Note

Tetravalent half-arc-transitive graphs of order a product of three primes

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A graph is \textit{half-arc-transitive} if its automorphism group acts transitively on its vertex set, edge set, but not arc set. Let \( n \) be a product of three primes. The problem on the classification of the tetravalent half-arc-transitive graphs of order \( n \) has been considered by Xu (1992), Feng et al. (2007) and Wang and Feng (2010), and it was solved for the cases where \( n \) is a prime cube or twice a product of two primes. In this paper, we solve this problem for the remaining cases. In particular, there exist some families of these graphs which have a solvable automorphism group but are not metacirculants.

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1. Introduction

In this paper, all graphs considered are finite, connected, undirected and simple. Given a graph \( X \), denote by \( V(X) \), \( E(X) \), \( A(X) \) and \( \text{Aut}(X) \) the vertex set, edge set, arc set and automorphism group of \( X \), respectively. A graph \( X \) is said to be \textit{vertex-transitive}, \textit{edge-transitive} and \textit{arc-transitive} (symmetric) if \( \text{Aut}(X) \) acts transitively on \( V(X) \), \( E(X) \) and \( A(X) \), respectively. The graph \( X \) is said to be \textit{half-arc-transitive} provided that it is vertex- and edge- but not arc-transitive. More generally, by a \textit{half-arc-transitive} action of a subgroup \( G \) of \( \text{Aut}(X) \) on \( X \) we shall mean a vertex- and edge-, but not arc-transitive action of \( G \) on \( X \). In this case we say that the graph \( X \) is \( G \)-\textit{half-arc-transitive}.

In 1947, Tutte [40] initiated the investigation of half-arc-transitive graphs by showing that a vertex- and edge-transitive graph with odd valency must be arc-transitive, and few years later, Bouwer [5] gave a construction of \( 2k \)-valent half-arc-transitive graph for every \( k \geq 2 \). Following these two classical articles, half-arc-transitive graphs have been extensively studied from different perspectives over decades by many authors. See, for example, [2,11,18,19,22,24,39,41–43].

One of the standard problems in the study of half-arc-transitive graphs is to classify such graphs of certain orders. Let \( p \) be a prime. It is well-known that there are no half-arc-transitive graphs of order \( p \) or \( p^2 \) [8], and by Cheng and Oxley [9], there are no half-arc-transitive graphs of order \( 2p \). Alspach and Xu [2] classified the half-arc-transitive graphs of order \( 3p \), Kutnar et al. [21] classified half-arc-transitive graphs of order \( 4p \), and Wang [41] classified half-arc-transitive graphs of order a product of two distinct primes. Despite all of these efforts, however, more classifications of half-arc-transitive graphs with general valencies seem to be very difficult.

In view of the fact that 4 is the smallest admissible valency for a half-arc-transitive graph, special attention has rightly been given to the study of tetravalent half-arc-transitive graphs. In particular, constructing and classifying the tetravalent half-arc-transitive graphs is currently one of active topics in algebraic graph theory (for example, see [1,10,12,14–16,23,25–34,38,44,48]).
Let $p$, $q$, $r$ be distinct odd primes and let $n$ be a product of three primes. Then $n \in \{8, 3p^2, 2p^2, 4p, p^2q, 2pq, pqr\}$. Let $X$ be a tetravalent half-arc-transitive graph of order $n$. Clearly, $n \neq 8$ since the smallest half-arc-transitive graph has order 27. If $n = p^2$, $2p^2$, $4p$ or $2pq$ then the graphs $X$ was determined in [13,16,44,47], respectively. In this paper, we give the classification of tetravalent half-arc-transitive graphs of order $pq$ and $p^2q$. Thus, the classification of tetravalent half-arc-transitive graphs of order a product of three primes is determined. In particular, there exist some families of tetravalent half-arc-transitive graphs of order $p^2q$ which have solvable automorphism groups but are not metacirculants. In fact, most of the known vertex-imprimitive half-arc-transitive graphs are metacirculants. Kutnar et al. [20] gave one family of half-arc-transitive graphs which are not metacirculant. It is therefore worth mentioning that some families of tetravalent half-arc-transitive graphs of order $p^2q$ contain precisely unknown half-arc-transitive graphs which are not metacirculants.

2. Preliminary results

Let $X$ be a tetravalent $G$-half-arc-transitive graph for a subgroup $G$ of $\text{Aut}(X)$. Then under the natural $G$-action on $V(X) \times V(X)$, the arc set $\text{A}(X)$ is partitioned into two $G$-orbits, say $A_1$ and $A_2$, which are paired with each other, that is, $A_2 = \{(v, u) \mid (u, v) \in A_1\}$. Each of two corresponding oriented graphs $(V(X), A_1)$ and $(V(X), A_2)$ has out-valency and in-valency which are equal to 2, and admits $G$ as a vertex- and arc-transitive group of automorphisms. Moreover, each of them has $X$ as its underlying graph. Let $D_{C}(X)$ be one of these two oriented graphs, fixed from now on. For an arc $(u, v)$ in $D_{C}(X)$, we say that $u$ and $v$ are the tail and the head of the arc $(u, v)$, respectively. An even length cycle $C$ in $X$ is called a $G$-alternating cycle if the vertices of $C$ are alternately the tail or the head in $D_{C}(X)$ of their two adjacent edges in $C$. It was shown in [25, Proposition 2.4(i)] that, first, all $G$-alternating cycles in $X$ have the same length – half of this length is called the $G$-radius of $X$ and second, that any two adjacent $G$-alternating cycles in $X$ intersect in the same number of vertices, called the $G$-attachment number of $X$. The intersection of two adjacent $G$-alternating cycles is called a $G$-attachment set. We say that $X$ is tightly $G$-attached if its $G$-attachment number coincides with $G$-radius. If $X$ is half-arc-transitive, the terms Aut($X$)-alternating cycle, Aut($X$)-radius, and Aut($X$)-attachment number are referred to as an alternating cycle of $X$, radius of $X$ and attachment number of $X$, respectively. Similarly, if $X$ is tightly Aut($X$)-attached, we say that $X$ is tightly attached. For the purpose of this paper, we introduce a result due to Marušič.

Let $m \geq 3$ be an integer, $n \geq 3$ an odd integer and let $r \in \mathbb{Z}_n^*$ satisfy $r^m = \pm 1$. The graph $X(r; m, n)$ is defined to have vertex set $V = \{u_i^j \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$ and edge set $E = \{(u_i^j, u_{i+1}^{j+1}) \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$.

**Proposition 2.1** ([25, Theorem 3.4]). A connected tetravalent graph $X$ is a tightly attached half-arc-transitive graph of odd radius $n$ if and only if $X \cong X(r; m, n)$, where $m \geq 3$, and $r \in \mathbb{Z}_n^*$ satisfying $r^m = \pm 1$, and moreover none of the following conditions is fulfilled:

1. $r^2 = \pm 1$;
2. $(r; m, n) = (2; 3, 7)$;
3. $(r; m, n) = (r; 6, 7k)$, where $k \geq 1$ is odd, $(7, k) = 1$, $r^6 = 1$, and there exists a unique solution $q \in \{r, -r, r^{-1}, -r^{-1}\}$ of the equation $x^2 + x - 2 = 0$ such that $7(q - 1) = 0$ and $q \equiv 5 \pmod{7}$.

The following proposition is due to Marušič and Praeger [31].

**Proposition 2.2** ([31, Lemma 3.5]). Let $X$ be a connected tetravalent $G$-half-arc-transitive graph for some $G \leq \text{Aut}(X)$, and let $A$ be a $G$-attachment set of $X$. If $|A| \geq 3$, then the vertex-stabilizer of $v \in V(X)$ in $G$ is of order 2.

Let $S$ be a Cayley subset of a finite group $G$. We call $S$ a CI-subset, if for any Cayley subset $T$ of $G$, Cay($G, S$) $\cong$ Cay($G, T$) implies that there is $\alpha \in \text{Aut}(G)$ such that $S^\alpha = T$. The following result is a well-known criterion for CI-subset due to Babai [3].

**Proposition 2.3** ([3]). Let $X = \text{Cay}(G, S)$ be a Cayley graph on a finite group $G$ with respect to $S$. Then $S$ is a CI-subset of $G$ if and only if for any $\sigma \in S_G$ with $\sigma^{-1}R(G)\sigma \leq \text{Aut}(X)$, there exists an $\alpha \in \text{Aut}(X)$ such that $\sigma^{-1}R(G)\alpha = \alpha^{-1}R(G)\sigma$, where $S_G$ denotes the symmetric group on $G$.

The following proposition is straightforward (see [16]).

**Proposition 2.4** ([16, Propositions 2.1 and 2.2]). Let $X = \text{Cay}(G, S)$ be half-arc-transitive. Then $S$ contains no involutions, and there is no $\alpha \in \text{Aut}(G, S)$ such that $s^\alpha = s^{-1}$ for some $s \in S$. In particular, there are no half-arc-transitive Cayley graphs on abelian group.

Let $p$, $q$ be distinct odd primes. To state the classification of connected tetravalent symmetric graphs of order $pq$, we need to define the following graphs. For each divisor $r$ of $p - 1$ we use $H_r$ to denote the unique subgroup of $\mathbb{Z}_p^*$, Define a graph $G(3p, r)$ by

$V(G(3p, r)) = \{x_i \mid i \in \mathbb{Z}_3, x \in \mathbb{Z}_p\}$,

$E(G(3p, r)) = \{xy_{i+1} \mid i \in \mathbb{Z}_3, x, y \in \mathbb{Z}_p \text{ and } y - x \in H_r\}$.

Then $G(3p, r)$ is a connected symmetric graph of order $3p$ and valency 2r. From [2,3,5,36,41,45], we have the following proposition.
Proposition 2.5. Let $X$ be a tetravalent graph of order $pq$ where $q < p$ are distinct odd primes. If $X$ is half-arc-transitive then $2q \mid p - 1$, $(q, p) \neq (3, 7), (5, 11), (11, 23)$, and $\text{Aut}(X) \cong \mathbb{Z}_p \times \mathbb{Z}_{2q}$. If $X$ is symmetric then one of the following holds:

1. $|V(X)| = 3 \cdot 7$ and $\text{Aut}(X) = \text{PGL}(2, 7)$;
2. $|V(X)| = 5 \cdot 7$ and $\text{Aut}(X) = S_7$;
3. $|V(X)| = 5 \cdot 11$ and $\text{Aut}(X) = \text{PGL}(2, 11)$;
4. $|V(X)| = 11 \cdot 23$ and $\text{Aut}(X) = \text{PGL}(2, 23)$;
5. $|V(X)| = 3 \cdot 5$ and $\text{Aut}(X) = S_5$;
6. $X \cong G(3p, 2)$ and $|\text{Aut}(X)| = 12p$;
7. $X$ is a normal Cayley graph on group $\mathbb{Z}_{pq}$ and $|\text{Aut}(X)| = 4pq$.

Let $X$ be a vertex transitive graph and $G = \text{Aut}(X)$. Assume that $N \triangleleft G$. The normal quotient graph $X_N$ of $X$ relative to $N$ is defined as the graph whose vertices are the orbits of $N$ in $V(X)$ and two orbits are adjacent if there is an edge in $X$ between vertices lying in these two orbits. If the valency of $X_N$ equals the valency of $X$, then $X$ is called a normal cover of $X_N$. From [23], we have the following proposition.

Proposition 2.6 ([23]). Let $G$ be a finite group of odd order, and let $X = \text{Cay}(G, S)$ be connected and of valency 4. Assume that $A = \text{Aut}(X)$ and $X$ is edge-transitive. Then one of the following holds:

1. $G$ is normal in $A$;
2. there is a subgroup $M \triangleleft G$ such that $M \triangleleft A$, and $X$ is a normal cover of $X_M$;
3. $A$ has a unique minimal subgroup $N \cong \mathbb{Z}_p^k$ with $p$ odd prime and $k \geq 2$ such that
   - (i) $G = N \rtimes R \cong \mathbb{Z}_p^k \times \mathbb{Z}_m$, where $m > 1$ is odd;
   - (ii) $A = N \rtimes (H \rtimes R) \cong \mathbb{Z}_p^k \rtimes (\mathbb{Z}_p^l \rtimes \mathbb{Z}_m)$, where $H \cong \mathbb{Z}_p^1$ with $2 \leq l \leq k$, satisfying the following statements:
     - (a) there exist $x_1, \ldots, x_k \in N$ and $\tau_1, \ldots, \tau_k \in H$ such that $N = \langle x_1, \ldots, x_k \rangle$ and $\langle \tau_1, \ldots, \tau_k \rangle \cong D_{2p}$ and $H = \langle \tau_i \rangle \times C_{p^i}(x_i)$ for $1 \leq i \leq k$;
     - (b) $R$ does not centralize $H$;
     - (c) $A/(NH) \cong \mathbb{Z}_m$ or $D_{2m}$, and $X$ is arc-transitive if and only if $A/(NH) \cong D_{2m}$;
4. $G \cong \mathbb{Z}_5, \mathbb{Z}_7 \times \mathbb{Z}_3, \mathbb{Z}_{11} \times \mathbb{Z}_5$, or $\mathbb{Z}_{23} \times \mathbb{Z}_{11}$.

The following propositions collect some results about group theory. By checking the orders of the non-abelian simple groups, we have the following proposition.

Proposition 2.7 ([17, pp. 12–14, 135–136]). Let $G$ be a non-abelian simple group and let $p > q > r$ be odd primes. If $|G|$ has at most three prime divisors then $G$ is isomorphic to one of the following groups:

- $A_5, A_6, \text{PSL}(2, 7), \text{PSL}(2, 8), \text{PSL}(2, 17), \text{PSL}(3, 3), \text{PSU}(3, 3), \text{PSU}(4, 2)$.

If $|G| = 2^npqr$ then $G \cong S(8), \text{PSL}(2, p)$, or $\text{PSL}(2, 2^t)$ with an integer $t \geq 4$.

The following proposition gives the structure of the Sylow 2-subgroups of $GL(2, p)$.

Proposition 2.8 ([7]). Let $p$ be a prime and $N$ a Sylow 2-subgroup of $GL(2, p)$. If $p \equiv 1 \pmod{4}$ then $N \cong \mathbb{Z}_{2^t} \rtimes \mathbb{Z}_2$ where $2^t \mid p - 1$ and $2^{t+1} \nmid p - 1$; if $p \equiv 3 \pmod{4}$ then $N \cong (a^2 = b^2 = 1, a^b = a^{q+1-1})$, where $s \geq 3$ such that $2^{s-1} \mid p + 1$ and $2^s \nmid p + 1$. In particular, $N$ has no elementary abelian subgroup of order 8.

It is well known that a finite non-abelian group whose all Sylow subgroups are cyclic is metacyclic defined by the following defining relations:

$G = \langle a, b \rangle, \quad a^n = b^n = 1, \quad a^b = a^t$,

where $(r - 1)n, m) \equiv 1 \pmod{m}, r \neq 1 \pmod{m}$ but $r^n \equiv 1 \pmod{m}$ and $|G| = mn$. It is easy to get the following proposition.

Proposition 2.9. Let $p > q > r$ be odd primes and $G$ a non-abelian group of order $pq$. Then $G$ is isomorphic to one of the following groups:

1. $G_1 = \langle a, b \mid a^p = b^q = 1, a^b = a^r \rangle$ where $s^p \equiv 1 \pmod{p}, s^q \equiv 1 \pmod{q}$ and $s^r \equiv 1 \pmod{p}$;
2. $G_2 = \langle a, b \mid a^d = b^r = 1, a^b = a^i \rangle$ where $s^d \equiv 1 \pmod{d}$ and $s^i \equiv 1 \pmod{i}$;
3. $G_3 = \langle a, b \mid a^d = b^r = 1, a^b = a^i \rangle$ where $s^d \equiv 1 \pmod{i}$ and $s^i \equiv 1 \pmod{i}$;
4. $G_4 = \langle a, b \mid a^p = b^r = 1, a^b = a^i \rangle$ where $s^i \equiv 1 \pmod{pq}$.
By [6, pp. 76–80], we have the following proposition.

**Proposition 3.10.** Let \( p, q \) be distinct odd primes and \( G \) a non-abelian group of order \( p^2 q \). Then \( G \) is isomorphic to one of the following groups:

1. \( H_1 = \{ a, b \mid a^3 = b^4 = 1, a^3 = a' \} \) where \( r^p \equiv 1 \pmod{q} \);
2. \( H_2 = \{ a, b \mid a^3 = b^4 = 1, a^6 = a' \} \) where \( r^p \equiv 1 \pmod{q} \) and \( r^q \not\equiv 1 \pmod{p} \);
3. \( H_3 = \{ a, b \mid a^4 = b^4 = 1, a^6 = a' \} \) where \( r^q \equiv 1 \pmod{p} \);
4. \( H_4 = \{ a, b, c \mid a^6 = b^4 = c^3 = [a, b] = [a, c] \equiv 1, c^2 = c' \} \) where \( r^p \equiv 1 \pmod{q} \);
5. \( H_5 = \{ a, b, c \mid a^6 = b^4 = c^3 = [a, b] = (a, c) \equiv 1, b^6 = b' \} \) where \( r^q \equiv 1 \pmod{p} \);
6. \( H_{6x} = \{ a, b, c \mid a^3 = b^4 = c^3 = [a, b] = 1, a^3 = a', b^6 = b' \} \) where \( r^q \equiv 1 \pmod{p} \) and \( x \neq 0 \); there are \( \frac{1}{2}(q+1) \) non-isomorphic groups;
7. \( H_2 = \{ a, b, c \mid a^6 = b^4 = c^3 = [a, b] = 1, a^6 = a', b^6 = a'^{-1} b^2 p \} \) where \( t^l \equiv 1 \pmod{p} \).

**Remark.** For group \( H_{6x} \), take \( u = b, v = a, w = c \) with \( xy \equiv 1 \pmod{q} \). It is easy to check that \( H_{6x} = \langle a, b, c \mid a^6 = b^4 = c^3 = [a, b] = 1, a^6 = a', b^6 = b' \rangle \equiv \langle u, v, w \mid u^b = v^b = w^s = [u, v] = 1, u^w = u', v^w = v' \rangle = H_{6y} \). Since \( x \in \mathbb{Z}_q^* \), \( x \) has \( 1 + \frac{q-3}{2} = \frac{q+1}{2} \) choices. For group \( H_2 \), \( l \) is a complex number such that \( l^l \equiv 1 \pmod{p} \).

### 3. Classification

In this section, we determine the classification of tetravalent half-arc-transitive graphs of order \( pqr \) and \( p^2q \) where \( p, q, r \) are distinct odd primes. First, we give the following lemma.

**Lemma 3.1.** Let \( G = \langle a, b \mid a^n = b^m = 1, a^6 = a' \rangle \) where \( s^n = 1 \pmod{m} \). For \( k \in \mathbb{Z}_n^* \), set \( C^k := Cay(G, \{b^k, b^{-k}, b^ka, (b^ka)^{-1}\}) \). Then \( C^k \cong X(s^k; n, m) \).

**Proof.** Set \( T_k = [b^k, b^{-k}, b^ka, (b^ka)^{-1}] \). Recall that \( X(s^k; n, m) \) has vertex set \( V = \{ u \mid i \in \mathbb{Z}_n, j \in \mathbb{Z}_m \} \) and edge set \( E = \{ [u_i, u_{i+j}] \mid i \in \mathbb{Z}_n, j \in \mathbb{Z}_m \} \). It is easy to see that \( a^ib^j = a^{i+j} \) for all integers \( i \) and \( j \). Also, one may easily check that the map \( u_i \mapsto (b^ka)^i \) \( i \in \mathbb{Z}_n, j \in \mathbb{Z}_m \) is an isomorphism from \( X(s^k; n, m) \) to \( Cay(G, T) \), where \( T = [b^ka^{-1}, (b^ka^{-1})^{-1}, b^ka, (b^ka)^{-1}] \). For any \( i \in \mathbb{Z}_n^* \), the map \( a \mapsto a^i, b \mapsto b \) induces an automorphism of \( G \). This implies that \( Aut(G) \) is 2-transitive on the set \( \{ b^ka^j \mid i \in \mathbb{Z}_n^* \} \). It follows that \( G \) has an automorphism \( \varphi \) such that \( (b^ka)^i = b^ka \) and \( (b^ka^{-1})^i = b^ka^{-1} \). Then \( \varphi^2 = T_k \), and hence \( \varphi \) is an isomorphism from \( Cay(G, T) \) to \( C^k \). Consequently, \( C^k \cong X(s^k; n, m) \). \( \square \)

Now, we give the classification of tetravalent half-arc-transitive graphs of order \( pqr \).

**Theorem 3.2.** Let \( 3 \leq r < q < p \) be distinct primes and let \( X \) be a connected tetravalent half-arc-transitive graph of order \( pqr \). Then \( X \cong X(s^k; q, r, p) \), \( X(s^k; r, p, q) \), or \( X(s^k; q, pr) \) and \( X \) is a normal Cayley graph on a Frobenius group.

Furthermore, the number of non-isomorphic connected tetravalent half-arc-transitive graphs of order \( pqr \) is equal to

\[
\left\lfloor \frac{q+1}{2} \right\rfloor - 1 \quad \text{if } qr \mid p-1, \\
\frac{q+r}{2} - 1 \quad \text{if } q \mid p-1, r \mid p-1, r \mid q-1, \\
\frac{q-1}{2} \quad \text{if } q \mid p-1, r \mid p-1, r \mid q-1, \\
\frac{r}{2} - 1 \quad \text{if } r \mid p-1, q \mid p-1, \\
\frac{r-1}{2} \quad \text{if } r \mid p-1, q \mid p-1, r \mid q-1, \\
0 \quad \text{otherwise.}
\]

**Proof.** Let \( X \) be a connected tetravalent half-arc-transitive graph of order \( pqr \). Note that the group of order \( pqr \) is a Frobenius group. By [37], we know that \( X \) is a normal Cayley graph on Frobenius group, say \( X = Cay(G, S) \) where \( S = \{x, y, x^{-1}, y^{-1}\} \) with \( o(x) = o(y) \) and \( G = \langle x, y \rangle \). By **Proposition 2.9**, \( G \cong G_1, G_2, G_3, \) or \( G_4 \). Assume that \( G \cong G_1 = \langle a, b \mid a^6 = b^4 = 1, a^3 = a' \rangle \) where \( s^p \equiv 1 \pmod{m} \), \( s^q \not\equiv 1 \pmod{m} \) and \( s^r \not\equiv 1 \pmod{m} \). Set \( T_j = \{ b^j, b^{-j}, b^ka, (b^ka)^{-1} \} \) with \( (j, qr) = 1 \). It is easy to check that \( S \cong T_j \). Note that \( a^{-1}b^j = b^ja^{-j} \). The automorphism of \( G_1 \) induced by \( a \mapsto a^{-j} \), \( b \mapsto b \) maps \( T_j \) to \( T_{qr-j} \). This implies that \( Cay(G_1, T_j) \cong Cay(G_1, T_{qr-j}) \). To complete the proof, it suffices to show that \( Cay(G_1, T_j) \) where \( 1 \leq j \leq \frac{qr-1}{2} \) and \( (j, qr) = 1 \) are pair-wise non-isomorphic.

Set \( A = Aut(X) \). By **Lemma 3.1**, \( Cay(G_1, T_j) \cong X(s^k; q, r, p) \). By **Proposition 2.2**, \( |A| = 2pqr \) and \( A_\circ \cong \mathbb{Z}_2 \) for \( u \in V(Cay(G_1, T_j)) \). It follows that \( R(G_1) < A \). Take \( \sigma \in S_{G_1} \) such that \( \sigma^{-1}R(G)\sigma \leq A \). Suppose that \( R(G)\sigma \neq R(G) \).
Then $R(G)R(G)\sim A$. Thus, $2 \mid \frac{|R(G)||R(G)|'}{|R(G)|R(G)'|'}$, it follows that $2 \mid \text{gcd}(m, n)$, a contradiction. Thus, $R(G)\sim R(G)$, by Proposition 2.3, $T_j$ is a CI-subset of $G$. Let $1 \leq j_1, j_2 \leq \frac{r-1}{2}$ and $(j_2, q) = 1$ with $j_1 \neq j_2$. Suppose that $\text{Cay}(G, T_j) \cong \text{Cay}(G, T_{j_2})$. Since $T_j = \langle b^i, b^{-i}, b^ja, (b^ja)^{-1} \rangle (i = 1, 2)$ are CI-subsets of $G$, $\text{Cay}(G, T_j) \cong \text{Cay}(G, T_{j_2})$ implies that there is a $\beta \in \text{Aut}(G)$ such that $T_{j_2}^\beta = T_{j_2}$. Now that $\beta$ must map $b$ to $a^n b$ for some $m \in \mathbb{Z}_p$. Thus, $(b_1)^\beta = b_1 a^n$ for some $n \in \mathbb{Z}_p$, it means that $b_1 a^n = b_2 b_3 a^n$ for which is impossible because $1 \leq j_1, j_2 \leq \frac{r-1}{2}$ and $(j_2, q) = 1$ with $j_1 \neq j_2$. Thus, $\text{Cay}(G, T_j) \cong \text{Cay}(G, T_{j_2})$. There are exactly $\frac{(q^2-1)(q-1)}{2}$ non-isomorphic such graphs, they are $\text{Cay}(G, T_j)$ for $1 \leq j \leq \frac{q-1}{2}$ and $(j, q) = 1$.

Similarly, if $G \cong G_2$, $G_3$, or $G_4$, then $X \cong X(s'; q, r, p), X(s'; q, p, r)$ or $X(s'; r, p, q)$. Furthermore, there are exactly $\frac{q-1}{2}, \frac{r-1}{2}, \frac{t-1}{2}$ non-isomorphic such graphs, respectively. It is easy to check that $\text{Cay}(G, T_j) \cong \text{Cay}(G, T_{j_2})$ for $1 \leq i, j \leq 4$ and $i \neq j$.

Thus, if $q \mid p - 1$, then $G \cong G_1$, $G_2$, $G_3$, or $G_4$ and there are exactly $\frac{q-1}{2} + \frac{q-1}{2} + \frac{r-1}{2} + \frac{t-1}{2} = \frac{q+r+t+1}{2}$ non-isomorphic such graphs. If $q \mid p - 1$, then $G \cong G_2$, and there are exactly $\frac{q-1}{2}$ non-isomorphic such graphs. If $r \mid p - 1$, $q \mid p - 1$, then $G \cong G_3$, and there are exactly $\frac{q-1}{2}$ non-isomorphic such graphs. If $r \mid p - 1$, $q \mid p - 1$, then $G \cong G_4$, and there are exactly $\frac{q-1}{2} + \frac{r-1}{2} + \frac{r-1}{2} = \frac{q-1}{2}$ non-isomorphic such graphs. If $r \mid p - 1$, $q \mid p - 1$, then $G \cong G_4$ and there are exactly $\frac{q-1}{2}$ non-isomorphic such graphs.

Now, we give the classification of tetravalent half-arc-transitive graphs of order $p^3$. First, we give the following lemmas.

**Lemma 3.3.** Let $p$, $q$ be distinct odd primes and $X$ a tetravalent half-arc-transitive graph of order $p^3 q$. Then $X$ is a normal Cayley graph.

**Proof.** Let $X$ be a tetravalent half-arc-transitive graph of order $p^3 q$ and $u \in V(X)$. Set $A = \text{Aut}(X)$. Note that $A_q$ is a 2-group. Then we may assume that $|A| = 2^m p^3 q$ with some integer $m \geq 1$. Let $N$ be a minimal normal subgroup of $A$. Suppose that $N$ is unsolvable. By Proposition 2.7, $N \cong A_5$, $A_6$, $PSL(2, 7)$, $PSL(2, 8)$ or $PGL(2, 17)$. By [46], there is no tetravalent half-arc-transitive graph of order 45 or 75, and $A_6 = \mathbb{Z}_2$ if $(p, q) = (3, 7)$ or $(7, 3)$. It follows that $N \cong PSL(2, 17)$, and $(p, q) = (3, 17)$. Furthermore, $N$ is transitive on $V(X)$. Let $C = C_A(N)$. Since $C \cap N \cong N$, we have $C \cap N = 1$, implying that $C$ is a 2-group. Clearly, $A$ has no non-trivial normal 2-subgroup. Thus, $C = 1$. By N/C-theorem, we have $A \cong A/C \leq \text{Aut}(N) \cong PGL(2, 17)$. Thus, $A = PSL(2, 17)$ or $PGL(2, 17)$. Note that $X \cong \text{Cos}(A, H, H \cdot (g^{-1} H))$, where $H = A_q$ for some $u \in V(X)$ and $g \in A$ such that $(H, g) = A$. Since $X$ has valency four, there exist $\frac{2}{p+q} = 2$. It follows that $|H \cap H^g| = 8$ or 16, which is impossible by Magma [4]. Thus, $N$ is solvable.

Then $N \cong \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_q$ or $\mathbb{Z}_q$, implying that $|A/N| = 2^m p^3 q, 2^m q$ or $2^m p^2$. Suppose that $A/N$ is unsolvable. By Proposition 2.7, $(p, q) = (3, 5)$ or $(3, 7)$. However, this is impossible by [46]. Thus, $A/N$ is solvable, implying that $A$ is solvable. It follows that the Hall 2-subgroup $G$ of $A$ is regular on $V(X)$, and $X$ is a Cayley graph on $G$. By Proposition 2.6, we have one of the following three cases:

1. $G$ is normal in $A$;
2. there is a subgroup $M < G$ such that $M \triangleleft A$, and $X$ is a normal cover of $X_M$;
3. $A$ has a unique minimal subgroup $N \cong \mathbb{Z}_p^2$ such that $G = N \times S \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$ and $A = N \times (H \times R) \cong \mathbb{Z}_p^2 \times (\mathbb{Z}_q^2 \times \mathbb{Z}_q)$.

For case (1), $X$ is normal. For case (2), there is a subgroup $M < G$ such that $M \triangleleft A$, and $X$ is a normal cover of $X_M$. Then $|X_M| \in \{pq, p, q, p^2\}$. Assume that $|X_M| = \ell$ where $\ell \mid p + q$. If $\ell \mid p - q$ then $X_M$ is a normal Cayley graph. It follows that $G/M \triangleleft A/M$, implying that $G \triangleleft A$, hence $X$ is normal. If $\ell = 5$ then $|A/M| = 10$ because $X_M$ is $A/M$-half-arc-transitive and $A/M \leq S_5$. Again, $G/M \triangleleft A/M$ and $G \triangleleft A$, hence $X$ is normal. Assume that $|X_M| = p^2$. Then $X_M$ is a Cayley graph on abelian group $G/M$. It is well known that $X_M$ is normal, that is, $G/M \triangleleft A/M$. Again, $G \triangleleft A$ and $X$ is normal. Assume that $|X_M| = pq$. Assume that $X_M$ is half-arc-transitive. By Proposition 2.5, $A/M \cong \mathbb{Z}_p \times \mathbb{Z}_q$. It follows that $G/M \triangleleft A/M$ and $G \triangleleft A$, implying that $X$ is normal.

For case (3), $|A| = 4^m p$ and $|A/N| = 4^m$. Then $G/N \triangleleft A/N \cong \mathbb{Z}_p \times \mathbb{Z}_q$ and $\text{gcd}(m, q) = 3$ and $G/N \not\cong A/N$. Then $|A/N| = 12$, it follows that $A/N \cong A_4$. Note that $C_4(N) = N$. Then $A_4 \cong A/N \cong A/C_4(N) \leq GL(2, p)$. Note that the center $Z(GL(2, p))$ of $GL(2, p)$ is $\mathbb{Z}_2$. Let $Z \cap A_4 \cong A_2$, implying that $N \cap A_4 = 1$ and $A_4 = Z \times A_4$. It follows that $GL(2, p)$ has a subgroup which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. By Proposition 2.8, it is impossible. Thus, $G/N \triangleleft A/N$. It follows that $G \triangleleft A$ and hence $X$ is normal. □

**Lemma 3.4.** Let $D_{x,1} = \text{Cay}(H_5, \{ac^i, (ac)^{-1}, abc^i, (abc)^{-1}\})$ and $D_{x,2} = \text{Cay}(H_5, \{ac^i, (ac)^{-1}, a^{-1}bc^i, (a^{-1}bc)^{-1}\})$ for $i \in \mathbb{Z}_q^*$. Then $D_{x,1}$ and $D_{x,2}$ are connected tetravalent half-arc-transitive graphs of order $p^2 q$, and there are exactly $q - 1$ non-isomorphic such graphs. Furthermore, $D_{x,1} \cong X(v; pq, p)$. □
Proof. Recall that $H_5 = \langle a, b, c \mid a^p = b^5 = c^q = [a, b] = [a, c] = 1, b^i = b^j \rangle \cong \mathbb{Z}_p \times F_{pq}$ where $r^q \equiv 1 \pmod{p}$. Set $S_i = \{ac, (ac)^{-1}, abc, (abc)^{-1}\}$ and $T_i = \{ac, (ac)^{-1}, a^{-1}bc, (a^{-1}bc)^{-1}\}$. Clearly, the map $a \mapsto a^{-1}, b \mapsto b^{-i}, c \mapsto c$ induces an automorphism of $H_5$. Thus, $S_i \cong S_{(1-i)}$ and $T_i \cong T_{(1-i)}$. Hence, we may assume that $1 \leq i \leq \frac{1}{2}(q - 1)$. Set $X = \mathcal{D}_{i,2}^i$ or $\mathcal{D}_{i,2}^i$ and $A = \text{Aut}(X)$. First, we prove the following claim.

Claim: $X$ is normal.

The maps $a \mapsto a^{-1}, b \mapsto b^{-i}, c \mapsto c$ such that $(c^i)^a = bc^i$, can induce automorphisms of $H_5$. This means that $X$ is edge transitive. By Proposition 2.6, we have one of the following three conditions:

(1) $H_5$ is normal in $A$;
(2) there exists $M < H_5$ such that $M \triangleleft A$ and $X$ is a normal cover of $X_M$;
(3) $A$ has a unique minimal subgroup $P \cong \mathbb{Z}_p \times \mathbb{Z}_q$ such that $A = P \rtimes ((H \rtimes R), O) \cong \mathbb{Z}_p^2 \times ((\mathbb{Z}_2 \times \mathbb{Z}_q)_T)$ with $t = 1$ or 2. Furthermore, $A/(PH) \cong \mathbb{Z}_p^2$ or $\mathbb{Z}_2^2$, and $X$ is arc-transitive if and only if $A/(PH) \cong \mathbb{Z}_2^2$.

If (1) holds, then $X$ is normal. If (2) holds, then $|M| = p, p^2$ or $pq$ because $M \triangleleft H_5$. It follows that $|X_M| = pq, q$ or $p$. Assume that $|X_M| = p$. Note that $q \mid p - 1$. It follows that $p \geq 7$. It is well known that $X_M$ is a normal Cayley graph. Then $H_5/M \triangleleft A/M$, implying that $H_5 \triangleleft A$. Hence, $X$ is normal. Assume that $|X_M| = q$. If $q > 5$ then $X$ is normal. If $q = 5$ then $X_M \cong S_5$ and $A/M \leq S_5$. It follows that $A/M \cong D_{10}$, $F_{20}$, $A_5$ or $S_5$ because $A/M$ is transitive on the edge set of $X_M$. Suppose that $A/M \cong A_5$ or $S_5$. Then $X_M$ is 2-arc-transitive, implying that $X$ is 2-arc-transitive. It follows that $A_1$ is 2-transitive on $N(1)$. However, it is impossible because there is one 5-cycle passing through 1, $c^i$ and $c^{-i}$, but there is no 5-cycle passing through 1, $c^i$ and $c^{ab}$. Thus, $A/M \cong D_{10}$ or $F_{20}$. It follows that $\mathbb{Z}_5 \cong H_5/M \triangleleft A/M$, implying that $H_5 \triangleleft A$. Hence, $X$ is normal. Now assume that $|X_M| = pq$. Note that $q \mid p - 1$. If $X_M$ is half-arc-transitive, then $A/M \cong \mathbb{Z}_p \times \mathbb{Z}_q$ by Proposition 2.5. It follows that $|A| = 2pq$ and $A$ is solvable. Then $H_5 \triangleleft A$ and $X$ is normal. If $X_M$ is symmetric, by Proposition 2.5, then $X_M$ is a normal Cayley graph and $|A/M| = 4pq$, or $|X_M| = 3 \cdot 7 \cdot 5 \cdot 11$ or $11 \cdot 23$. For the latter cases, that is $(p, q) = (7, 3), (11, 5)$ or $(23, 11)$, by Magma, they are half-arc-transitive. For the former case, it is easy to see that $G_5 \triangleleft A$. Thus, $X$ is normal. If (3) holds then $|H_5/P| = q$ and $|A/P| = 4q$ or $8q$. If $|A/P| = 4q$ then $H_5/P \triangleleft A/P$ because $q \geq 5$. Hence, $X$ is normal. If $|A/P| = 8q$ then $A/(PH) \cong \mathbb{Z}_2^2$. It follows that there exists a subgroup $B$ of $A$ such that $B/(PH) \cong \mathbb{Z}_2^2$. Then $|B/P| = 4q$. It follows that $H_5/P \triangleleft A/P$, and hence $H_5/P \triangleleft A/P$, implying that $H_5 \triangleleft A$. Thus, $X$ is normal.

Thus, the claim is true. It is easy to see that $\text{Aut}(H_5, S_5) = \text{Aut}(H_5, T_1) \cong \mathbb{Z}_2$. Thus, $A_1 \cong \mathbb{Z}_2$ and $X$ is half-arc-transitive. In the same argument to Theorem 3.2, $S_1$ and $T_1$ are CI-subsets of $H_5$, it means that $\mathcal{D}_{i,1}$ and $\mathcal{D}_{i,2}$ are pairwise non-isomorphic, for $1 \leq k \leq \frac{1}{2}(q - 1)$. Furthermore, it is easy to see that $\mathcal{D}_{i,1} \cong X(r^k; pq, p)$. □

Lemma 3.5. Let $\mathcal{D}_{6k} = \text{Cay}(H_{6k}, \{c^i, c^{-i}, c^{ab}, (c^{ab})^{-1}\})$ for $i \in \mathbb{Z}_6$. If $q \geq 5$ and $x \neq 1$ then $\mathcal{D}_{6k}$ is a connected tetravalent half-arc-transitive graph of order $p^2q$, and there are exactly $\frac{1}{2}(q - 1)$ non-isomorphic such graphs.

Proof. Recall that $H_{6k} = \langle a, b, c \mid a^p = b^5 = c^q = [a, b] = 1, a^i = a^j, b^i = b^j \rangle \cong \mathbb{Z}_p \times F_{pq}$ where $r^q \equiv 1 \pmod{p}$ and $x \neq 0$. Set $X = \mathcal{D}_{6k}$ and $A = \text{Aut}(X)$. One may check that there exists an automorphism $\alpha$ of $H_{6k}$ which interchanges $c^i$ and $c^{ab}$, $c^{-i}$ and $(c^{ab})^{-1}$. Clearly, $\alpha \in A_1$, it follows that $X$ is edge-transitive. In the same argument to Lemma 3.4, $X$ is normal. It follows that $A_1 = \text{Aut}(H_{6k}, S)$ where $S = \{c^i, c^{-i}, c^{ab}, (c^{ab})^{-1}\}$. It is easy to check that $\text{Aut}(H_{6k}, S) \cong \mathbb{Z}_2$. Thus, $X$ is half-arc-transitive. Furthermore, $\mathcal{D}_{6k}$ are pairwise non-isomorphic, for $1 \leq k \leq \frac{1}{2}(q - 1)$. □

Remark. An $(m, n)$-metacirculant is a graph of order $mn$ which has an automorphism $\sigma$ with a cycle decomposition

$$\sigma = (v_1, v_2, \ldots, v_{mn}) = (v_{1n}, v_2, \ldots, v_{mn})$$

and an automorphism $\tau$ normalizing $\sigma$ and cyclically permuting the orbits

$$V_i = \{v_{1i}, v_{2i}, \ldots, v_{mi}\}, \quad i = 1, 2, \ldots, n,$$

such that $\tau$ has a cycle of size m in its disjoint cycle decomposition; also refer to [1]. It is easy to know that $\mathcal{D}_{i,2}$ and $\mathcal{D}_{6k}$ are not metacirculants.

Now the main result follows.

Theorem 3.6. Let $p, q$ be distinct odd primes and $X$ a tetravalent half-arc-transitive graph of order $p^2q$. Then $X \cong \mathcal{D}_{i,1}^k$ for $i = 1, 2, 3, 4$, or $X \cong \mathcal{D}_{3,2}^k$ for $j = 1, 2, 3, 4$, or $X \cong \mathcal{D}_{6k}$. Furthermore, the number of non-isomorphic connected tetravalent half-arc-transitive graphs of order $p^2q$ is equal to

$$\begin{align*}
p - 1 & \quad \text{if } p \mid q - 1, \quad p^2 \mid q - 1, \\
p(p + 1) - 1 & \quad \text{if } p^2 \mid q - 1, \\
\frac{(q - 1)(q + 7)}{4} & \quad \text{if } 5 \leq p \mid q - 1, \\
3 & \quad \text{if } 3 = q \mid p - 1, \\
0 & \quad \text{otherwise.}
\end{align*}$$
First assume that $G = H_1 = \langle a, b \ | \ a^d = b^e = 1, a^d = a' \rangle$ where $r^p \equiv 1 \pmod{q}$. Then $a(a') = q, o(b^p) = p$ and $b^p \in Z(G), o(b^p) = p = p^2$, where $(i, q) = (j, p) = 1$ and $k \in \mathbb{Z}_q$. Since $x$ is connected, we have $o(a) = o(b) = p^2$. Thus, we may assume that $S = \{ b^a, (b^a)^{-1}, b^a, (b^a)^{-1} \}$ with $(j, p) = (t, p) = 1$ and $i \neq 0$. Note that $(b^a)^{-1} = b^{-a-b^{-i}}$, and any element in $\text{Aut}(G)$ must map $b$ to $b^{1+kp}a^m$ for $m \in \mathbb{Z}_q$ and $k \in \mathbb{Z}_q$. Since $x$ is half-arc-transitive and $x$ is a normal Cayley graph, there exists an automorphism $\alpha \in \text{Aut}(G)$ which interchange $b^a$ and $b^a$ or interchange $b^a$ and $(b^a)^{-1}$, implying that $j = 0$ or $t$. Since $\text{Aut}(G)$ is transitive on the set $\{ b^a \ | \ i \in \mathbb{Z}_q \}$ for a given $j \in \mathbb{Z}_q$, we may assume that $b^i \in S$. Then $S \cong \{ b^i, b^{-i}, b^a, (b^a)^{-1} \}$ with $k \neq 0$. Note that the map $a^d \mapsto a$ and $b \mapsto b$ induces an automorphism of $G$. It follows that $S \cong \{ b^i, b^{-i}, b^a, (b^a)^{-1} \} \cup \{ b^1 \}$. By Lemma 3.1, $X \cong X(s^2; p^2, q)$, and by Proposition 2.1, $X$ is half-arc-transitive. Similarly, there are exactly $\frac{1}{2}(p - 1)$ non-isomorphic such graphs.

Now assume that $G = H_2$ or $H_3$. Similarly, we have $X \cong X(s^2; p^2, q)$ or $X(s^3; q, p^2)$. And, there are exactly $\frac{1}{2}(q - 1)$ non-isomorphic graphs, respectively.

Now assume that $G = H_4 = \langle a, b, c \ | \ a^d = b^e = c^f = [a, b] = [a, c] = 1, a^b = c^e \rangle \cong \mathbb{Z}_p \times \mathbb{F}_p$ where $r^p \equiv 1 \pmod{q}$. Similarly, we have $S \cong \{ b^i, b^{-i}, b^a, (b^a)^{-1} \}$. Since $p | q - 1$, one may assume that $q = kp + 1$. Set $S = \{ r + (p + 1) - r \}$. Then $s^d = (r + (p + 1) - r)(kp - 1)p^3 = (kp - 1)p^3 = 1 \pmod{p}$, and $s^d = 2 \times (p + 1 - r)p^3 = 1 \pmod{p}$. Clearly, $\langle a, b, c \ | \ a^d = b^e = c^f = [a, b] = [a, c] = 1, a^b = c^e \rangle \cong \langle a, b, c \ | \ a^d = b^e = [a, b] = [a, c] = 1, a^b = c^e \rangle$. One may easily check that the map $\phi : u \mapsto (b^a)^{c^e} (i \in \mathbb{Z}_p, j \in \mathbb{Z}_p)$ is an isomorphism from $X(s^2; p, q)$ to $\text{Cay}(H_4, T)$, where $T = \langle b^a, (b^a)^{-1}, b^{-a}, (b^{-a})^{-1} \rangle$. Similarly, $H_4$ has an automorphism $\psi$ such that $(b^a)^{c^e} = b^{-a}$ and $(b^{-a})^{-1} = b^{a}$. Then $T = S$, and hence $\psi$ is an isomorphism from $\text{Cay}(H_4, T)$ to $\text{Cay}(H_4, S)$. Consequently, $X(H_4, S) \cong X(s^2; p, q)$ is a connected tetravalent tightly attached half-arc-transitive graph of order $p^2q$ by Proposition 2.1. Furthermore, there are exactly $\frac{1}{2}(q - 1)$ non-isomorphic such graphs.

Now assume that $G = H_5$. Similarly, we may assume that $S \cong \{ ac, (ac)^{-1}, ab^c, (abc)^{-1} \}$ (or $\{ a^c, (ac)^{-1}, a^{-1}bc, (a^{-1}bc)^{-1} \}$) for $i \in \mathbb{Z}_q^*$. Thus, $X \cong D_{5,1} = X(s^2; pq)$ or $X \cong D_{5,2} = X(s^3; pq)$. Furthermore, there are exactly $q - 1$ non-isomorphic such graphs.

Now assume that $G = H_6 = \langle a, b, c \ | \ a^d = b^e = c^f = [a, b] = 1, a^d = a', b' = b' \rangle$ where $r^q \equiv 1 \pmod{p}$ and $x \neq 0$; or $x \neq 0$. There are exactly $\frac{1}{2}(q + 1)$ non-isomorphic groups. Then $o(c^ab^b) = q$ and $o(a^b) = p$, where $(i, q) = 1$ and $st \neq 0 \pmod{p}$. Similarly, we may assume that $S = \{ c^i, c^{-i}, c^iab, (c^iab)^{-1} \}$. Suppose that $x = 1$. Then $X$ is not connected because $|S| = pq < q^2$. Thus, $x \neq 1$. Suppose that $q = 3$. Then $X = 2$. Define the map $\rho : c^p = c^{-1}, a^p = b^{-1-2i}, b^p = a^{-r-1}$. Clearly, $\rho$ is an automorphism of $G$ and $\rho \rho = S$. By Proposition 2.4, $X = \text{Cay}(G, S)$ is not half-arc-transitive. Hence, $\rho \rho \neq 1$ and $X \cong D_{6,3}$.

Now assume that $G = H_7 = \langle a, b, c \ | \ a^d = b^e = c^f = [a, b] = 1, a^d = b', b' = b^{-1} \rangle$ where $k = p^l + 1$ with $l = 1 \pmod{p}$. It is easy to know that for any automorphism $\alpha$ of $G$, we have that $c^i \alpha = c^ia^b$ or $c^i \alpha = c^i a^b$ where $s, t \in \mathbb{Z}_q$. Similarly, we may assume that $S = \{ c^i, c^{-i}, c^iab, (c^iab)^{-1} \}$. Define the map $\beta : c \mapsto c^{-1}, a \mapsto a^{-1}b^{-1}, b \mapsto a^{-1}b^{-1}$. It is easy to check that $\beta$ is an automorphism of $G$ and $\beta \beta = S$. By Proposition 2.4, $X$ is not half-arc-transitive. Thus, if $p \mid q - 1$ and $p^2 \mid q - 1$, then $X \cong X(H_1, H_2, O_4)$ and there are exactly $p^2 - 1 - p - 1$ non-isomorphic such graphs.

If $p^2 \mid q - 1$, then $G \cong X(H_1, H_2, O_4)$ and there are exactly $p^2 - 1 + p^{2-1} + p^{2-1} = p^2 - 1$ non-isomorphic such graphs.

If $5 \leq q \mid p - 1$, then $G \cong X(H_5, H_6, H_7)$ and there are exactly $q - 1 + q^{2-1} + q^{2-1} = (q^{2-1} + q^{2-1})$ non-isomorphic such graphs.

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