Nonlocal nonlinear differential equations with a measure of noncompactness in Banach spaces

Xingmei Xue*
Department of Mathematics, Southeast University, Nanjing 210096, China

A R T I C L E   I N F O

Article history:
Received 25 September 2007
Accepted 20 March 2008

MSC:
34G20
34G25
47H20

Keywords:
Nonlinear nonlocal initial condition
Equicontinuous semigroup
m-dissipative operator
Hausdorff’s measure of noncompactness

A B S T R A C T

In this paper we give the existence of integral solutions for nonlinear differential equations with nonlocal initial conditions under the assumptions of the Hausdorff measure of noncompactness in separable and uniformly smooth Banach spaces.

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction and preliminaries

In this paper we are concerned with the existence of integral solutions of the nonlinear nonlocal initial value problem (IVP for short)

\[ u'(t) \in Au(t) + f(t, u(t)), \quad t \in (0, T) \text{ a.e.} \]  (1.1)
\[ u(0) = g(u), \]  (1.2)

where \( A \) is a nonlinear \( m \)-dissipative operator (it may be multi-valued) which generates a contraction semigroup \( S(t) \) on \( D(A) \) in a Banach space \( X \) and \( f : [0, T] \times D \rightarrow X, \ g : C([0, T]; D) \rightarrow D(A) \) are given \( X \)-valued functions with \( D(A) \subset D \subset X \).

The semilinear nonlocal IVP (i.e., when \( A \) generates a \( C_0 \)-semigroup) was initiated by Byszewski. The importance of the problem consists in the fact that it is more general and has a better effect than the classical initial condition; it has been studied extensively under various conditions on \( A, f \) and \( g \) by several authors (cf. [4,6,8–13,17,19,20,23,24] and references therein). We refer to some of the papers below. Byszewski [8,9], Byszewski and Lakshmikantham [10] give the existence and uniqueness of a mild solution when \( f \) and \( g \) satisfy Lipschitz type conditions. Ntouyas and Tsamatos [23,24] study the case with compactness conditions. In [19] Lin and Liu discuss semilinear integro-differential equations under Lipschitz type conditions. Byszewski and Akca [11] give the existence of mild solutions of a functional differential equation when \( S(t) \) is compact, and \( g \) is convex and compact on a given ball of \( C([0, T]; X) \). In [13] Fu and Ezzinbi study neutral functional differential equations with nonlocal initial conditions. Benchohra and Ntouyas [6] discuss second-order differential equations with nonlocal conditions under compact conditions.

* This research was supported by NNSF of China (No. 10674024).
* Tel.: +86 13851701937; fax: +86 25 83792316.
E-mail address: xmxue@seu.edu.cn.

0362-546X/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2008.03.046
In [1] Aizicovici and Gao discuss the nonlinear nonlocal problem (1.1) and (1.2) when \( g \) and \( f \) satisfy Lipschitz conditions. In [2] Aizicovici and McKibben discuss the nonlinear nonlocal problem (1.1) and (1.2) when \( g \) and \( S(t) \) are compact. They also give the existence of integral solutions when \( g : L(0, T; X) \to X \) is continuous and \( S(t) \) is compact but without the assumption of compactness of \( g \). Recently, in [14] García-Falset discusses the nonlinear inclusions with nonlocal initial conditions for the case when \( f \) is compact and \( g \) is condensing; the cases when \( g : C([0, T]; X) \to D(A) \) and \( f : C([0, T]; X) \to L(0, T; X) \) are condensing.

In this paper we give the existence of an integral solution of the nonlinear IVP (1.1) and (1.2) when \( g \) is compact or Lipschitz, \( S(t) \) is equicontinuous and the hypotheses of the Hausdorff measure of noncompactness of \( f \) are assumed. We also suppose that \( f \) and \( g \) are locally defined on \( D \) rather than on \( X \).

Let \((X, \| \cdot \|)\) be a real Banach space. We denote by \( C([0, T]; X) \) the space of \( X \)-valued continuous functions on \([0, T]\) with the norm \( \| u \| = \sup \{ \| u(t) \|, t \in [0, T] \} \) and by \( L(0, T; X) \) the space of \( X \)-valued Bochner integrable functions on \([0, T]\) with the norm \( \| u \|_1 = \int_0^T \| u(t) \|\, dt \) (we denote as \( \| \cdot \| \) the norm of space \( L(0, T; X) \)).

The Hausdorff measure of noncompactness \( \beta \), defined by \( \beta_r(B) = \inf \{ r > 0, B \) can be covered by a finite number of balls with radii \( r \} \) for bounded subset \( B \) in Banach space \( Y \). We recall the following properties of the Hausdorff measure of noncompactness \( \beta \):

**Lemma 1.1** ([3]). Let \( Y \) be a real Banach space and \( B, C \subseteq Y \) be bounded: then the following properties are satisfied:

1. \( B \) is precompact if and only if \( \beta_r(B) = 0 \);
2. \( \beta_r(B) = \beta_r(B) = \beta_r(\text{conv} B) \), where \( \text{conv} B \) is the closure and convex hull of \( B \) respectively;
3. \( \beta_r(B) \leq \beta_r(C) \) when \( B \subseteq C \);
4. \( \beta_r(B + C) \leq \beta_r(B) + \beta_r(C) \), where \( B + C = \{ x + y; x \in B, y \in C \} \);
5. \( \beta_r(B \cup C) \leq \max \{ \beta_r(B), \beta_r(C) \} \);
6. \( \beta_r(\lambda B) = |\lambda| \beta_r(B) \) for any \( \lambda \in \mathbb{R} \);
7. if the map \( Q : D(Q) \subseteq Y \to Z \) is Lipschitz continuous with constant \( k \) then \( \beta_r(QB) \leq k \beta_r(B) \) for any bounded subset \( B \subseteq D(Q) \), where \( Z \) is a Banach space;
8. \( \beta_r(B) = \inf \{ d_r(B, C); C \subseteq Y \text{ is precompact} \} = \inf \{ d_r(B, C); C \subseteq Y \text{ is finite valued} \} \), where \( d_r(B, C) \) means the nonsymmetric (or symmetric) Hausdorff distance between \( B \) and \( C \) in \( Y \);
9. if \( \{ W_n \}_{n=1}^{+\infty} \) is a decreasing sequence of bounded closed nonempty subsets of \( Y \) and \( \lim_{n \to +\infty} \beta_r(W_n) = 0 \), then \( \bigcap_{n=1}^{+\infty} W_n \) is nonempty and compact in \( Y \).

The map \( Q : W \subseteq Y \to Y \) is said to be a \( \beta_r \)-contraction if there exists a positive constant \( k < 1 \) such that \( \beta_r(Q(C)) \leq k \beta_r(C) \) for any bounded subset \( C \subseteq W \), where \( Y \) is a Banach space.

**Lemma 1.2** ([3]; Darbo–Sadovskii). If \( W \subseteq Y \) is bounded closed and convex, the continuous map \( Q : W \to W \) is a \( \beta_r \)-contraction; then the map \( Q \) has at least one fixed point in \( W \).

In this paper we denote by \( \beta \) the Hausdorff measure of noncompactness of \( X \) and denote by \( \beta_k \) the Hausdorff measure of noncompactness of \( C([0, T]; X) \). Denote as \( d(\cdot, \cdot) \) (resp. \( d_s(\cdot, \cdot) \)) the nonsymmetric (or symmetric) Hausdorff distance of \( X \) (resp. \( C([0, T]; X) \)). To discuss the existence we also need the following lemmas in this paper.

**Lemma 1.3** ([3]). If \( W \subseteq C([0, T]; X) \) is bounded, then

\[
\beta(W(t)) \leq \beta_t(W)
\]

for all \( t \in [0, T] \), where \( W(t) = \{ u(t); u \in W \} \subseteq X \). Furthermore if \( W \) is equicontinuous on \([0, T]\), then \( \beta(W(t)) \) is continuous on \([0, T]\) and

\[
\beta_t(W) = \sup \{ \beta(W(t)), \ t \in [0, T] \}.
\]

A multi-valued map \( A \) with domain \( D(A) \) is said to be \( m \)-dissipative if \( \| x_1 - x_2 \| \leq \| x_1 - x_2 - \lambda(y_1 - y_2) \| \) for all \( \lambda > 0, [x_1, y_1] \in A, i = 1, 2 \) and \( R(I - A) = X \), where \( I \) is the identical operator on \( X \). If \( A \) is \( m \)-dissipative with domain \( D(A) \) and \( X \) is uniformly smooth, then \( D(A) \) must be convex.

By an integral solution of

\[
\begin{align*}
  u'(t) &\in Au(t) + h(t), & t \in (0, T) \text{ a.e.} \\
  u(0) &= u_0
\end{align*}
\]

we mean a function \( u \in C([0, T]; X) \) with \( u(0) = u_0 \) and the inequality

\[
\| u(t) - x \|^2 \leq \| u(s) - x \|^2 + 2 \int_s^t \langle u(r) - x, h(t) + y \rangle \, dr
\]

holding for all \( [x, y] \in A \) and \( 0 \leq s \leq t \leq b \). Here \( \langle x, y \rangle = \sup \{(x^\ast, y); x^\ast \in J(x) \} \) for \( x, y \in X \) with \( J : X \to X^\ast \) being the duality mapping defined by \( J(x) = \{ x^\ast \in X^\ast; \langle x^\ast, x \rangle = \| x \|^2 = \| x \|^2 \} \). If \( X \) is uniformly smooth then \( J \) is single valued and uniformly continuous on bounded subsets of \( X \).
A continuous $X$-valued function $u$ is said to be an integral solution to (1.1) and (1.2) if it is an integral solution of (1.3) and (1.4) with $u_0 = g(u)$ and $h(t) = f(t, u(t))$ for a.e. $t \in [0, T]$.

If $A$ is $m$-dissipative, $h \in L(0, T; X)$ and $u_0 \in D(A)$ then there exists a unique integral solution $u$ to (1.3) and (1.4) with $u(t) \in D(A)$ for $t \in [0, T]$. In addition, if $v$ is the integral solution to

$$
\begin{align*}
v'(t) & \in Av(t) + \overline{h}(t), \quad t \in (0, T) \\
v(0) & = v_0.
\end{align*}
$$

then the inequalities

$$
\|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_0^t \|h(\tau) - \overline{h}(\tau)\|d\tau,
$$

and

$$
\|u(t) - v(t)\|^2 \leq \|u(s) - v(s)\|^2 + 2 \int_0^t (u(\tau) - v(\tau), h(\tau) - \overline{h}(\tau))d\tau
$$

hold for $0 \leq s \leq t \leq T$.

If $h(t) = 0$ for $t \in (0, T)$ a.e., then $u(t) = S(t)u_0$, where $S(t)$ is the nonexpansive semigroup generated by $A$ on $\overline{D(A)}$. The semigroup $S(t)$ is said to be equicontinuous if $\{S(t)x : x \in B\}$ is equicontinuous at any $t > 0$ for any bounded subset $B \subset X$ (cf. [5,25]).

**Lemma 1.4** ([27]). If $A$ generates an equicontinuous semigroup $S(t), B \subset L(0, T; X)$ is uniformly integrable and $C \subset \overline{D(A)}$ is compact, then the set $\Theta(B, C) = \{u : u$ is an integral solution of (1.3) and (1.4) for some $h \in B$ and some $u_0 \in C\}$ is bounded and equiconitnuous in $C([0, T]; X)$.

**Lemma 1.5** ([27]). If $X$ is separable and uniformly smooth, $A$ generates an equicontinuous semigroup $S(t), B \subset L(0, T; X)$ is uniformly integrable, $C \subset \overline{D(A)}$ is compact and $\Theta = \Theta(B, C)$ is defined as above, then

$$
\beta(\Theta(t)) \leq \int_0^t \beta(B(s))ds, \quad t \in [0, T].
$$

**Lemma 1.6** ([22]). If $X$ is separable, then there exists an increasing sequence of finite dimensional subspaces $\{X_n\}_{n=1}^\infty$ such that $X = \bigcup_{n=1}^\infty X_n$ and for any countable bounded set $\{X_n\}_{n=1}^\infty$, 

$$
\beta(\{X_n\}_{n=1}^\infty) = \lim_{n \to \infty} \lim_{m \to \infty} \sup d(x_m, X_n),
$$

where $d(x, Y) = \inf\{\|x - y\| : y \in Y\}$.

In this paper we always suppose that $X$ is a real separable and uniformly smooth Banach space, $D$ is closed and convex. Denote by $E$ the closed and convex subset $C([0, T]; D) = \{u \in C([0, T]; X) : u(t) \in D$ for all $t \in [0, T]\}$ and $B_r = \{u \in E : \|u\| \leq r\}$.

This paper is organized as follows. In Section 2 we establish sufficient conditions for the existence of integral solutions for (1.1) and (1.2) when $g$ is compact. In Section 3 we study (1.1) and (1.2) when $g$ is Lipschitz. In Section 4 we present an example illustrating the abstract theory of the previous sections.

### 2. The case when $g$ is completely continuous

In this section, by using the techniques of nonlinear semigroups, the Hausdorff measure of noncompactness and its applications in differential equations in Banach spaces (see, e.g., [3,7,14,15,18,27]), we give some existence results for the nonlocal IVP (1.1) and (1.2) when $g : C([0, T]; D) \to \overline{D(A)}$ is continuous and compact. Here we list the following hypotheses:

- (H$_0$): The semigroup $S(t)$ generated by $A$ is equicontinuous.
- (H$_1$): $g : E \to \overline{D(A)}$ is continuous and compact.
- (2): There exist constants $c$ and $d$ such that for all $u \in E$

$$
\|g(u)\| \leq c\|u\| + d.
$$

(2): There exist functions $a, b \in L([0, T]; R^+)$ such that $\|f(t, x)\| \leq a(t)\|x\| + b(t)$ for a.e. $t \in [0, T], x \in D$. 

(3): There exist $\alpha \in L([0, T]; R^+)$ such that for any bounded set $B \subset D$

$$
\beta(f(t, B)) \leq \alpha(t)\beta(B) \quad t \in [0, T] \text{ a.e.,}
$$

where $f(t, B) = \{f(t, y) : y \in B\}$.

Now we give an existence result under the above hypotheses.
Theorem 2.1. If $X$ is separable and uniformly smooth, the hypotheses ($H_4$), ($H_5$) and ($H_6$) are true. Then the nonlocal IVP (1.1) and (1.2) has at least one integral solution provided that $c + \|a\|_1 < 1$.

Proof. We define $K : E \to E$ by $Kv$ being the integral solution to (1.3) and (1.4) with $h(t) = f(t, v(t))$ a.e. $t \in [0, T]$ and $v_0 = g(v)$ for any $v \in E$. Clearly $K$ is continuous due to ($H_8$), ($H_9$) (cf. [2]). And for any fixed $[x, y] \in A$

$$
\|Kv(t)\| \leq c|v| + d + 2\|x\| + T\|y\| + \|b\|_1 + \int_0^t a(s)\|v(s)\|\, ds
$$

for $t \in [0, T]$. As $c + \|a\|_1 < 1$, let

$$
r = \frac{2\|x\| + d + T\|y\| + \|b\|_1}{1 - c - \|a\|_1}
$$

and then $|Kv| \leq r$ when $|v| \leq r$.

Let $W_0 = \{v \in E; |v| \leq r\}$ and $W_1 = \text{conv } kW_0$. Then $W_1 \subseteq W_0$, where $\text{conv}$ means the closure of the convex hull. As $g$ is compact and $f$ satisfies ($H_4$), this implies that $Kw_0 \subseteq \Theta(B, C)$, where $\Theta(B, C)$ is defined as in Lemma 1.4 with $B = \{f(t, v(t)) \in L(0, T; X); v \in W_0\}$ and $C = g(W_0)$. So we know that $W_1 \subseteq \Theta([0, T]; X)$ is equicontinuous and convex.

Define $W_{n+1} = \text{conv } Kw_n$ for $n = 1, 2, \ldots$; we know that $W_{n+1} \subseteq W_n$ for $n = 1, 2, \ldots$ as $W_1 \subseteq W_0$. Now it is implied that $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded closed convex and equicontinuous subsets of $E \subseteq C([0, T]; X)$.

By Lemma 1.5, we know that $W_n \subseteq \Theta(B_n, C_n)$, where $\Theta(B_n, C_n)$ is defined as in Lemma 1.4 with $B_n = \{f(\cdot, v(\cdot)) \in L(0, T; X); v \in W_n\}$ and $C_n = g(W_n)$. Then

$$
\beta(W_{n+1}(t)) \leq \int_0^t \beta(f(s, W_n(s)))\, ds \leq \int_0^t \alpha(s)\beta(W_n(s))\, ds
$$

for all $t \in [0, T]$, where $f(s, W_n(s)) = \{f(s, v(s)) \in E; v \in W_n\}$. Let $\eta(t) = \lim_{n \to \infty} \beta(W_n(t))$; then $\eta \in L(0, T; R)$ and $\eta(0) = 0$ as $W_{n+1}(0) = \text{conv } g(W_n)$ is precompact. Let $n \to \infty$. This implies that

$$
\eta(t) \leq \int_0^t \alpha(s)\eta(s)\, ds.
$$

From Gronwall’s inequality we know that $\eta(t) = 0$ for all $t \in [0, T]$. This means that $\beta_c(W_n) \to 0$. Thus, by Lemma 1.1, we know that $W = \cap_{n=1}^{\infty} W_n$ is nonempty, compact and convex and $K : W \to W$.

By the famous Schauder fixed point theorem, we may conclude that the integral solution $u$ exists, and $u$ is a fixed point of the continuous map $K : W \to W$. □

Theorem 2.2. If $X$ is separable and uniformly smooth, the hypotheses ($H_4$), ($H_5$)(1) and ($H_6$)(1)(3) are true. Then the nonlocal IVP (1.1) and (1.2) has at least one integral solution when the following hypotheses are satisfied:

(1) There exists a constant $M$ such that $\|g(u)\| \leq M$ for all $u \in E$.

(2) There exists a function $h : [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(\cdot, s) \in L(0, T; R)$ for any $s \geq 0$, $f(t, \cdot)$ is continuous and increasing for a.e. $t \in [0, T]$, and

$$
\|f(t, x)\| \leq h(t, \|x\|)
$$

for a.e. $t \in [0, T]$ and $x \in D$. Furthermore we suppose that there exists at least one solution to the following equation:

$$
m(t) = M + 2\|x_0\| + T\|y_0\| + \int_0^t h(s, m(s))\, ds, \quad t \in [0, T]
$$

for some fixed $[x_0, y_0] \in A$.

Proof. Let $m(t)$ be a solution to (2.1). Define $W_0 = \{u \in E; \|u(t)\| \leq m(t) \text{ for all } t \in [0, T]\}$. Then, for any $v \in W_0$, we know that

$$
\|Kv(t)\| \leq M + 2\|x_0\| + T\|y_0\| + \int_0^t h(s, \|v(s)\|)\, ds \leq m(t)
$$

for $t \in [0, T]$. This implies that $Kv \in W_0$. Let $W_{n+1} = \text{conv } Kw_n$ for $n = 1, 2, \ldots$. We can complete the proof just like the proof of Theorem 2.1. □

Remark 2.1. The hypothesis (H)(2) is weaker than ($H_6$)(1), (2) and it is given extensively in references.

Remark 2.2. From the proof of Theorem 2.1 we can prove that Theorems 2.1 and 2.2 are also true if we replace the hypothesis ($H_6$(3)) by the following hypothesis in Banach space $X$ without the separability and uniform smoothness assumptions:
(H'): There exists $\alpha \in L(0, T; R^+)$ such that for any bounded equicontinuous set $B \subset E$

$$\beta(KB(t)) \leq \int_0^t \alpha(s)\beta(B(s))ds, \quad t \in [0, T].$$

**Remark 2.3.** If the semigroup $S(t)$ is compact then the hypothesis (H') is true with $\alpha \equiv 0$ as $K$ is compact (cf. [15, Theorem 3.1]). If $f(t, \cdot)$ is compact for a.e. $t \in [0, T]$, then the hypotheses (H') and ($H_f'(3)$) are satisfied with $\alpha \equiv 0$. If $f = f_1 + f_2$ where $f_1(t, \cdot)$ is compact and $f_2(t, \cdot)$ is Lipschitz continuous with constant $\alpha(t)$ for a.e. $t \in [0, T]$, then the hypotheses (H') and ($H_f'(3)$) are satisfied. This means that the hypotheses (H') and ($H_f'(3)$) are the unification and generalization of the Lipschitz property and compactness assumptions for $f$ and the compactness assumption for $S(t)$.

If $X$ is a Hilbert space, and $\varphi$ is a proper, convex and lower semicontinuous function from $X$ into $(-\infty, \infty]$, then its subdifferential $\partial \varphi$ is $m$-accretive. Let $A = -\partial \varphi$; then $A$ generates an equicontinuous nonlinear contraction semigroup (cf. [5, 25]). From above we can get the following existence result.

**Corollary 2.3.** If $X$ is a separable Hilbert space, the hypotheses ($H_g'$) and ($H_f'$) are true, and $A = -\partial \varphi$ with $\varphi$ is proper, convex and lower semicontinuous from $X$ into $(-\infty, \infty]$. Then the nonlocal IVP (1.1) and (1.2) has at least one integral solution provided that $c + \|a\|_1 < 1$.

### 3. The case when $g$ is Lipschitz

In this section, by giving a new estimation of the Hausdorff measure of noncompactness for bounded subsets of $C([0, T]; X)$ failing to be equicontinuous, we get the existence of an integral solution when $g$ satisfies the following Lipschitz hypothesis:

($H_g'$): $g : E \rightarrow \overline{D(A)}$ is Lipschitz continuous with constant $k < 1$.

**Theorem 3.1.** Let $X$ be separable and uniformly smooth, the hypotheses ($H_A$), ($H_g'$) and ($H_f'$) be true. Then the nonlocal IVP (1.1) and (1.2) has at least one integral solution when $k + \|a\|_1 < 1$ and $k + \|a\|_1 < 1$.

The proof of Theorem 3.1 is based on the following sequence of lemmas:

**Lemma 3.2 ([1,28]).** If the hypotheses ($H_g'$) is true, then for any $h \in L(0, T; X)$ there exists one and only one integral solution to

$$u'(t) \in Au(t) + h(t), \quad t \in [0, T] \text{ a.e.,}$$

$$u(0) = g(u).$$

Define $G : L(0, T; X) \rightarrow E$ by $G(h)$ being the integral solution of (3.1) and (3.2) for $h \in L(0, T; X)$.

**Lemma 3.3.** If the hypotheses ($H_A$) and ($H_g'$) are true, then $GB \subset E \subset C([0, T]; X)$ is bounded and equicontinuous at any $t > 0$ for $B \subset L(0, T; X)$ satisfying $\|h(t)\| \leq p(t)$ for all $h \in B$ and $t \in [0, T]$ a.e. with given $p \in L(0, T; R)$.

**Proof.** For any $h \in B$, we know that for any fixed $x \in D$

$$\|G(h(t))\| \leq \|S(t) |G(h(t)| \| + \int_0^t \|h(s)\|ds$$

$$\leq \|G(h(t))\| + \|S(t)|G(h(t)| \| + \|p(t)\|_1$$

for any $t \in [0, T]$, where $x \in C([0, T]; X)$ is defined by $\hat{x}(t) = x$ for all $t \in [0, T]$. This implies that $GB$ is bounded as $k < 1$.

From (1.8) and the uniqueness of the integral solution of (1.3) and (1.4) we know that

$$\|G(h(t) - S(s)|G(h(t)| \| \leq \int_t^{t+s} \|h(t)| \|dt$$

for all $t, s \geq 0$.

Let $t > 0$ be fixed. For any $\varepsilon > 0$ there exist $\delta \in (0, t)$ such that

$$\int_0^t \|h(s)\|ds < \varepsilon/3$$

for any interval $I'$ with $m(I') \leq \delta$.

For any $h \in B$, we know that for any $0 < s < \delta$

$$\|G(h(t) - S(s)|G(h(t)| \| \leq \|G(h(t) - S(s)|G(h(t)| \| + \|S(s + \delta)|G(h(t) - \delta) - S(s)|G(h(t)| \|$$

$$+ \|S(s + \delta)|G(h(t) - \delta) - S(\delta)|G(h(t) - \delta)\| + \|G(h(t) - S(\delta)|G(h(t) - \delta)\|$$

$$\leq \varepsilon + \|S(s + \delta)|G(h(t) - \delta) - S(\delta)|G(h(t) - \delta)\|.$$


As the semigroup $S(t)$ is equicontinuous then for all $h \in B$

$$\|S(s + \delta)Gh(t - \delta) - S(\delta)Gh(t - \delta)\| \leq \epsilon(\delta, s)$$

with $\epsilon(\delta, s) \to 0$ when $s \to 0$. This means that $t \mapsto (GB_t)(t)$ is equicontinuous at any $t > 0$. □

For any $C \subseteq X$ define $\tilde{C} = \{h \in L(0, T; X) : h(t) \in C \ a.e. \ t \in [0, T]\}$. Thanks to the techniques and lemmas of Gutman [16], we can prove that:

**Lemma 3.4.** If $X$ is separable and uniformly smooth, the hypotheses $(H_\Lambda)$ and $(H'_\Lambda)$ are true. Then $G\tilde{C} \subseteq E \subseteq C([0, T]; X)$ is precompact for any precompact subset $C \subseteq X$.

**Proof.** To prove this lemma, it is enough to show that there exists a subsequence of $\{G(h_n)\}$ such that it is convergence in $C([0, T]; X)$ for any sequence $\{h_n\} \subseteq \tilde{C}$.

From above we know that $G\tilde{C} \subseteq C([0, T]; X)$ is bounded. We define $c' = \sup\{\|x\| : x \in C\}$ and $d' = \sup\{\|u\| : u \in G\tilde{C}\}$.

From [12, Theorem 2], we get that $\tilde{C} \subseteq L(0, T; X)$ is relatively weakly compact. So $\tilde{C}$ is metrizable. This means that the convergence is equivalent to the convergence of sequences in $\tilde{C}$. Hence, without loss of generality, we can suppose that there exists $h \in L(0, T; X)$ and $h_n \rightharpoonup h$ weakly in $L(0, T; X)$. As $X$ is uniformly smooth then the duality map $J$ is single valued and uniformly continuous on bounded subsets of $X$.

Let $u_n = G(h_n)$ and $u = G(h)$; then we know that

$$\|u_n(t) - u(t)\|^2 \leq \|u_n(0) - u(0)\|^2 + 2 \int_0^t (h_n(s) - h(s), u_n(s) - u(s))ds$$

$$\leq k^2|u_n - u|^2 + 2 \int_0^t (h_n(s) - h(s), J(u_n(s) - u(s)))ds$$

for $t \in [0, T]$. This implies that

$$|u_n - u|^2 \leq \frac{2}{1 - k^2} \sup_{t \in [0, T]} \int_0^t (h_n(s) - h(s), J(u_n(s) - u(s)))ds.$$  \hspace{1cm} (3.3)

For any $\epsilon > 0$ and $t \in [0, \epsilon]$ then

$$\left|\int_0^t (h_n(s) - h(s), J(u_n(s) - u(s)))ds\right| \leq 4c'd\epsilon.$$  \hspace{1cm} (3.4)

From Lemma 3.3 we get that $G\tilde{C} \subseteq C([\epsilon, T]; X)$ is equicontinuous. Define $Jv = w \in C_w$ by $w(t) = Jv(t)$ for all $t \in [\epsilon, T]$ for $v \in G\tilde{C}$, where $C_w = C([\epsilon, T]; X^*)$. $X^*$ is the space of bounded continuous $X^*$-valued functions from $[\epsilon, T]$ to $X^*$, and $X^*$ is equipped with its weak-star topology $\sigma(X^*, X)$. From [16, Lemma 2.6], we have that $J(G\tilde{C} - Gh) \subseteq C_w$ is relatively compact. We choose a subsequence of $\{h_n\}$ (also denote it by $\{h_n\}$) such that $J(G\tilde{C} - Gh) \rightharpoonup v \in (L(\epsilon, T; X))^*$ in $C_w$ and

$$c_n \equiv \sup\{\|J(\tilde{C}(\epsilon, t) - Gh(t)) - v(t, x)\| : x \in C, t \in [\epsilon, T]\} \to 0.$$  \hspace{1cm} (3.6)

For $t \in [\epsilon, T]$ we can get that

$$\int_0^t (h_n(s) - h(s), J(u_n(s) - u(s)))ds = \int_0^\epsilon (h_n(s) - h(s), J(u_n(s) - u(s)))ds + \int_\epsilon^t (h_n(s) - h(s), v(s))ds$$

$$+ \int_\epsilon^t (h_n(s) - h(s), J(u_n(s) - u(s)) - v(s))ds.$$  \hspace{1cm} (3.5)

The first integral is estimated as (3.4). For the second, as $v \in (L(\epsilon, T; X))^*$ and $h_n \to h$ weakly in $L(\epsilon, T; X)$, then it implies that

$$\int_\epsilon^t (h_n(s) - h(s), v(s))ds \to 0$$

uniformly for $t \in [\epsilon, T]$. The last integral is estimated as follows:

$$\left|\int_\epsilon^t (h_n(s) - h(s), J(u_n(s) - u(s)) - v(s))ds\right| \leq c_n T \to 0$$

(3.6)

for all $t \in [\epsilon, T]$.

From (3.3)-(3.6) we know that

$$\limsup_{n \to \infty} |u_n - u| \leq \frac{8c'd\epsilon}{1 - k^2}.$$  \hspace{1cm} (3.7)

We complete the proof as $\epsilon$ is given arbitrarily. □
Lemma 3.5. If $X$ is separable and uniformly smooth, the hypotheses $(H_A)$ and $(H'_A)$ are true and $p \in L(0, T; R^+)$, Then

$$\beta_c(GB) \leq \frac{1}{1-k} \int_0^T \beta(B(t)) dt$$

for any subset $B \subseteq L(0, T; X)$ which satisfies $\|h(t)\| \leq p(t)$ a.e. $t \in [0, T]$ and for all $h \in B$.

Proof. From Lemma 3.3 we know that $GB$ is equicontinuous on $(0, T)$, but this fails at 0. So we cannot estimate its Hausdorff measure of noncompactness by Lemma 1.3.

From (1.8) and $(H'_A)$ it is easy to prove that $G : L(0, T; X) \rightarrow C([0, T]; X)$ is Lipschitz with constant $1/(1-k)$.

As $X$ is separable, by Lemma 1.6, there exists an increasing sequence of finite dimensional subspaces $\{X_n\}_{n=1}^{\infty}$ such that $X = \overline{\bigcup_{n=1}^{\infty} X_n}$ and

$$\beta((x_n)_{m=1}^{\infty}) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} d(x_m, X_n)$$

for any countable set $\{x_n\}_{n=1}^{\infty}$.

As $X$ is separable, without loss of generality, we can suppose that $GB = ([Gh_n]_{m=1}^{\infty}. Then \beta_c(GB) = \beta_c([Gh_n]_{m=1}^{\infty})$ for any $l \geq 1$.

Define the projection $P_n : X \rightarrow X_n$ by $P_n(x) = \{y \in X_n : \|x-y\| = d(x, X_n)\}$ (it may be multivalued if $X$ fails to be strictly convex). Then $\|y\| \leq 2\|x\|$ for any $y \in P_n(x), P_n(x)$ is compact and convex for any $x \in X$ and $t \rightarrow P_n(h(t))$ is measurable for any $h \in B$. This implies that $Q_n h = \{v \in L(0, T; X_n) : v(t) \in P_n(h(t)) \in [0, T] \}$ a.e. is nonempty, convex and closed.

For any $\varepsilon > 0$ there exist a constant $N > 0$ and a measurable subset $I' \subseteq [0, T]$ such that

$$\int_{I'} p(t) dt < \varepsilon/2 \quad \text{and} \quad p(t) \leq N \quad \text{for all} \ t \in [0, T] \setminus I'.$$

Define $B' = \{v \in L(0, T; X_n) : v(t) = h(t) \in [0, T] \setminus I' \}$ and $v(t) = 0$ when $t \in I'$ for some $h \in Q_nB$. This implies that $B'$ is an $\varepsilon/(1-k)$-net of $Q_nB$ in $L(0, T; X_n)$. Then $GB'$ is an $\varepsilon/(1-k)$-net of $Q_nB$ in $L(0, T; X_n)$ and $GB'$ is compact in $X$.

By Lemma 1.1 we know that

$$\beta_c([Gh_n]_{m=1}^{\infty}) \leq d_l([Gh_n]_{m=1}^{\infty}, Q_nB)$$

for any $l, n \geq 1$.

For any $m, n \geq 1$ and $t \in [0, T]$, it follows that

$$\|Gh_m(t) - Gh_n(t)\| \leq \|Gh_m(0) - Gh_n(0)\| + \int_0^t \|h_m(s) - h_n(s)\| ds,$$

where $h_n \in Q_n h_m$. This implies that

$$|Gh_m - Gh_n| \leq \frac{1}{1-k} \int_0^T \|h_m(s) - h_n(s)\| ds$$

This means that

$$\beta_c(GB) \leq d_l([Gh_n]_{m=1}^{\infty}, Q_nB) \leq \sup_{m \geq l} \frac{1}{1-k} \int_0^T d(h_m(s), X_n) ds.$$

Let $l, n \rightarrow \infty$; due to Lebesgue’s convergence theorem and Fatou’s theorem, we know that

$$\beta_c(GB) \leq \frac{1}{1-k} \int_0^T \beta([h_m(s)]_{m=1}^{\infty}) ds \leq \frac{1}{1-k} \int_0^T \beta(B(s)) ds.$$

We complete the proof. □

Proof of Theorem 3.1. Define $F : E \rightarrow E$ by $Fv$ being the integral solution of (3.1) and (3.2) with $h(t) = f(t, v(t))$ for any $v \in E$. Then $F$ is continuous as $G : L(0, T; X) \rightarrow E$ is continuous and the Nemyckii operator defined by $u \mapsto f(\cdot, u(\cdot))$ from $E$ to $L(0, T; X)$ is continuous under the hypotheses $(H_f)(1), (2)$.

To prove the existence of an integral solution we need only show that there exists at least one fixed point of $F$.

Let $(x, y) \in A$ be fixed. For $v \in E$ let $u = Fv$, then

$$\|u(t)\| \leq k|u(t)| + (k+1)\|x\| + \|s(t)g(\hat{x})\| + \|v\|= 1 + \|b\|_1$$

for all $t \in [0, T]$, where $\hat{x}$ defined in the proof of Lemma 3.3. So there exists a positive number $r$ such that we have the continuous map $F : B_r \rightarrow B_r$ as $k + \|a\|_1 < 1$. 

\[\text{X. Xue / Nonlinear Analysis 70 (2009) 2593–2601} 2599\]
For any \( W \subseteq B_i \), by Lemma 3.5, it is implied that
\[
\beta_i(FW) \leq \frac{1}{1-k} \int_0^T \beta(f(s, W(s)))ds
\]
\[
\leq \frac{1}{1-k} \int_0^T \alpha(s) \beta(W(s))ds
\]
\[
\leq \frac{1}{1-k} \|\alpha\|_1 \beta_i(W).
\]
As \( k + \|\alpha\|_1 < 1 \), then the continuous map \( F : B_i \rightarrow B_i \) is a \( \beta_i \)-contraction. By Lemma 1.2 we complete the proof. \( \square \)

From above we can get the following corollaries:

**Corollary 3.6.** If \( X \) is a separable Hilbert space, the hypotheses \((H_1')\) and \((H_1)\) are true, and \( A = -\partial \phi \) with \( \phi \) is proper, convex and lower semicontinuous from \( X \) into \((-\infty, \infty]\), then the nonlocal IVP (1.1) and (1.2) has at least one integral solution when \( k + \|a\|_1 < 1 \) and \( k + \|\alpha\|_1 < 1 \).

**Corollary 3.7.** If \( X \) is separable and uniformly smooth, the hypotheses \((H_1)\), \((H_1')\) and \((H_1)(1)2\) are true, and \( f(t, \cdot) : D 
\rightarrow X \) is compact for a.e. \( t \in [0, T] \). Then the nonlocal IVP (1.1) and (1.2) has at least one integral solution when \( k + \|a\|_1 < 1 \).

### 4. An example

As an application of our results we consider the following nonlinear parabolic systems of the form

\[
\frac{\partial u_i(t, x)}{\partial t} - c_i \Delta u_i(t, x) = f_i(t, u_1(t, x), u_2(t, x)) \quad \text{in } (0, T) \times \Omega
\]

\[
-\frac{\partial u_i(t, x)}{\partial \nu} \in \partial j_i(u_i(t, x)) \quad \text{on } (0, T) \times \Gamma
\]

\[
u_i(0, x) = g_i(u_1, u_2) \quad \text{in } \Omega,
\]

where \( i = 1, 2, \Omega \) is a bounded domain \( \mathbb{R}^p, p \geq 1 \), with smooth boundary \( \Gamma, c_i \) are positive constants, \( f_i : \mathbb{R}^3 \rightarrow \mathbb{R} \) are given mappings and \( j_i : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\} \) are proper, convex and lower semicontinuous functions with \( j_i(0) = 0, i = 1, 2 \). Here

\[
g_i(u, v)(x) = \int_0^T \int_0^T h_i(t, x, z, u(z), v(z))dtdz,
\]

where \( i = 1, 2 \) and

\[
f_2(t, u, v)(x) = \int_\Omega h_3(t, x, z, u(z), v(z))dz.
\]

Let \( X = L^2(\Omega) \times L^2(\Omega) \) be endowed with the inner product \( \langle \cdot, \cdot \rangle \) defined by

\[
\langle (u, v), (\tilde{u}, \tilde{v}) \rangle = \langle u, \tilde{u} \rangle_{L^2(\Omega)} + \langle v, \tilde{v} \rangle_{L^2(\Omega)}
\]

for each \( (u, v), (\tilde{u}, \tilde{v}) \in X \). Obviously \( X \) is a separable real Hilbert space.

We define \( A : D(A) \subset X \rightarrow X \) by \( A(u, v) = (-c_1 \Delta u, -c_2 \Delta v) \) for each \( (u, v) \in D(A) \), where

\[
D(A) = \left\{ (u, v) \in X, -\frac{\partial u}{\partial \nu} \in \partial j_1(u), -\frac{\partial v}{\partial \nu} \in \partial j_2(v), \text{ a.e. on } \Gamma \right\}.
\]

It is well known that \( A \) is the subdifferential of a proper, convex and lower semicontinuous functions \( \psi : X \rightarrow \mathbb{R} \cup \{+\infty\} \) (see [26]). In addition, if \( j_1 \) and \( j_2 \) are positive, then \( A \) generates a compact semigroup \( S(t) \). Otherwise, \( S(t) \) may be only equicontinuous.

Now we define \( f : [0, T] \times X \rightarrow X \) and \( g : C([0, T]; X) \rightarrow X \) by

\[
f(t, (u, v)) = (f_1(t, (u, v)), f_2(t, (u, v))), \quad (u, v) \in X,
\]

\[
g(u, v) = (g_1(u, v), g_2(u, v)), \quad (u, v) \in X,
\]

where \( f_i \) and \( g_i \) are superposition mappings associated with \( f_i \) and \( g_i \) defined by

\[
f_i(t, (u, v)) = \{ h \in L^2(\Omega); h(x) = f_i(t, u(x), v(x)), \text{ a.e. for } x \in \Omega \},
\]

and

\[
g_i(u, v) = \{ h \in L^2(\Omega); h(x) = g_i(u(x), v(x)), \text{ a.e. for } x \in \Omega \}.
\]
Let us observe that (4.1)–(4.3) may be rewritten as

\[ U'(t) \in AU(t) + f(t, U(t)), \quad t \in (0, T) \text{ a.e.} \tag{4.4} \]

\[ U(0) = g(U), \tag{4.5} \]

where \( U(t) = (u(t), v(t)) \), while \( A, f \) and \( g \) are as above.

We suppose that:

(1) There exists \( k(t) \in L(0, T) \) such that \( f_1 : [0, T] \times R \times R \to R \) is a Carathéodory type function and, for \( u, u', v, v' \in R \),

\[ |f_1(t, u, v) - f_1(t, u', v')| \leq k(t)(|u - u'|^2 + |v - v'|^2)^{\frac{1}{2}}. \]

(2) For \( i = 1, 2, 3 \), \( h_i : [0, T] \times \Omega \times \Omega \times R \times R \to R \) satisfies Carathéodory conditions. In addition:

(i) \( |h_i(t, x, x', z, r, s) - h_i(t', x', z', r, s)| \leq w_i^1(t, x, x', z) \) for \( (t, x, z) \), \( (t', x', z') \in [0, T] \times \Omega \times \Omega \) and \( |r|, |s| \leq k \), where \( w_i^1 \in L([0, T] \times \Omega)^3 \) are such that

\[ \lim_{x \to \Omega} \int_0^T w_i^1(t, x, x', z) \, dt \, dz = 0 \]

uniformly for \( x \in \Omega \) (i = 1, 2), and for \( t \in [0, T] \)

\[ \lim_{x \to \Omega} \int_0^T w_i^2(t, x, x', z) \, dz = 0, \]

uniformly for \( x \in \Omega \);

(ii) \( |h_i(t, x, x, z, r, s)| \leq c_i(t)(|r|^2 + |s|^2)^{\frac{1}{2}} + w_i(t, x, z) \), where \( c_i \in L(0, T) \) and \( d_i = \int_0^T \int_{\Omega \times \Omega} (w_i(t, x, z))^2 \, dx \, dz \, dt < +\infty \) for \( i = 1, 2, 3 \).

Adapting the arguments given in [21] it is not difficult to show that \( g \) satisfies the hypothesis (H2) with \( c^2 = 2m(\Omega)^{\\frac{1}{2}} + ||c_2||^2 \) and \( d^2 = 2m(\Omega)^2(d_1 + d_2) \), and \( f \) satisfies the hypothesis (H3) with \( a(t) = k(t) \), \( a^2(t) = k^2(t) + 2m(\Omega)c_2^2(t) \) and \( ||b||^2 = 2m(\Omega)d_3 \), where \( m(\Omega) \) means the Lebesgue measure of \( \Omega \) in \( R^3 \). By Corollary 2.3, we conclude that (4.1)–(4.3) has at least one generalized solution \( (u_1, u_2) \in C([0, T]; L^2(\Omega) \times L^2(\Omega)) \) provided that

\[ (2m(\Omega)^{\\frac{1}{2}} + ||c_2||^2 + ||c_1||^2) + \int_0^T (k^2(t) + 2m(\Omega)c_2^2(t))^{\frac{1}{2}} \, dt < 1. \]

References