
\]

(5)

This is a Jacobi differential equation, provided \( l \equiv k \pmod{2} \). One polynomial solution of (5) is given by the ultraspherical (or Gegenbauer) polynomial

\[
w(t) = C_{\frac{k+l}{2}}^{\frac{k-l}{2}}(t).
\]

Hence,

\[
u_{k,l} = \rho^k \sin \frac{\phi}{2} C_{\frac{k-l}{2}}^{\frac{k+l}{2}}(\cos \phi) (A \cos l\theta + B \sin l\theta).
\]

(6)
Finally, using the orthogonality properties of spherical harmonics $Y_l(\theta)$ and the orthogonality property of the Gegenbauer polynomial $C^\lambda_l(x)$ (see [19]), we obtain that (6) are a basis of all homogeneous G-harmonic polynomials of degree $k$ in $\mathbb{R}^3$.

Now, we study nodal sets of these homogeneous G-harmonic polynomials in (6). Because of the homogeneity, it is convenient to consider the restriction of the nodal sets of $u_{k,l}$ to the unit gauge sphere $S_\rho$, which are called the nodal lines. It is obvious that nodal lines divide $S_\rho$ into a certain finite number of domains which indicate the topological structure of $u_{k,l}$.

**Theorem 2.1.** There are $\frac{k+l}{2}$ nodal lines of the spherical G-harmonics $u_{k,l}$. These nodal lines can be divided into two groups, one is consisted of $l$ great circles through the pole $\theta = 0$, and another one is consisted of latitude circles.

**Proof.** (i) If $l = 0$, the corresponding polynomial $u_{k,l}$ is a constant multiple of the Gegenbauer polynomial $C^\frac{1}{2}_l(\cos \phi)$.

Using the theory of zeros of orthogonal polynomials (see [19]), it can show that $u_{k,0}$ has $\frac{k}{2}$ distinct zeros between 0 and $\pi$, arranged symmetrically about $\phi = \frac{\pi}{2}$. Accordingly on unit sphere $S_\rho$, the function $u_{k,0}$ vanishes on $\frac{k}{2}$ latitude circles, and there nodal lines divide $S_\rho$ into $\frac{k}{2} + 1$ nodal domains. We note that $l = 0$ means $k$ is an even number, so $\frac{k}{2}$ is integer (see Fig. 1).

(ii) If $0 < l < k$, $l \equiv k \pmod{2}$, the spherical G-harmonics $u_{k,l}$ are of the form

$$\sin^{l/2} \phi C^\frac{l+1}{k+1}_k(\cos \phi)(A \cos l\theta + B \sin l\theta).$$

The first factor vanishes at the points $\phi = 0$ and $\phi = \pi$ and the second on $\frac{k-l}{2}$ latitude circles. The third factor vanishes when $A \cos l\theta + \sin l\theta = 0$, i.e. when $\tan l\theta = -A/B$, and on $S_\rho$ this corresponds to $l$ great circles through the pole $\phi = 0$ (see Fig. 2).

(iii) Finally, if $l = k$, the spherical G-harmonics $u_{k,l}$ are of the form

$$\sin^k \phi (A \cos k\theta + B \sin k\theta)$$

that is $\text{Re}(x + iy)^k$ and $\text{Im}(x + iy)^k$, which vanishes when $\phi = 0$, $\phi = \pi$, or when $\tan k\theta = -A/B$. This corresponds on $S_\rho$ to the points $\phi = 0$, $\phi = \pi$, and to $k$ great circles through these points, the angle between the planes of any two consecutive being $\frac{\pi}{k}$. These great circles divide $S_\rho$ into $2k$ domains (see Fig. 3).

At last, we show an upper bound of numbers of the nodal domains of the spherical G-harmonics. Let $N_k(u)$ denote the number of nodal domains of the spherical G-harmonics $u$ with degree $k$.

We can follow the same argument in [3] (or [2]) to prove the Courant’s nodal domain theorem for the eigenfunctions of $L^*_G$ on the unit gauge sphere.
Theorem 2.2. Assume that $u_i$ is an eigenfunction associated with the $i$th eigenvalue of $L^*_G$ and $N_i$ is the number of nodal domains of $u_i$, then $N_i \leq i + 1$.

Notice that homogeneous G-harmonic polynomials of degree $k$ in $\mathbb{R}^3$, when restricted to $S_\rho$, become eigenfunctions of $L^*_G$ on $S_\rho$, with eigenvalue $\lambda = \frac{k(k+2)}{4}$ of multiplicity $k + 1$, ($k = 0, 1, \ldots$) Therefore, by applying Theorem 2.2, we obtain a upper bound of numbers of nodal domains of spherical G-harmonics.

Theorem 2.3. Let $u_{k,l}$ be the spherical G-harmonics of degree $k$ as in (6), then $N_k(u_{k,l}) \leq \frac{k^2 + k + 2}{2}$.

3. Minimum number of nodal domains

In this section, we prove the lower bounds for $N_k$, and construct spherical G-harmonics of degree $k$ whose nodal lines divide $S_\rho$ into two domains in cases $k = 4m + 1$, $k = 4m + 2$, and $4m + 3$. Moreover, we prove $N_k \geq 3$ for $k = 4m$, and present some examples of spherical harmonics of degree $4m$ such that the number of their nodal domains is actually 3.

3.1. The case $k = 4m + 2$

Theorem 3.1. For $k = 4m + 2$, $m \in \mathbb{N}$, there exists a spherical G-harmonic of degree $k$, such that the nodal lines of this function divide $S_\rho$ into two domains.

In order to prove this theorem, we need the following lemmas.

Lemma 3.2. Let $\Psi(x, y)$ and $f(x, y)$ both are continuous functions in a domain $D$. If $\Psi(x, y) \neq 0$ in $D$, then for small $\epsilon > 0$, $\Psi(x, y) - \epsilon f(x, y) \neq 0$ in $D$. 
Lemma 3.3. Let A be an open Jordan arc on which a polynomial $\Psi(x, y) = 0$. Let $\Psi$ be nonsingular on A, and $f$ satisfy $f(x, y) \neq 0$ in a neighborhood $D$ of A. Then there exists an open connected portion of A, which is denoted by $A'$, such that $A' \subseteq D' \subseteq D$, and the curve $A' = \{(x, y) | \Psi - \epsilon f = 0, (x, y) \in D'\}$ is a nonsingular arc which tends to $A'$ as $\epsilon \to 0$.

Lemma 3.4. If $k > 2$, $\Psi(x, y) = \text{Im}(x + iy)^k$ and $f(x, y)$ is analytic in the neighborhood of the origin such that $f(0, 0) = 1$, then there exists a $\rho > 0$ such that for $\epsilon > 0$ small enough, the equation

$$
\Psi(x, y) - \epsilon f(x, y) = 0
$$

has exactly one solution $y(x)$ in $0 < y \leq x \tan \frac{\pi}{2k}$ for each $x$ in $x_1 \leq x \leq \rho$, where $x_1$ is solution of $\Psi(x_1, x_1 \tan \frac{\pi}{2k})$, and no solution of the Eq. (7) exists in $0 \leq x \leq x_1$.

Here Lemmas 3.2 and 3.3 are taken directly from [15]. Lemma 3.4 is similar to Lemma 4 in [15]. Basing on the above three lemmas, we then will give the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $F(x, y, t) = f(x, y, t) - \epsilon g(x, y, t)$, where $f(x, y, t) = \text{Im}(x + iy)^k$, and $g(x, y, t) = u_{k,0}(x, y, t)$, here $u_{k,0}$ is the basis in (6). We show that $F$ is the desired function for $\epsilon$ sufficiently small.

We cover the sphere $S_{\rho}$ by finitely many closed neighborhoods $\Omega$, each contained in a hemisphere. Each neighborhood is one of the three cases.

1. $f > 0$ through $\Omega$ or $f < 0$ through $\Omega$.
2. $\Omega$ intersects the nodal line of $f$ in one nonsingular arc.
3. $\Omega$ contains $(0, 0, 1)$ or $(0, 0, -1)$.

In each of the $k$ sectors $0 < \theta < \frac{\pi}{k}, \frac{2\pi}{k} < \theta < \frac{3\pi}{k}, \ldots, \frac{2k-2\pi}{k} < \theta < \frac{2k-\pi}{k}$, the $\Omega$ of the third case containing the point $(0, 0, 1)$, for $\epsilon > 0$ small enough, $F(x, y, t)$ defines a U-shaped curve tending to the two sides of the sector as $\epsilon \to 0$ from the above lemmas. Similarly $\Omega$ containing $(0, 0, -1)$ intersects with $F(x, y, t) = 0$ in the sectors $\frac{\pi}{k} < \theta < \frac{2\pi}{k}, \ldots, \frac{2k-1\pi}{k} < \theta < 2\pi$, because $u_{k,0}$ changes the sign when $t$ changes the sign. We see that $F = 0$ is a nonsingular curve of $S$ starting in the sector $0 < \theta < \frac{\pi}{k}$ near $(0, 0, 1)$. Because $g(x, y, t) = u_{k,0}(x, y, t)$ has odd latitudes, the nodal curve will cross the line $\theta = \frac{\pi}{k}$ odd times and arriving in $\frac{\pi}{k} < \theta < \frac{2k-1\pi}{k}$ near $(0, 0, -1)$. Continuing this process, the nodal curve of $F$ will finally come back in $0 < \theta < \frac{\pi}{k}$ near $(0, 0, 1)$. Thus we get a closed Jordan curve on $S_{\rho}$ which divides it into two domains.

Remark 1. In Fig. 4, we show the case $k = 6$ as an example. Here the real lines mean the nodal lines of $f(x, y, t)$, the dotted lines mean the nodal lines of $g(x, y, t)$, and the thick line means the nodal curve of $F$. The origin is the point $(0, 0, 1)$.

3.2. The case $k = 4m + 3$

Theorem 3.5. For $k = 4m + 3, m \in \mathbb{N}$, there exists a spherical G-harmonic $F$ of degree $k$ such that the nodal curve of $F$ divides $S_{\rho}$ into two parts.

Proof. Let $f(x, y, t) = \text{Im}(x + iy)^k$, $g = u_{k,1}(x, y, t)$. One see that the longitude of $g$ is $\{x = 0\}$. Let $F(x, y, t) = f(x, y, t) - \epsilon g(x, y, t)$. We show that for $\epsilon$ small enough, the nodal line of $F$ will divides $S_{\rho}$ into two domains.

In fact, $g(x, y, t) = x \cdot \tilde{g}(x, y, t)$ and $\tilde{g}(0, 0, 1) > 0$, $\tilde{g}(0, 0, -1) < 0$. As in the proof of Theorem 3.1, we will cover $S_{\rho}$ by finitely many closed neighborhoods, and each neighborhood $\Omega$ is one of the three cases.
In each of the sectors $0 < \theta < \frac{\pi}{k}$, $\frac{k-3}{2k}\pi < \theta < \frac{k-1}{2k}\pi$, $\frac{k+3}{2k}\pi < \theta < \frac{k+5}{2k}\pi$, $\frac{3k-3}{2k}\pi < \theta < \frac{3k-1}{2k}\pi$, $\frac{3k+3}{2k}\pi < \theta < \frac{3k+5}{2k}\pi$, $\frac{2k-2}{k}\pi < \theta < \frac{2k-1}{k}\pi$, $\frac{3k-2}{2k}\pi < \theta < \frac{3k-1}{2k}\pi$, $\frac{3k+2}{2k}\pi < \theta < \frac{3k+3}{2k}\pi$ of $\Omega$ of the third case containing $(0, 0, 1)$, for $\epsilon$ sufficiently small, \(F(x, y, t) = 0\) defines a U-shaped curve tending as $\epsilon \to 0$ to the two sides of the sector. In the sectors $\frac{\pi}{k}\pi < \theta < \frac{k+1}{2k}\pi$ and $\frac{\pi}{k}\pi < \theta < \frac{3k+1}{2k}\pi$ in the neighborhood $\Omega$ containing $(0, 0, 1)$ and the sectors $\frac{k-3}{2k}\pi < \theta < \frac{\pi}{k}$ and $\frac{3k-3}{2k}\pi < \theta < \frac{3k-2}{2k}\pi$ in the neighborhood $\Omega$ containing $(0, 0, -1)$, the situations are different. It is obvious that $(0, 0, 1)$ and $(0, 0, -1)$ are zero points of $F$. Next we will show that the nodal line in these sectors will be regular and pass through the points $(0, 0, 1)$ and $(0, 0, -1)$.

Without loss of generality, we only prove the case near $(0, 0, 1)$ and in the sector $\frac{\pi}{2k}\pi < \theta < \frac{k+1}{2k}\pi$. For each fixed $t$ in $\Omega$, consider the equation

\[ F(x, y, t) = f(x, y, t) - \epsilon g(x, y, t) = 0. \]

Here $g(x, y, t) = x \cdot \tilde{g}(x, y, t), f(x, y, t) = y \cdot \tilde{f}(x, y, t)$, and $\tilde{g}(0, 0, 1) > 0, \tilde{f}(0, 0, 1) = 0$. Thus, let $y = a \cdot x$, the above equation becomes

\[ ax \cdot \tilde{f}(a) - \epsilon x \cdot \tilde{g}(a) = 0, \tag{8} \]

where $\tilde{f}(a) = \tilde{f}(x(t), ax(t), t)$, and $\tilde{g}(a) = \tilde{g}(x(t), ax(t), t)$. Notice that $x = 0$ is a solution of (8), but it is not in this sector. Besides this solution, Eq. (8) becomes

\[ a\tilde{f}(a) - \epsilon \tilde{g}(a) = 0. \]

It is obvious that $a = y/x < \tan(\frac{k+1}{2k}\pi)$, $\tilde{f}(a) = 0$ when $a = \tan(\frac{k+1}{2k}\pi)$, $\tilde{f}(a) < 0, \lim_{a \to -\infty} \tilde{f}(a)$ exists, $\tilde{g}(a) < 0$ and $\tilde{g}(a) \to 0$ when $a \to -\infty$. Thus for $\epsilon$ small enough, this equation has exactly one solution. So for each fixed $t$ in the $\Omega, F = 0$ has exactly one solution in the sector $\frac{\pi}{k}\pi < \theta < \frac{k+1}{2k}\pi$. Thus the nodal line of $F$ is a Jordan curve near the point $(0, 0, 1)$ in the sector $\frac{\pi}{2k}\pi < \theta < \frac{k+1}{2k}\pi$ and passes through the point $(0, 0, 1)$. That is what we want.

Then by the same process in the proof of Theorem 3.1, the nodal line of $F$ must be a closed Jordan curve for $\epsilon > 0$ small enough and thus divides $S_\rho$ into two domains.

Remark 2. In Fig. 5, we show the case $k = 3$ as an example.

### 3.3. The case $k = 4m + 1$

**Theorem 3.6.** For $k = 4m + 1, m \in \mathbb{N}$, there exists a spherical G-harmonic $F$ of degree $k$ such that the nodal curve of $F$ divides $S_\rho$ into two domains.
Proof. Let \( F(x, y, t) = f(x, y, t) - \epsilon g(x, y, t) \), where \( f(x, y, t) = u_{k,3}(x, y, t) \) with the longitudes \( \{ y = 0 \}, \{ y = \sqrt{3}x \} \) and \( \{ y = -\sqrt{3}x \} \), \( g(x, y, t) = u_{k,1}(x, y, t) \) with the longitude \( \{ x = 0 \} \). Then for \( \epsilon \) small enough, the nodal curve of \( F \) is a closed Jordan curve and divides \( S_\rho \) into two parts. The proof is similar to the proof of Theorem 3.5.

Remark 3. The Fig. 6 is the example for \( k = 5 \) to explain this theorem.

3.4. The case \( k = 4m \)

Theorem 3.7. For \( k = 4m \), \( m \in \mathbb{N} \), there are no G-harmonic functions \( F \) with degree \( k \) such that the nodal curve of \( F \) divides \( S_\rho \) into two parts.

Proof. If the statement is not true, then one can find a G-harmonic function \( F(x, y, t) \) in degree \( k \) such that the nodal curve of \( F \) divides \( S_\rho \) into two parts. One may write \( F(x, y, t) \) as follows:

\[
F(x, y, t) = \alpha u_{k,0}(x, y, t) + \cdots
\]

If \( \alpha = 0 \), it is easy to check that \( (0, 0, 1) \) and \( (0, 0, -1) \) are singular zero points of \( F(x, y, t) \). Then the nodal curve of \( F \) cannot be a close Jordan curve and thus cannot divides \( S_\rho \) into two parts. So \( \alpha \neq 0 \).

Without loss of generality, we may assume \( \alpha > 0 \). Then \( F(0, 0, 1) > 0 \) and \( F(0, 0, -1) > 0 \) and thus these two points both are in the positive part of \( S_\rho \). So there exists a Jordan curve \( \gamma \) jointing these two points. Noting that \( F(x, y, t) \) is an even function with respect to \( (x, y) \), i.e.,

\[
F(x, y, t) = F(-x, -y, t),
\]
we have another Jordan curve $\gamma'$ which is symmetric to $\gamma$ with respect to the $t$-axis. Then $\gamma \cup \gamma'$ is at least a big circle which is consisted by two longitudes. On the other hand, there exists a point $p_0 = (x_0, y_0, t_0)$ in the negative part, i.e., $F(x_0, y_0, t_0) < 0$. Then from (9), the point $p'_0 = (-x_0, -y_0, t_0)$ is also in the negative part. So there exists a Jordan curve $\beta$ joints these two points $p_0$ and $p'_0$. Again from (9), the curve $\beta'$ which is symmetric to $\beta$ with respect to the $t$-axis also joints these two points. Then $\beta \cup \beta'$ is at least a latitude. It is obvious that $(\gamma \cup \gamma') \cap (\beta \cup \beta') \neq \emptyset$, and this is a contradiction.

**Remark 4.** We have already find that for $k = 4, k = 8,$ and $k = 12$, there exist spherical G-harmonic functions with degree $k$, whose nodal curves divide $S_\rho$ into three parts. We can construct spherical G-harmonics of degree $k$ ($k = 4, 8, 12$) whose nodal lines divide $S_\rho$ into three domains, the corresponding spherical G-harmonics $F_4, F_8$ and $F_{12}$ and the figure of $F_{12}$ (see Fig. 7) are as follows:

\[
\begin{align*}
F_4(x, y, t) & = 8t^2 - (x^2 + y^2)^2, \\
F_8(x, y, t) & = u_{8,8} + 0.1u_{8,2} + 0.01u_{8,0}, \\
\end{align*}
\]

and

\[
F_{12}(x, y, t) = u_{12,12} + 0.1u_{12,6} + 0.01u_{12,2} + 0.001u_{12,0}.
\]

References


