DIFFERENTIAL INVERSE VARIATIONAL INEQUALITIES IN FINITE DIMENSIONAL SPACES*

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Abstract In this article, a new differential inverse variational inequality is introduced and studied in finite dimensional Euclidean spaces. Some results concerned with the linear growth of the solution set for the differential inverse variational inequalities are obtained under different conditions. Some existence theorems of Carathéodory weak solutions for the differential inverse variational inequality are also established under suitable conditions. An application to the time-dependent spatial price equilibrium control problem is also given.

Key words Differential inverse variational inequality; differential variational inequality; linear growth; Carathéodory weak solutions

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1 Introduction

Let $K \subset \mathbb{R}^n$ be a nonempty, closed, and convex set and $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function. An inverse variational inequality (denoted by IVI($K, g$)) is formulated as follows: find $x^* \in \mathbb{R}^n$, such that

$$g(x^*) \in K, \quad \langle \dot{g} - g(x^*), x^* \rangle \geq 0, \quad \forall \dot{g} \in K. \tag{1.1}$$

Let SOLIVI($K, g$) denote the solution set of this problem. We write $\dot{x} := \frac{dx}{dt}$ for the time-derivative of a function $x(t)$. In this article, we introduce and study the following differential

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inverse variational inequality (denoted by DIVI):

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t)) + B(t, x(t))u(t), \\
u(t) &\in \text{SOLIVI}(K, G(t, x(t)) + F(\cdot)), \\
x(0) &= x_0,
\end{align*}
\]

where \( \Omega := [0, T] \times \mathbb{R}^m \), \((f, B, G) : \Omega \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \) are given functions and \( F : \mathbb{R}^n \to \mathbb{R}^n \) is a single-valued linear function. A point \((x, u)\) is called a Carathéodory weak solution of DIVI (1.2) if and only if \( x \) is an absolutely continuous function on \([0, T]\) and \( u \) is an integrable function on \([0, T]\) such that the differential equation satisfied for almost all \( t \in [0, T] \) and \( u(t) \in \text{SOLIVI}(K, G(t, x(t)) + F(\cdot)) \) for almost all \( t \in [0, T] \). The set of all Carathéodory weak solutions \((x, u)\) of the initial-value DIVI (1.2) is denoted by \( \text{SOLDIVI}(K, G + F) \).

It is well known that the variational inequality theory has wide applications in optimization, engineering, economics, and transportation (see, for example, [2–4, 6, 8, 26, 28]) and ordinary differential equations with smooth input functions are a classical paradigm in applied mathematics that have existed for centuries. Yet, as evidenced by the growing literature that has surfaced in recent years on multi-rigid-body dynamics with frictional contacts and on hybrid engineering systems, ordinary differential equations are inadequate to deal with many naturally occurring engineering problems that contain inequalities and disjunctive conditions. For solving these problems, Pang and Stewart [19] introduced and studied differential variational inequality (DVI) in finite-dimensional Euclidean spaces which significantly extends these differential equations and open up a broad paradigm for the enhanced modeling of complex engineering system. For some related work, we refer to [1, 5, 7, 15, 20, 21, 23, 24]. Recently, Li et al [17] introduced and investigated a class of differential mixed variational inequalities in finite-dimensional spaces. Very recently, Wang and Huang [25] introduced and studied differential vector variational inequalities in finite-dimensional spaces.

On the other hand, He et al [9, 10] first introduced and studied the inverse variational inequalities in finite dimensional Euclidean spaces. They pointed out that there are many control problems appearing in economics, transportation, and management science and energy networks can be modeled as the inverse variational inequalities, but they are difficult to be formulated as the classical variational inequalities. Furthermore, He et al [11] developed a proximal point based algorithm for solving the inverse variational inequality. He and Liu [12] proposed two projection-based methods for solving the inverse variational inequality. Yang [27] considered the dynamic power price problem and characterized the optimal price as a solution of an inverse variational inequality. Scrima [22] studied the time-dependent spatial price equilibrium control problem and modeled it as an evolutionary inverse variational inequality. Some related work concerned with the inverse variational inequalities; we refer to [13, 14, 16] and the references therein. Obviously, if the function \( f \) is single-valued, setting \( u = f(x) \) and \( g(u) = f^{-1}(u) \), then the inverse variational inequality is transformed into the classical variational inequality. However, this transformation fails when \( f \) is set-valued. Moreover, in many real applications, explicit forms of function cannot be obtained which also causes failure of this transformation (see [11]). Therefore, it is important and interesting to consider an ordinary differential equation whose right-hand function is parameterized by an algebraic variable that is required to be a solution of an inverse variational inequality containing the state variable of
the system.

In this article, we give the linear growth of the solution set for the differential inverse variational inequality (1.2) under various conditions. Moreover, we show the existence theorems concerned with the Carathéodory weak solutions for the differential inverse variational inequality (1.2) in finite-dimensional spaces. We also give an application to the time-dependent spatial price equilibrium control problem under some suitable conditions.

2 Preliminaries

In this section, we will introduce some basic notations and preliminary results.

**Definition 2.1** ([19]) A map \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is said to be
(i) para-monotone on a convex set \( K \subset \mathbb{R}^n \) if \( f \) is monotone on \( K \), that is,
\[
\langle f(v) - f(u), v - u \rangle \geq 0, \quad \forall v, u \in K,
\]
and the following property holds: for any \( v, u \in K \), we have
\[
\langle f(v) - f(u), v - u \rangle = 0 \Rightarrow f(v) = f(u).
\]
(ii) strongly monotone on \( K \) if there exists a constant \( \alpha > 0 \) such that, for any \( v, u \in K \), we have
\[
\langle f(v) - f(u), v - u \rangle \geq \alpha \| v - u \|^2.
\]

**Definition 2.2** A map \( F : \Omega \rightarrow \mathbb{R}^n \) (respectively, \( B : \Omega \rightarrow \mathbb{R}^n \times \mathbb{R}^m \)) is said to be Lipschitz continuous if there exists a constant \( L_{F} > 0 \) (respectively, \( L_{B} > 0 \)) such that, for any \( (t_1, x), (t_2, y) \in \Omega \), we have
\[
\| F(t_1, x) - F(t_2, y) \| \leq L_{F}(|t_1 - t_2| + \| x - y \|),
\]
(respectively, \( \| B(t_1, x) - B(t_2, y) \| \leq L_{B}(|t_1 - t_2| + \| x - y \|) \)).

In the rest of this article, we assume that the following conditions (A) and (B) hold:
(A) \( f, B, \) and \( G \) are Lipschitz continuous functions on \( \Omega \) with Lipschitz constants \( L_{f} > 0, L_{B} > 0, \) and \( L_{G} > 0 \), respectively;
(B) \( B \) is bounded on \( \Omega \) with \( \sigma_B := \sup_{(t, x) \in \Omega} \| B(t, x) \| < \infty. \)

Let
\[
\mathcal{F}(t, x) := \{ f(t, x) + B(t, x)u : u \in SOLIVI(K, G(t, x) + F) \}.
\]

**Lemma 2.3** ([19], Theorem 5.1) Let \( \mathcal{F} : \Omega \rightrightarrows \mathbb{R}^m \) be an upper semicontinuous set-valued map with nonempty closed convex values. Suppose that there exists a scalar \( \rho_F > 0 \) satisfying
\[
\sup_{(t, x) \in \Omega} \{ \| y \| : y \in \mathcal{F}(t, x) \} \leq \rho_F(1 + \| x \|), \quad \forall (t, x) \in \Omega.
\]

Then, for every \( x^0 \in \mathbb{R}^n, DI : \dot{x} \in \mathcal{F}(t, x), x(0) = x^0 \) has a weak solution in the sense of Carathéodory.

**Lemma 2.4** ([19], Lemma 6.3) Let \( h : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) be a continuous function and \( U : \Omega \rightrightarrows \mathbb{R}^m \) be a closed set-valued map such that for some constant \( \eta_U > 0 \),
\[
\sup_{u \in U(t, x)} \| u \| \leq \eta_U(1 + \| x \|), \quad \forall (t, x) \in \Omega.
\]
Let $v : [0, T] \to \mathbb{R}^n$ be a measurable function and $x : [0, T] \to \mathbb{R}^n$ be a continuous function satisfying $v(t) \in h(t, x(t), U(t, x(t)))$ for almost all $t \in [0, T]$. Then, there exists a measurable function $u : [0, T] \to \mathbb{R}^m$ such that $u(t) \in U(t, x(t))$ and $v(t) = h(t, x(t), u(t))$ for almost all $t \in [0, T]$.

**Lemma 2.5** Let $(f, G, B)$ satisfy conditions (A) and (B), and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Suppose that there exists a constant $\rho > 0$ such that, for all $q \in G(\Omega)$,

$$\sup\{\|u\| : u \in SOLIVI(K, q + F)\} \leq \rho(1 + \|q\|).$$

Then, there exists a constant $\rho^G > 0$ such that (2.2) holds for the map $F > 0$ defined by (2.1). Hence, $F$ is an upper semicontinuous closed-valued map on $\Omega$.

**Proof** Because $f$ and $G$ are Lipschitz continuous on $\Omega$, we know that $f, G$ have linear growth on $\Omega$ in $x$, that is, for some positive constants $\rho_f$ and $\rho_G$ and for any $(t, x) \in \Omega$,

$$\|f(t, x)\| \leq \rho_f (1 + \|x\|)$$

and

$$\|G(t, x)\| \leq \rho_G (1 + \|x\|).$$

In a similar way of Lemma 6.2 in [19], from (2.3), (2.4), and (2.5), we can obtain the fact that there exists $\rho^G > 0$ such that (2.2) holds. Thus, $F$ has linear growth.

Next, we prove that $F$ is upper semicontinuous on $\Omega$. We need only to prove that $F$ is closed. Let sequence $\{(t_n, x_n)\} \subset \Omega$ be a sequence converging to some vector $(t_0, x_0) \in \Omega$ and $\{f(t_n, x_n) + B(t_n, x_n)u_n\}$ converges to some vector $z_0 \in \mathbb{R}^m$ as $n \to \infty$, where $u_n \in SOLIVI(K, G(t_n, x_n) + F(\cdot))$ for every $n$. It follows that the sequence $\{u_n\}$ is bounded, and has a convergent subsequence, denoted again by $\{u_n\}$, with a limit $u_0 \in \mathbb{R}^n$. As $F$ is continuous and $K$ is nonempty, closed, and convex, it is easy to obtain

$$f(t_n, x_n) + B(t_n, x_n)u_n \to z_0 = f(t_0, x_0) + B(t_0, x_0)u_0 \in F(t_0, x_0)$$

and so $F$ is closed. \hfill $\Box$

**Lemma 2.6** Let $(f, G, B)$ satisfy conditions (A) and (B), and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and para-monotone map on $\mathbb{R}^n$. Suppose that $SOLIVI(K, q + F(\cdot)) \neq \emptyset$ for any $q \in G(\Omega)$. Then, $SOLIVI(K, q + F(\cdot))$ is closed and convex for all $q \in G(\Omega)$.

**Proof** Let $\{u_n\} \subset SOLIVI(K, q + F(\cdot))$ with $u_n \to u_0$. Applying the closedness and convexity of $K$ and the continuity of $F$, we deduce that $u_0 \in SOLIVI(K, q + F(\cdot))$ and so $SOLIVI(K, q + F(\cdot))$ is closed for all $q \in G(\Omega)$. Next, we prove that $SOLIVI(K, q + F(\cdot))$ is convex for all $q \in G(\Omega)$. Let $u_1, u_2 \in SOLIVI(K, q + F(\cdot))$. Then,

$$q + F(u_1) \in K, \quad q + F(u_2) \in K.$$ \hspace{1cm} (2.6)

Moreover, for any $\tilde{F} \in K$, we have

$$\langle \tilde{F} - q - F(u_1), u_1 \rangle \geq 0$$ \hspace{1cm} (2.7)

and

$$\langle \tilde{F} - q - F(u_2), u_2 \rangle \geq 0.$$ \hspace{1cm} (2.8)

It follows from (2.6) that, for every $\lambda \in [0, 1]$, we have

$$\lambda(q + F(u_1)) + (1 - \lambda)(q + F(u_2)) = q + \lambda F(u_1) + (1 - \lambda)F(u_2)$$
\[ = q + F(\tilde{u}) \in K, \quad (2.9) \]

where
\[ \tilde{u} = \lambda u_1 + (1 - \lambda)u_2. \]

Letting \( \tilde{F} = q + F(u_2) \) in (2.7) and \( \tilde{F} = q + F(u_1) \) in (2.8), respectively, one has
\[ \langle F(u_2) - F(u_1), u_1 - u_2 \rangle \geq 0. \quad (2.10) \]

Because \( F \) is para-monotone, we know that \( F(u_2) = F(u_1) \). It follows from (2.7) and (2.8) that
\[ \langle \tilde{F} - q - F(u_1), \lambda u_1 + (1 - \lambda)u_2 \rangle \geq 0, \]
which means that
\[ \langle \tilde{F} - q - F(\tilde{u}), \tilde{u} \rangle \geq 0. \]

This shows that \( \tilde{u} \in SOLIVI(K, q + F(\cdot)) \) and so \( SOLIVI(K, q + F(\cdot)) \) is convex for any \( q \in G(\Omega) \).

**Lemma 2.7** Let \((f, G, B)\) satisfy conditions (A) and (B), and \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a continuous and para-monotone map. Suppose that there exists a constant \( \rho > 0 \) such that (2.3) holds for any \( q \in G(\Omega) \), and \( SOL(K, q + F) \neq \emptyset \) for any \( q \in G(\Omega) \). Then, DIVI(1.2) has a weak solution in the sense of Carathéodory.

**Proof** Similar to the proof of Proposition 6.1 in [19], by Lemmas 2.3 and 2.4, we can deduce that DIVI(1.2) has a weak solution in the sense of Carathéodory. \( \Box \)

### 3 Main Results

**Theorem 3.1** Let \( K \subset \mathbb{R}^n \) be a nonempty compact convex subset and \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a continuous and para-monotone map. Suppose that \( q + F \) is invertible and \( (q + F)^{-1} \) is continuous on \( \mathbb{R}^n \). Then, \( SOLIVI(K, q + F(\cdot)) \) is a nonempty compact convex set in \( K \) for any \( q \in \mathbb{R}^n \), and there exists \( \rho > 0 \) such that (2.3) holds for any \( q \in \mathbb{R}^n \).

**Proof** For any \( u \in \mathbb{R}^n \), let
\[ g(u) = (q + F)^{-1}(u) = y. \]

Then,
\[ \langle g(u_1) - g(u_2), u_1 - u_2 \rangle = \langle y_1 - y_2, q + F(y_1) - q - F(y_2) \rangle = \langle y_1 - y_2, F(y_1) - F(y_2) \rangle. \]

Now, the monotonicity of \( F \) implies that \( g \) is monotone on \( \mathbb{R}^n \). For any \( q \in \mathbb{R}^n \), by Theorem 8.1 in [18], we know that \( SOL(K, q) \) is nonempty and so there exists \( u \in K \) such that
\[ \langle \tilde{u} - u, g(u) \rangle \geq 0, \quad \forall \tilde{u} \in K. \quad (3.1) \]

It follows from (3.1) that there exists \( y \in \mathbb{R}^n \) such that \( q + F(y) \in K \) and
\[ \langle \tilde{u} - q - F(y), y \rangle \geq 0, \quad \forall \tilde{u} \in K, \]
which means that \( SOLIVI(K, q + F(\cdot)) \) is nonempty for any \( q \in \mathbb{R}^n \). Thus, Lemma 2.6 yields that \( SOLIVI(K, q + F(\cdot)) \) is a nonempty, closed and convex set for every \( q \in \mathbb{R}^n \). Because \( K \)
is compact, it follows that \( SOLIVI(K, q + F(\cdot)) \) is a nonempty compact convex set for any \( q \in R^n \). This shows that there exists a constant \( \rho > 0 \) such that (2.3) holds for any \( q \in R^n \). □

**Theorem 3.2** Let \( K \subset R^n \) be nonempty compact convex set. Assume that \( F : R^n \to R^n \) be a continuous and strictly monotone map such that \( q + F \) is surjective for any \( q \in R^n \). Then, \( SOLIVI(K, q + F(\cdot)) \) is a singleton for any \( q \in R^n \) and there exists a constant \( \rho > 0 \) such that (2.3) holds for any \( q \in R^n \).

**Proof** Because \( F \) is continuous and strictly monotone on \( R^n \), it is easy to see that \( q + F \) is continuous and strictly monotone on \( R^n \). This implies that \( (q + F)^{-1} \) is strictly monotone and continuous on \( R^n \). By Theorem 8.1 in [18], we know that \( SOL(K, (q + F)^{-1}) \) is nonempty. From Theorem 3.1, it yields that \( SOLIVI(K, (q + F)) \) is nonempty. For any \( u_1, u_2 \in SOLIVI(K, (q + F)) \), we have

\[
q + F(u_1) \in K, \quad (\tilde{F} - q - F(u_1), u_1) \geq 0, \quad \forall \tilde{F} \in K
\]

and

\[
q + F(u_2) \in K, \quad (\tilde{F} - q - F(u_2), u_2) \geq 0, \quad \forall \tilde{F} \in K.
\]

It follows that

\[
(F(u_1) - F(u_2), u_1 - u_2) \leq 0.
\]

Now, the strictly monotonicity of \( F \) shows that \( u_1 = u_2 \) and so there exists a constant \( \rho > 0 \) such that (2.3) holds for any \( q \in R^n \). □

**Theorem 3.3** Let \( F : R^n \to R^n \) be a continuous and para-monotone map. Suppose that there exist \( u_0, y_0 \in R^n \) such that, for any \( u, y \in R^n \),

\[
\frac{\langle q + F(u), u - u_0 \rangle - \langle u, y_0 \rangle + \langle u_0, y \rangle}{\|u\|^2 + \|y\|^2} \to +\infty \quad \text{as} \quad \|u\|^2 + \|y\|^2 \to +\infty. \quad (3.2)
\]

Moreover, assume that there exists \( F^0 \in R^n \) such that

\[
\liminf_{\|u\| \to \infty} \frac{\langle F(u) - F^0, u \rangle}{\|u\|^2} > 0. \quad (3.3)
\]

Then, \( SOLIVI(R^n, q + F(\cdot)) \) is a nonempty, closed, and convex set for all \( q \in R^n \) and there exists a constant \( \rho > 0 \) such that (2.3) holds for any \( q \in R^n \).

**Proof** It follows from Lemma 4.1 in [9] that the problem \( IVI(R^n, q + F) : \) find \( u \in R^n \) such that \( q + F(u) \in R^n \) and

\[
(\tilde{F} - q - F(u), u) \geq 0, \quad \forall \tilde{F} \in R^n,
\]

is equivalent to the problem \( VI(R^{2n}, P) : \) find \( v \in R^{2n} \) such that

\[
(\tilde{v} - v, P(v)) \geq 0, \quad \forall \tilde{v} \in R^{2n},
\]

where

\[
v = \begin{pmatrix} u \\ y \end{pmatrix}, \quad P(v) = \begin{pmatrix} q + F(u) - y \\ u \end{pmatrix}.
\]

By the monotonicity of \( F \), one has

\[
\langle P(v_1) - P(v_2), v_1 - v_2 \rangle = \begin{pmatrix} F(u_1) - y_1 - F(u_2) + y_2 \\ u_1 - u_2 \end{pmatrix}^T \begin{pmatrix} u_1 - u_2 \\ y_1 - y_2 \end{pmatrix}
\]

which implies that $P$ is monotone on $\mathbb{R}^{2n}$. Thus, there exists $v_0 = \begin{pmatrix} u_0 \\ y_0 \end{pmatrix}$ such that
\[
\frac{\langle P(v), v - v_0 \rangle}{\|v\|} = \frac{\langle q + F(u) - y, u - u_0 \rangle + \langle u, y - y_0 \rangle}{\left(\|u\|^2 + \|y\|^2\right)^{\frac{1}{2}}} \rightarrow +\infty \text{ as } \|u\|^2 + \|y\|^2 \rightarrow +\infty,
\]
which means that
\[
\frac{\langle P(v), v - v_0 \rangle}{\|v\|} \rightarrow +\infty \text{ as } \|v\| \rightarrow +\infty.
\]
By Theorem 3.2 in [17], we know that \( \text{SOL}(\mathbb{R}^{2n}, P) \) is a nonempty set and so \( \text{SOLIV}(\mathbb{R}^n, q + F(\cdot)) \) is nonempty. It follows from Lemma 2.6 that \( \text{SOLIV}(\mathbb{R}^n, q + F(\cdot)) \) is a nonempty closed convex set for every $q \in \mathbb{R}^n$.

Next, we prove the second assertion. Suppose to the contrary, there exist \( \{q^k\} \subset \mathbb{R}^n \) and \( \{u^k\} \subset \mathbb{R}^n \) such that, for any $\tilde{F} \in \mathbb{R}^n$,
\[
\langle \tilde{F} - q^k - F(u^k), u^k \rangle \geq 0,
\]
and
\[
\|u^k\| > k(1 + \|q^k\|).
\]
Obviously, \( \{u^k\} \) is unbounded. It follows from (3.4) that
\[
\langle F^0 - q^k - F(u^k), u^k \rangle \geq 0
\]
and so
\[
\langle F(u^k) - F^0, u^k \rangle \leq \langle -q^k, u^k \rangle.
\]
Dividing by $\|u^k\|^2$, we have
\[
\liminf_{k \to \infty} \frac{\langle F(u^k) - F^0, u^k \rangle}{\|u^k\|^2} \leq 0,
\]
which contradicts (3.3). This shows that there exists a constant $\rho > 0$ such that (2.3) holds for any $q \in \mathbb{R}^n$. \( \square \)

**Theorem 3.4** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and para-monotone map. Suppose that \( \text{SOLIV}(\mathbb{R}^n, q + F(\cdot)) \neq \emptyset \) for any $q \in \mathbb{R}^n$ and there exists $F^0 \in \mathbb{R}^n$ such that
\[
\frac{\langle F(u) - F^0, u \rangle}{\|u\|} \rightarrow +\infty \text{ as } \|u\| \rightarrow +\infty.
\]
Then, \( \text{SOLIV}(\mathbb{R}^n, q + F(\cdot)) \) is a nonempty closed convex set for all $q \in \mathbb{R}^n$ and there exists a constant $\rho > 0$ such that (2.3) holds for all $q \in S$, where $S$ is bounded set.

**Proof** Similar to the proof of Theorem 3.1, we know \( \text{SOLIV}(\mathbb{R}^n, q + F(\cdot)) \) is a nonempty closed convex set for all $q \in \mathbb{R}^n$.

Now, we prove the second assertion. If the assertion is not true, then there exist \( \{q^k\} \subset S \) and \( \{u^k\} \subset \mathbb{R}^n \) such that for any $\tilde{F} \in \mathbb{R}^n$,
\[
\langle \tilde{F} - q^k - F(u^k), u^k \rangle \geq 0,
\]

\[
\langle F(u^k) - F^0, u^k \rangle \leq \langle -q^k, u^k \rangle.
\]
Dividing by $\|u^k\|^2$, we have
\[
\liminf_{k \to \infty} \frac{\langle F(u^k) - F^0, u^k \rangle}{\|u^k\|^2} \leq 0,
\]
which contradicts (3.3). This shows that there exists a constant $\rho > 0$ such that (2.3) holds for any $q \in \mathbb{R}^n$. \( \square \)
and
\[ \|u^k\| > k(1 + \|q^k\|). \]

It is clear that \( \{u^k\} \) is unbounded. From (3.6), one has
\[ \langle F^0 - q^k - F(u^k), u^k \rangle \geq 0, \]
which means
\[ \langle F(u^k) - F^0, u^k \rangle \leq \langle -q^k, u^k \rangle. \]

Dividing by \( \|u^k\| \), we have
\[ \frac{\langle F(u^k) - F^0, u^k \rangle}{\|u^k\|} \leq \frac{\langle -q^k, u^k \rangle}{\|u^k\|}. \]

Because \( \{q^k\} \) is bounded, there exists a constant \( C \) such that
\[ \frac{\langle F(u^k) - F^0, u^k \rangle}{\|u^k\|} \leq C, \]
which contradicts (3.5).

In the rest of this article, let
\[ S := \{v \in \mathbb{R}^n : \langle Fv, v \rangle = 0\}. \]

Obviously, \( S \) is a linear subspace of \( \mathbb{R}^n \) and \( S^\perp \) is also a linear subspace of \( \mathbb{R}^n \).

**Theorem 3.5** Let \( F_{n \times n} \) be a positive semi-defined matrix. Suppose that for any \( n \in \mathbb{N} \), we have
\[ SOLIVI \left( \mathbb{R}^n, q + \left( 1 - \frac{1}{n} \right)F + \frac{1}{n}I \right) \neq \emptyset, \]
where \( I \) is the identity map on \( \mathbb{R}^n \). Then,
(i) \( SOLIVI(\mathbb{R}^n, q + F(\cdot)) \) is a nonempty closed convex set for all \( q \in S^\perp \).
(ii) there exists a constant \( \rho > 0 \) such that
\[ \sup\{\|u\| : u \in SOLIVI(\mathbb{R}^n, q + F(\cdot))\} \leq \rho(1 + \|q\|). \]

**Proof** We denote \( SOLIVI(\mathbb{R}^n, q + (1 - \frac{1}{n})F + \frac{1}{n}I) \) by \( SOLIVI_n(F_n) \). Assume for the sake of contrary that the contrary holds. Suppose that \( \bigcup_{n \in \mathbb{N}} SOLIVI_n(F_n) \) is unbounded. Then, there exists a sequence \( \{u_n\} \subset \mathbb{R}^n \) such that, for any \( \tilde{F} \in \mathbb{R}^n \),
\[ \left\langle \tilde{F} - q - (1 - \frac{1}{n})F(u_n) - \frac{1}{n}I(u_n), u_n \right\rangle \geq 0, \]
where \( \|u_n\| \to \infty \). Let
\[ \lim_{n \to \infty} \frac{u_n}{\|u_n\|} = u_\infty. \]

Dividing by \( \|u_n\|^2 \) and taking \( n \to \infty \) in (3.7), we have
\[ \langle F(u_\infty), u_\infty \rangle \leq 0. \]

As \( F \) is positive semi-defined, one has \( \langle F(u_\infty), u_\infty \rangle = 0 \) and so \( u_\infty \in S \). Because
\[ \left\langle (1 - \frac{1}{n})F(u_n) + \frac{1}{n}I(u_n), u_n \right\rangle \geq 0, \]
it follows from (3.7) that
\[ \langle \tilde{F} - q, u_n \rangle \geq 0 \]
and so
\[ \left\langle \tilde{F} - q, \frac{u_n}{\|u_n\|} \right\rangle \geq 0. \]
Letting \( n \to \infty \) in the above inequality, we have
\[ \langle \tilde{F} - q, u_\infty \rangle \geq 0. \]
It follows from \( u_\infty \in S \) and \( q \in S_\perp \) that
\[ \langle \tilde{F}, u_\infty \rangle \geq 0. \]
Taking \( \tilde{F} = -u_\infty \), we obtain a contradiction. Therefore, \( \bigcup_{n \in \mathbb{N}} SOLIVI_n(F_n) \) is bounded and so there exists a convergent subsequence with a limit \( u_0 \). It follows from (3.7) that for any \( \tilde{F} \in \mathbb{R}^n \),
\[ \langle \tilde{F} - q - F(u_0), u_0 \rangle \geq 0, \]
which implies that \( u_0 \in SOLIVI(\mathbb{R}^n, q + F(\cdot)) \) and so \( SOLIVI(\mathbb{R}^n, q + F(\cdot)) \) is nonempty for all \( q \in S_\perp \). Similar to the proof of Lemma 2.6, we can prove that \( SOLIVI(\mathbb{R}^n, q + F(\cdot)) \) is nonempty, closed and convex set.

Next, we prove the second assertion. If not, then there exist \( \{q_k\} \subset S_\perp \) and \( \{u_k\} \) such that, for any given \( \tilde{F} \in \mathbb{R}^n \),
\[ \langle \tilde{F} - q_k - F(u_k), u_k \rangle \geq 0 \]
and
\[ \|u_k\| > k(1 + \|q_k\|). \]
It follows that
\[ \lim_{k \to \infty} \|u_k\| = \infty, \quad \lim_{k \to \infty} \frac{\|q_k\|}{\|u_k\|} = 0. \]
Because \( \{q_k\} \subset S_\perp \) is bounded, without loss of generality, we can assume that
\[ \lim_{n \to \infty} q_k = q_\infty \in S_\perp \]
and
\[ \lim_{k \to \infty} \frac{u_k}{\|u_k\|} = u_\infty. \]
From (3.8), we have
\[ \left\langle \frac{\tilde{F} - q_k - F(u_k)}{\|u_k\|}, \frac{u_k}{\|u_k\|} \right\rangle \geq 0. \]
Letting \( k \to \infty \) in the above inequality, one has
\[ \langle F(u_\infty), u_\infty \rangle \leq 0. \]
As \( F \) is semi-defined, we obtain
\[ \langle F(u_\infty), u_\infty \rangle = 0 \]
and so \( u_\infty \in S \). Moreover, it follows from (3.8) that for any \( \tilde{F} \in \mathbb{R}^n \),
\[ \langle \tilde{F} - q_k, u_k \rangle \geq 0. \]
This means that
\[ \langle \tilde{F} - q^k, \frac{u^k}{\|u^k\|} \rangle \geq 0 \]
and so
\[ \langle \tilde{F} - q^\infty, u_\infty \rangle \geq 0. \]

As \( u_\infty \in S \) and \( q^\infty \in S^\perp \), we have
\[ \langle \tilde{F}, u_\infty \rangle \geq 0, \]
which is a contradiction. \( \square \)

**Lemma 3.6** Let \( K \) be a nonempty closed convex set and \( F : R^n \to R^n \) be a para-monotone and continuous map. Assume that \( SOLIVI(K, q + F(\cdot)) \neq \emptyset \) for any \( q \in R^n \) and the linear growth (2.3) holds. Then, \( A : R^n \to R^n \) is continuous, where \( A \) is defined by \( A(q) = F(u) \) for any \( q \in R^n \) and \( u \in SOLIVI(K, q + F(\cdot)) \).

**Proof** Let \( q_n \to q \) and \( u_n \in SOLIVI(K, q_n + F(\cdot)) \). Then, \( q_n + F(u_n) \in K \) and for any \( \tilde{F} \in K \),
\[ \langle \tilde{F} - q_n - F(u_n), u_n \rangle \geq 0. \]

It follows that \( \{u_n\} \) is bounded and so there exists a convergent subsequence of \( \{u_n\} \), denoted again by \( \{u_n\} \), with a limit \( u_0 \). Because \( K \) is closed and \( F \) is continuous, we have \( q + F(u_0) \in K \) and
\[ \langle \tilde{F} - q_0 - F(u_0), u_0 \rangle \geq 0, \quad \forall \tilde{F} \in K. \]

This means that \( u_0 \in SOLIVI(K, q + F(\cdot)) \). Suppose that there exists another convergent subsequence of \( \{u_n\} \), denoted again by \( \{u_n\} \), with a limit \( u_1 \). Then, \( u_1 \in SOLIVI(K, q + F(\cdot)) \) and so \( F(u_0) = F(u_1) \). It follows that
\[ A(q_n) = F(u_n) \to F(u_1) = A(q) \]
and so \( A : R^n \to R^n \) is continuous. \( \square \)

**Theorem 3.7** Let \( F \in R^{n \times n} \) be a psd-plus matrix. Suppose that \( SOLIVI(R^n, q + F(\cdot)) \neq \emptyset \) for all \( q \in R^n \) and there exists a constant \( \rho > 0 \) such that (2.3) holds. Let \( D : R^n \to R^n \) be a continuous map such that
\[ \|D(u)\| \leq L_D \|u\|, \quad \forall u \in R^n \]
for some constant \( L_D \in (0, \frac{1}{\rho}) \). Then, for any \( q \in R^n \), \( SOLIVI(R^n, q + H) \) is a nonempty closed set, where \( H = F + D \), and
\[ \sup\{\|u\| : u \in SOLIVI(R^n, q + H)\} \leq \frac{\rho(1 + \|q\|)}{1 - \rho L_D}. \] (3.10)

Assume further that there exist constants \( L_A > 0 \) and \( L \in (0, \frac{1}{L_A}) \) such that
\[ \begin{cases} \|A(q_1) - A(q_2)\| \leq L_A \|q_1 - q_2\|, & \forall q_1, q_2 \in R^n, \\ \|D(u_1) - D(u_2)\| \leq L \|F(u_1) - F(u_2)\|, & \forall u_1, u_2 \in R^n, \end{cases} \] (3.11)
where $A$ is defined as that in Lemma 3.6. Then, for any $q^i \in R^n$ and $u^i \in SOLIVI(R^n, q + H)$ with $i = 1, 2$,
\[
\|Fu_1 - Fu_2\| \leq \frac{L_A\|q_1 - q_2\|}{1 - L_AL},
\]  
(3.12)
and for every $q \in R^n$,
\[
SOLIVI(R^n, q + H) = F^{-1}v(q) \bigcap \{v : \langle F' - w(q), v \rangle \geq 0, \forall F' \in R^n \},
\]
where $v(q) = F\hat{u}, w(q) = q + H(\hat{u})$ for any $\hat{u} \in SOLIVI(R^n, q + H)$, and $F^{-1}v(q)$ is the inverse image of $v(q)$. Consequently, $SOLIVI(R^n, q + H)$ is a convex set.

**Proof** Similar to the proof of Theorem 3.5 in [17], we can obtain all the results except for the last one. Now, we prove the last result. For any $u_1, u_2 \in SOLIVI(R^n, q + H)$, by the inequality (3.12), we know that $\|Fu_1 - Fu_2\| = 0$. This means that $Fu$ is a constant vector for all $u \in SOLIVI(R^n, q + H)$. Furthermore, it follows from (3.11) that $\|Du_1 - Du_2\| = 0$. Thus, $Du$ is a constant vector and so is $H(u)$ for all $u \in SOLIVI(R^n, q + H)$.

For any $u \in SOLIVI(R^n, q + H)$ and $\hat{F} \in R^n$, one has
\[
Fu = F\hat{u} = v(q)
\]
and so $u \in F^{-1}v(q)$. As
\[
w(q) = q + H(\hat{u}), \quad \hat{u} \in SOLIVI(R^n, q + H),
\]
we know that $v(q)$ and $w(q)$ are constants. Moreover, for any $u \in SOLIVI(R^n, q + H)$, we have
\[
\langle \hat{F} - q - H(u), u \rangle \geq 0,
\]
which implies that
\[
\langle \hat{F} - q - H(\hat{u}), u \rangle \geq 0
\]
and so
\[
\langle \hat{F} - w(q), u \rangle \geq 0.
\]
It follows that
\[
SOLIVI(R^n, q + H) \subset F^{-1}v(q) \bigcap \{v : \langle \hat{F} - w(q), v \rangle \geq 0, \forall \hat{F} \in R^n \}.
\]
Conversely, for any $u \in F^{-1}v(q) \bigcap \{v : \langle \hat{F} - w(q), v \rangle \geq 0, \forall \hat{F} \in R^n \}$, we have
\[
Fu = v(q) = F\hat{u},
\]
where $\hat{u} \in SOLIVI(R^n, q + H)$. It follows from (3.11) that $Du = D\hat{u}$ and so $H(u) = H(\hat{u})$.

Consequently, we have
\[
0 \leq \langle \hat{F} - w(q), u \rangle = \langle \hat{F} - q - H\hat{u}, u \rangle = \langle \hat{F} - q - Hu, u \rangle
\]
and so $u \in SOLIVI(R^n, q + H)$. This shows that
\[
SOLIVI(R^n, q + H) = F^{-1}v(q) \bigcap \{v : \langle \hat{F} - w(q), v \rangle \geq 0, \forall \hat{F}' \in R^n \}.
\]

Next, we show that $SOLIVI(R^n, q + H)$ is a convex set. In fact, for any $u_1, u_2 \in SOLIVI(R^n, q + H)$, we only need to show that $\tilde{u} = \lambda u_1 + (1 - \lambda)u_2 \in SOLIVI(R^n, q + H)$ for all $\lambda \in [0, 1]$. Because $F(u_1) = F(u_2) = v(q)$, one has
\[
F(\lambda u_1 + (1 - \lambda)u_2) = \lambda F(u_1) + (1 - \lambda)F(u_2) = v(q),
\]
which means that \( \hat{u} \in F^{-1}v(q) \). Moreover, for any \( \tilde{F} \in R^n \), we have
\[
\langle \tilde{F} - w(q), u_1 \rangle \geq 0, \quad \langle \tilde{F} - w(q), u_2 \rangle \geq 0.
\]
It follows that
\[
\langle \tilde{F} - w(q), \hat{u} \rangle \geq 0
\]
and so
\[
\hat{u} \in F^{-1}v(q) \cap \{ v : \langle \tilde{F} - w(q), v \rangle \geq 0, \forall \tilde{F} \in R^n \},
\]
which shows that \( \hat{u} \in SOLIVI(R^n, q + H) \). \( \square \)

**Theorem 3.8** Let \( F : R^n \to R^n \) be a given linear map and \( (f, G, B) \) satisfy conditions (A) and (B). Then, DIVI (1.2) has a weak solution in the sense of Carathéodory under any one of the following conditions:

(a) \( K \subset R^n \) is a nonempty compact convex set, and \( F : R^n \to R^n \) is continuous and para-monotone such that \( q + F \) is invertible and \( (q + F)^{-1} \) is continuous on \( R^n \) for all \( q \in R^n \);

(b) \( K \subset R^n \) is a nonempty, compact and convex set, and \( F : R^n \to R^n \) is surjective, continuous, and strictly monotone;

(c) \( K = R^n \), \( F : R^n \to R^n \) is continuous and para-monotone, and there exist \( u_0, y_0, F^0 \in R^n \) such that (3.2) and (3.3) hold;

(d) \( K = R^n \), \( F : R^n \to R^n \) is continuous and para-monotone, and there exist \( u_0, y_0, F^0 \in R^n \) such that (3.2) and (3.5) hold;

(e) \( F \) is a positive semi-define matrix such that, for any \( n \in N \),
\[
SOLIVI \left( R^n, q + \left(1 - \frac{1}{n}\right)F + \frac{1}{n}I \right) \neq \emptyset,
\]
where \( I \) is the identity map on \( R^n \);

(f) \( F = \tilde{F} + D \), where \( \tilde{F} \in R^{n \times n} \) is a psd-plus matrix such that (3.2) and (3.3) hold and \( D \) is a continuous map such that (3.9) and (3.11) hold.

**Proof** It follows from Theorems 3.1–3.7 that \( SOLIVI(K, q + F) \) is a nonempty, closed and convex set and satisfies condition (2.3). By Lemma 2.7, we know that DIVI (1.2) has a weak solution in the sense of Carathéodory. \( \square \)

### 4 An Application

In this section, we will give an application of the DIVI to the time-dependent spatial price equilibrium control problem.

As discussed by Scrimali [22], we consider the time-dependent spatial price equilibrium control problem. Assume that a single commodity is produced at \( m \) supply markets, with typical supply market denoted by \( i \) and is consumed at \( n \) demand markets, with typical demand market denoted by \( j \), during the time interval \([0, T]\) with \( T > 0 \). \((i, j)\) denotes the typical pair of producers and consumers for \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \). Let \( S_i(t) \) be the supply of the commodity produced at supply market \( i \) at time \( t \in [0, T] \) and group the supplies into a column vector
\[
S(t) = (S_1(t), S_2(t), \ldots, S_m(t)) \in R^m.
\]
Let $D_j(t)$ be the demand of the commodity associated with demand market $j$ at time $t \in [0, T]$ and group the demands into a column vector

$$D(t) = (D_1(t), D_2(t), \cdots, D_n(t)) \in \mathbb{R}^n.$$  

Let $x_{ij}(t)$ be the commodity shipment from supply market $i$ to demand market $j$ at time $t \in [0, T]$ and group the commodity shipments into a column vector $x(t) \in \mathbb{R}^{mn}$.

Suppose that for all $t \in [0, T]$, 

$$S_i(t) = \sum_{j=1}^n x_{ij}(t), \quad D_j(t) = \sum_{i=1}^m x_{ij}(t).$$

Now, we consider the problem from the policy-maker’s point of view and present the time-dependent optimal control equilibrium problem. Under this perspective, by adjusting taxes $u(t)$, it is possible to control the resource exploitations $S(x(t), u(t))$ at supply markets and the consumption $D(x(t), u(t))$ at demands markets. It is known that the tax adjustment is an efficient means of regulating production and consumption. Specifically, if the policy-maker is concerned with restricting production or consumption of a certain commodity, then higher taxes will be imposed; whereas if the government aims to encourage production or consumption of some commodities, subsidies will be imposed.

Similar to Scrimali [22], we introduce the function of commodity shipments $x(t)$ and regulatory taxes $u(t)$ as follows:

$$W(t, x(t), u(t)) = (S(x(t), u(t)), D(x(t), u(t)))^T, \quad \forall t \in [0, T].$$

Obviously, the map $W$ is defined as $W : [0, T] \times \mathbb{R}^{mn} \times \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$. We assume that the map $W(t, x, u)$ can be written as

$$W(t, x(t), u(t)) = G(t, x(t)) + F(u(t)), \quad \forall t \in [0, T]$$

such that $G(t, x)$ is a Carathéodory function (that is, it is measurable in $t$ for all $x \in \mathbb{R}^{mn}$ and continuous with respective to $x$) and $F(u)$ is Lipschitz continuous. Moreover, assume that there exists $\gamma(t) \in L^2(0, T)$ such that

$$\|G(t, x)\| \leq \gamma(t) + \|x\|.$$  

Thus, it is easy to know that

$$W : [0, T] \times L^2([0, T], \mathbb{R}^{mn}) \times L^2([0, T], \mathbb{R}^{m+n}) \to L^2([0, T], \mathbb{R}^{m+n}).$$

Finally, we suppose that the following lower and upper capacity constrains are satisfied:

$$\underline{\omega}(t) = (\underline{S}(t), \underline{D}(t)), \quad \overline{\omega}(t) = (\overline{S}(t), \overline{D}(t)),$$

where $\underline{S}(t), \overline{S}(t) \in L^2([0, T], \mathbb{R}^n), \underline{D}(t), \overline{D}(t) \in L^2([0, T], \mathbb{R}^n)$, $0 \leq \underline{S}(t) < \overline{S}(t)$ for almost all $t \in [0, T]$ and $0 \leq \underline{D}(t) < \overline{D}(t)$ for almost all $t \in [0, T]$. We note that the capacity constrains are assumed to be independent of $x$ and $u$.

We introduce the set of feasible states as follows:

$$\mathcal{K} = \{ w \in L^2([0, T], \mathbb{R}^{m+n}) : \underline{\omega}(t) \leq w(t) \leq \overline{\omega}(t) \quad \text{for almost all } t \in [0, T] \}.$$  

Similar to the definition of Scrimali [22], we say that $u^*(t)$ is an optimal regulatory tax if it makes the corresponding state $W(t, x(t), u^*(t))$ satisfying the constraint $W(t, x(t), u^*(t)) \in \mathcal{K}$
Employing Theorem 2 of Scrimalli [22], it is easy to see that a regulatory tax vector $u^*(t) \in L^2([0,T], R^{m+n})$ is optimal if and only if it solves the following inverse variational inequality:

$$W(t, x(t), u^*(t)) \in \mathcal{K}, \quad \int_0^T \langle w(t) - W(t, x(t), u^*(t)), u^*(t) \rangle \, dt \leq 0, \quad \forall w(t) \in \mathcal{K}. \tag{4.1}$$

On the other hand, we know that there is a relationship between the change rate of commodity shipments $x(t)$ and regulatory taxes $u(t)$ with the commodity shipments $x(t)$. We require that

$$\dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t), \quad \text{for almost all } t \in [0, T], \tag{4.2}$$

where $f : [0, T] \times R^{mn} \to R^{mn}$ and $B : [0, T] \times R^{mn} \to R^{m \times (m+n)}$ are two maps satisfying some suitable conditions.

Combining (4.1) and (4.2), we know that $(x(t), u(t))$ is a Carathéodory weak solution of the following DIVI problem:

$$\begin{cases}
\dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t), \\
u(t) \in \text{SOLIVI}(-\mathcal{K}, -G(t, x(t)) - F(\cdot)), \\
x(0) = x_0.
\end{cases} \tag{4.3}$$

Specially, suppose that $\underline{w}_r(t)$ and $\overline{w}_r(t)$ are constants for $r = 1, 2, \cdots, m+1$ and

$$f(t, x) = \begin{pmatrix}
\alpha_1 t \\
\vdots \\
\alpha_m t
\end{pmatrix} + \beta x,$$

$$B(t, x) = \begin{pmatrix}
t \sin(x_1) & 0 & \cdots & 0 \\
t \sin(x_2) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
t \sin(x_m) & 0 & \cdots & 0
\end{pmatrix},$$

$$G(t, x) = \begin{pmatrix}
\lambda_1 e^t + x_1 \\
\vdots \\
\lambda_m e^t + x_m \\
0
\end{pmatrix},$$

$$F(u) = u,$$

where $x = (x_1, \cdots, x_m)^T$. Then, all the conditions of (b) in Theorem 3.8 are satisfied and so it shows that DIVI (4.3) has a Carathéodory weak solution $(x(t), u(t))$. 

and for almost all $t \in [0, T]$, the following three conditions hold:

$$W_r(t, x(t), u^*(t)) = \underline{w}_r(t) \quad \Rightarrow u_r^*(t) \geq 0, \quad r = 1, 2, \cdots, m+n,$$

$$W_r(t, x(t), u^*(t)) = \overline{w}_r(t) \quad \Rightarrow u_r^*(t) \leq 0, \quad r = 1, 2, \cdots, m+n,$$

$$\underline{w}_r(t) < W_r(t, x(t), u^*(t)) < \overline{w}_r(t) \quad \Rightarrow u_r^*(t) = 0, \quad r = 1, 2, \cdots, m+n.$$
5 Conclusions

It is known that the inverse variational inequality is equivalent to the special case of the general variational inequality and it has broad applications in various disciplines ([9–12]). ScrimaI [22] showed that time-dependent spatial price equilibrium problem can be formulated as the evolutionary parametric variational inequality and the time-dependent optimal control equilibrium problem can be presented as the time-dependent inverse variational inequality when we formulate the problem from the policy-maker’s point of view. On the other hand, since Pang and Stewart [19] introduced and studied differential variational inequality in finite-dimensional Euclidean spaces, many authors have studied the differential variational inequalities under different conditions (see, for example, [1, 5, 7, 15, 17, 20, 21, 23–25] and the references therein). However, to our best knowledge, there is no paper to study the differential inverse variational inequality.

The purpose of this article is to introduce and study a new differential inverse variational inequality in finite-dimensional spaces. We proved the linear growth of the solution set for the differential inverse variational inequality under various conditions and the existence theorems concerned with the Carathéodory weak solutions for the differential inverse variational inequality in finite-dimensional spaces. Moreover, we formulated the time-dependent spatial price equilibrium control problem as the time-dependent parametric inverse variational inequality. The example given in Section 4 shows that it is suitable and convenient to employ the differential inverse variational inequality to the study concerned with the ordinary differential equation whose right-hand function is parameterized by an algebraic variable that is required to be a solution of the time-dependent spatial price equilibrium control problem.

It is worth noting that the capacity constraints in Section 4 are assumed to be independent of the $x$ and $u$. Therefore, it is interesting and important to explore the more general case in which they depend on the commodity shipments $x(t)$ and regulatory taxes $u(t)$.

References