Cluster synchronization in complex networks of nonidentical dynamical systems via pinning control

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1. Introduction

In the past years, the complex networks have become a hot topic because of their extensive existence in nature and society, such as immune systems, social networks, World Wide Web and electrical power grids. Synchronization, as an interesting phenomenon of complex networks, has attracted many researchers to investigate because of its potential applications in many fields, such as neural networks [1], image processing [2], and secure communication [3]. Researchers have investigated a lot of different synchronization protocols, for example, complete synchronization, cluster synchronization, phase synchronization, lag synchronization, generalized synchronization, and so on.

If all the nodes of the network finally converge to the same trajectory, the complete synchronization is called to be reached, see [4–13]. If the nodes’ dynamical behavior is ignored further, the synchronization problem turns into the consensus problem. So far, many graph theories have been introduced to investigate the complete synchronization phenomenon. For example, the master stability function (MSF) method was set up to investigate the stability of the synchronization state in [4]; while the left eigenvector corresponding to the zero eigenvalue of the coupling matrix was used by the authors in [5–8] to investigate synchronization problem; [10,11] studied the consensus problem with or without time delay. Despite these protocols’ effectiveness in making the networks achieve synchronization, complex networks cannot synchronize to any given trajectory without external control. In practice, it is difficult to add the controllers on every node of the networks because of the network’s scale is usually large. Therefore, to solve the problem, the pinning control strategy is proposed, which means the controllers are just added on partial nodes, see [14–20]. For example, [14] proved rigorously that a complex network can be pinned to synchronize by adding a single controller on just one node.

On the other hand, the cluster synchronization is a more practical phenomenon than the complete synchronization, which is significant in communication engineering [21], biological sciences [22], and so on. The cluster synchronization is characterized by that the nodes in the same cluster achieve a uniform state while the nodes in different clusters have different states. That is to say, the complete synchronization occurs in every cluster, while there is no synchronization between different clusters. Specially, if there is only one cluster in the network, the cluster synchronization problem becomes the complete synchronization.

Investigations have shown that the cluster synchronization depends heavily on the choice of coupling schemes. Generally speaking, there are two main schemes to investigate the cluster synchronization. For the first scheme, nodes in the same cluster only have cooperative connections, while nodes in different clusters can have cooperative or competitive connections, see [23–31]. For example, Ma, Liu and Zhang were the first to propose such a new coupling scheme to realize cluster synchronization in [23]; the

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authors in [24] generalized the scheme to asymmetric matrices to realize the cluster synchronization; [25] investigated the cluster synchronization under pinning control with symmetric coupling matrix; [26] generalized the coupling matrix to be asymmetric and the pinning control can be periodically intermittent; [27] studied the cluster synchronization under nonlinear coupling and periodically intermittent pinning control. All these works were based on the construction of the coupling matrix, which played a key role in the analysis of cluster synchronization problem.

The other scheme for the cluster synchronization is to assume that all the coupling nodes only have cooperative connections, i.e., the coupling matrix is a Metzler matrix with zero row sums, where a Metzler matrix is a matrix in which all the off-diagonal components are nonnegative. For example, in [32,33], for a given nearest-neighborhood network with zero-flux or periodic boundary conditions, the authors proposed an effective method to determine some possible states of cluster synchronization and their stability; in [34], for linearly and symmetrically coupled networks, the authors discussed the connection between the cluster synchronization and the complete synchronization by analyzing the invariant synchronization manifold and proposed several criteria for the global attractivity of the invariant synchronization manifold. Recently, [35] studied the cluster synchronization in networks of coupled nonidentical systems and indicated that the common intercluster coupling condition and the intracluster communication played key roles for cluster synchronization. An interesting phenomenon is reported, the network can realize cluster synchronization even if there is no connections between nodes of the same cluster. Refs. [36,37] studied the cluster synchronization and cluster consensus for discrete-time coupled systems respectively, and [38] investigated the cluster consensus problem for continuous-time coupled systems. Ref. [39] showed how different mechanisms may lead to clustering behavior in connected networks consisting of diffusively coupled agents by considering self-dynamics, delays, both positive and negative couplings.

Until now, most existing works only discuss the cluster synchronization by mutual coupling, however, in some cases, the final synchronization trajectories should also be considered. That is to say, if we want the nodes finally converge to some targets, the external control should be added. Moreover, for constant coupling strength, the cluster synchronization may not be realized, while by adding external control, the cluster synchronization can be achieved, see [40–42]. However, all these works realize the synchronization by adding the controllers on all clusters, that is to say, any cluster should be finally synchronized to these given targets.

To our best knowledge, there are few works discussing the cluster synchronization by adding the external controllers on just partial clusters. Motivated by the above discussions, we will investigate the cluster synchronization of complex networks via pinning control. There are three contributions in this paper: (1) a new network model for cluster synchronization is set up, only partial clusters are controlled while others are not; (2) criteria are obtained by using pinning control and synchronization techniques; (3) a new adaptive scheme for coupling strength to realized cluster synchronization is proposed and rigorously proved.

The rest of this paper is organized as follows. In Section 2, the network model is described and some necessary definitions, lemmas, assumptions and notations are given. In Section 3, we investigate the cluster synchronization problem with pinning control and some sufficient criteria are proposed to guarantee the cluster synchronization. The adaptive technique is also applied on the coupling strength and its effectiveness is rigorously proved in Section 4. In Section 5, some numerical simulations are presented to show the validity of theoretical results. Finally, this paper is concluded in Section 6.

2. Preliminaries

The graph $\mathcal{G}$ can be denoted by a double set $\{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V}$ represents the vertex set numbered by $\{1, \ldots, N\}$ and $\mathcal{E}$ denotes the edge set with $e(i, j) \in \mathcal{E}$ if and only if there is an edge from vertex $j$ to $i$. Assume the set of nodes in the network can be divided into $m$ clusters, i.e., $\{1, \ldots, N\} = C_1 \cup C_2 \cup \cdots \cup C_m$ where

$$C_1 = \{1, \ldots, r_1\}, C_2 = \{r_1 + 1, \ldots, r_2\}, \ldots, C_m = \{r_{m-1} + 1, \ldots, N\}. \tag{1}$$

The network of coupled dynamical systems is defined on the graph $\mathcal{G}$. The individual uncoupled system on the vertex $i$ is denoted by an $n$-dimensional ordinary differential equation

$$\dot{x}_i(t) = f_i(x_i(t)), \quad \forall i \in C_k, \quad k = 1, \ldots, m \tag{2}$$

In [35], the cluster synchronization of the above model has been investigated, but the synchronization heavily depends on the value of the coupling strength $c$. So we will investigate the linearly coupled systems by adding some external controllers in order to realize the cluster synchronization with a small value of $c$.

Before our discussion, we first present some useful definitions, notations and lemmas, which will be used throughout the whole paper.

We make the following assumption for function $f_k(\cdot)$.

**Assumption 1** (QUAD condition, Lu and Chen [5,6], Liu and Chen [7,8]). The function $f_k(\cdot), k = 1, 2, \ldots, m$, is said to satisfy the QUAD condition, denoted as $f_k(\cdot) \in \text{QUAD}(\alpha, \eta)$. That is, if for some $\alpha > 0$, there exists $\eta > 0$ such that

$$(x - y)^T [f_k(x) - f_k(y) - \alpha (x - y)] \leq -\eta (x - y)^T (x - y) \tag{3}$$

holds for all $x, y \in \mathbb{R}^n$.

**Definition 1** (Lu and Chen [26]). Matrix $A = [a_{ij}]_{i,j=1}^N$ of order $N$ is said to belong to class $A_1$, denoted as $A \in A_1$, if

1. $a_{ij} \geq 0, i \neq j, a_{ii} = -\sum_{j=1}^N a_{ij}, i = 1, \ldots, N$,
2. $A$ is irreducible.

Furthermore, if $A \in A_1$ and $a_{ij} = a_{ji}, i \neq j$, then we say $A \in A_2$.

**Lemma 1** (Lu et al. [35]). If a matrix $A_{N \times N} \in A_2$, its eigenvalues are all real and can be sorted as

$$0 = \lambda_1(A) > \lambda_2(A) \geq \cdots \geq \lambda_N(A). \tag{4}$$

**Definition 2** (Wu and Chen [34]). Matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ is said to belong to class $A_3$, denoted as $A \in A_3$, if its each row-sum is equal, i.e., $\sum_{i=1}^N a_{ij} = \sum_{i=1}^N a_{ij}, j = 1, \ldots, N$.

Now, using the above definitions of matrices, we can define a new type of coupling matrix $A$ for the following cluster synchronization analysis.

**Definition 3** (Wu and Chen [34]). Suppose $A \in \mathbb{R}^{N \times N}$, the indexes $\{1, \ldots, N\}$ can be divided into $m$ clusters as defined in (1), and the
following form holds:

\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1m} \\
A_{21} & A_{22} & \cdots & A_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
A_{m1} & A_{m2} & \cdots & A_{mm}
\end{pmatrix}
\]  
(5)

where \( A_{ij} \in \mathbb{R}^{(r_i-\tau_i)\times(r_j-\tau_j)} \), \( A \in A1 \) and \( A_{ij} \in A3 \), \( i,j \in \{1, \ldots, m\} \). Then the matrix is said to belong to class A4, denoted as \( A \in A4 \).

**Remark 1.** In previous works [24–27], the coupling matrix \( A \) is assumed to satisfy: \( A_{ii} \in A1, A_{ij} \in A3, i \neq j \in \{1, \ldots, m\} \). Therefore, the coupling matrix in this paper is different with that in [24–27].

**Remark 2.** From the definition of \( A4 \), we have \( \sum_{j \in C} A_{ij} = \sum_{j \in C} a_{ij}, \) \( \forall i, j \in C_k, k \neq k' \), which leads \( \sum_{j \in C} A_{ij} = \sum_{j \in C} a_{ij}, \) \( \forall i, j \in C_k \). In order to facilitate the writing, for \( \forall i \in C_k \), we assume

\[
A_{k,k'} = \sum_{j \in C_k} a_{ij}.
\]  
(6)

**Definition 4.** In this paper, the set \( \mathcal{M} = \{ x = (x_1^T, \ldots, x_N^T) : x_i \in \mathbb{R}^n, \forall i, j \in C_k, k = 1, \ldots, m \} \) is called \( m \)-cluster synchronization manifold.

**Notation 1.** Throughout this paper, we denote the identity matrix by \( I \) with appropriate dimensions. \( A^T \) denotes the transpose of the matrix \( A \) and the symmetrical part of matrix \( A \) as \( A' = (A + A^T)/2 \). A symmetric real matrix \( A \) is positive definite (semidefinite) if \( x^T A x > 0 \) (\( \geq 0 \)) for all nonzero \( x \), denoted as \( A > 0 \) (\( A \geq 0 \)). \( \# Z \) denotes the number of the set \( Z \) with finite elements. The Kronecker product of an \( N \) by \( M \) matrix \( A = [a_{ij}] \) and a \( p \) by \( q \) matrix \( B \) is the \( Np \) by \( Mq \) matrix \( A \otimes B \), denoted as

\[
A \otimes B = \begin{pmatrix}
a_{11}B & \cdots & a_{1M}B \\
\vdots & \ddots & \vdots \\
a_{N1}B & \cdots & a_{NM}B
\end{pmatrix},
\]  
(7)

and the Kronecker product has the property

\[
(A \otimes B)(C \otimes D) = (AC) \otimes (BD).
\]  
(8)

### 3. Cluster synchronization with a static coupling strength

#### 3.1. Network model

In this paper, we study cluster synchronization with a static coupling strength. Unlike those cluster synchronization schemes existing in the literature, we just control partial clusters to synchronize to the target trajectories, while the synchronized trajectories of other clusters can be unknown. Therefore, the \( N \) linearly coupled systems of static control strength with clusters \( C_k \), \( k = 1, \ldots, m \) defined in (1), can be described as

\[
\dot{x}_i(t) = f_i(x_i(t)) + \sum_{j=1}^{N} a_{ij} f_j(x_j(t) - x_i(t)) + u_i(t), \quad i \in C_k, \; k = 1, \ldots, m,
\]  
(9)

where the matrix \( A = [a_{ij}] \in A4 \) has the form (5), and the inner matrix \( I' \) is diagonal and positive definite. The controllers can be added on one cluster or partial clusters, but in this paper, for simplicity we assume that only the first cluster is controlled. Thus, (9) becomes the following equation:

\[
\begin{align*}
\dot{x}_i(t) &= f_i(x_i(t)) + \sum_{j=1}^{N} a_{ij} f_j(x_j(t) - x_i(t)) + u_i(t), \quad i \in C_1; \\
\dot{x}_i(t) &= f_i(x_i(t)) + \sum_{j=1}^{N} a_{ij} f_j(x_j(t) - x_i(t)), \quad i \in C_k, \; k = 2, \ldots, m.
\end{align*}
\]  
(10)

Then the cluster synchronization can be defined as follows.

**Definition 5.** Network (10) with \( N \) nodes is said to realize cluster synchronization, if nodes can be divided into \( m \) clusters as defined by (1), and the following conditions are satisfied:

1. \( \lim_{t \to +\infty} \| x_i(t) - s(t) \| = 0, \) for \( i \in C_1 \).
2. \( \lim_{t \to +\infty} \| x_i(t) - x_j(t) \| = 0, \) for \( i, j \in C_k, k = \{2, \ldots, m\} \).
3. \( \lim_{t \to +\infty} \sup \| x_i(t) - s(t) \| > 0, \) and \( \lim_{t \to +\infty} \sup \| x_i(t) - x_j(t) \| > 0, \) \( i \in C_k, j \in C_k', k \neq k' \in \{2, \ldots, m\} \).

where \( \| \cdot \| \) is some norm, and \( s(t) = (s_1(t), \ldots, s_m(t))^T \in \mathbb{R}^m \) is a target trajectory for the first cluster defined by

\[
\dot{s}_i(t) = f_i(s(t)).
\]  
(11)

Let \( p = [p_1, \ldots, p_N]^T \) be a vector with \( p_i > 0 \) for all \( i = 1, \ldots, N \) and \( p = \text{diag}[p_1, \ldots, p_N] \). We use the vector to construct a projection of \( x = [x_1^T, \ldots, x_N^T]^T \) onto the cluster synchronization manifold \( \mathcal{M} = \{ x = [x_1^T, \ldots, x_N^T]^T \in \mathbb{R}^{Nm} : x_i(t) \in \mathbb{R}, \forall i \in C_k; x_j(t) \in \mathbb{R}^n, \forall i, j \in C_k, i \neq j \} \). From the theory of synchronization literature, one can define an average (weighted) state \( x_k(t) \) with respect to \( p \) in each cluster \( C_k, k = 2, \ldots, m \)

\[
x_k(t) = \frac{1}{\sum_{i \in C_k} p_i} \sum_{i \in C_k} p_i x_i(t),
\]  
(12)

then the synchronization definition in cluster \( C_k \) is equivalent to prove that

\[
\lim_{t \to +\infty} \| x_i(t) - x_k(t) \| = 0 \quad i \in C_k, \; k = 2, \ldots, m.
\]  
(13)

Denote the projection of \( x(t) \) on the cluster synchronization manifold \( \mathcal{M} \) with \( p \) as \( \bar{x}_k(t) = [\bar{x}_1(t), \ldots, \bar{x}_N(t)]^T \), where \( \bar{x}_i(t) = s_k(t), \) if \( i \in C_k \), and

\[
s_k(t) = \begin{cases} s(t) & \text{if } k = 1; \\ \bar{x}_k(t) & \text{if } k \neq 1. \end{cases}
\]  
(14)

Then, the variations \( x_i(t) - s_k(t) \) can compose the transverse space

\[
\mathcal{L} = \{ v = [v_1^T, \ldots, v_N^T]^T \in \mathbb{R}^{Nm} : v_i = x_i(t) - s_k(t) \in \mathbb{R}^n, \forall i \in C_1, i = 1, \ldots, N \}.
\]  
(15)

With those notations, the cluster synchronization is equivalent to the transverse stability of the cluster synchronization manifold \( \mathcal{M} \), i.e., the projection of \( x(t) \) on the transverse space \( \mathcal{L} \) converges to zero as time goes to infinity.

A key problem is to design the form of external control \( u_i(t), i \in C_1 \). Now, assume all the nodes are in the manifold \( \mathcal{M} \), in this case, the manifold

\[
\mathcal{M} = \{ s_1(t), \ldots, s_k(t), \ldots, s_2(t), \ldots, s_m(t), \ldots, s_m(t) \}
\]  

should be invariant, where \( s_k(t) \) is defined by (14). For any node \( i \in C_k, k = 2, \ldots, m \), its equation is governed by

\[
\dot{s}_k(t) = f_k(s_k(t)) + \sum_{j \neq k} \alpha_{k,j} f_j(s_k(t) - s_j(t));
\]  
(16)
while for any node $i \in C_1$, its equation is governed by
\begin{equation}
    s(t) = f_1(s(t)) + c \sum_{k=1}^{m} A_{ik} f_1(x_k(t)) + u_i(t) = f_1(s(t));
\end{equation}

therefore, from (14), (16) and (17), we can define the control function $u_i(t)$ as
\begin{equation}
    u_i(t) = \begin{cases} 
    c \Gamma s(t) - x_i(t) - c \alpha_1, \Gamma s(t) - c \sum_{k=2}^{m} A_{ik} \Gamma x_k(t) & \text{if } i \in C_1, \\
    0 & \text{if } i \notin C_1,
    \end{cases}
\end{equation}

where $c$ is the control strength, and $\epsilon > 0$.

**Remark 3.** From the definition of $u_i$ defined in (18), one can easily find that any node in the first cluster $C_1$ should be added with the external control. In fact, we can also give other forms of the pinning control.

Moreover, if $m = 1$, the cluster synchronization becomes the complete synchronization problem, in this case, $x_i(t) = s(t)$, and with the definition of (19), $u_i$ can be defined by
\begin{equation}
    u_i(t) = \begin{cases} 
    c_0 \Gamma s(t) - x_i(t) & \text{if } c_0 > 0, \\
    0 & \text{otherwise},
    \end{cases}
\end{equation}

this is also the usual definition of pinning control in previous works, like [14]. Therefore, the pinning control technique (18) is a natural generalization of previous pinning control for cluster synchronization.

Denote $e_i(t) = x_i(t) - s_i(t), i \in C_k, E(t) = (e_1(t), \ldots, e_N(t))^\top$, and
\begin{equation}
    \hat{A} = \begin{pmatrix}
    \hat{A}_{11} & \cdots & \hat{A}_{1m} \\
    \vdots & \ddots & \vdots \\
    \hat{A}_{m1} & \cdots & \hat{A}_{mm}
    \end{pmatrix}
\end{equation}
where $\hat{A}_{ij}$ are defined as
\begin{equation}
    \hat{A}_{ij} = \begin{cases} 
    A_{i1} - c, & \text{if } i = j = 1, \\
    A_{ij} & \text{otherwise},
    \end{cases}
\end{equation}

Then, we have
\begin{equation}
    \begin{cases}
    \dot{e}_i(t) = f_1(x_i(t)) - f_1(s_i(t)) + c \sum_{j=1}^{N} a_{ij} \Gamma x_j(t) \\
    - c \Gamma e_i(x_i(t) - s_i(t)) - c \alpha_1, \Gamma s_i(t) - c \sum_{k=2}^{m} A_{ik} \Gamma x_k(t) & \text{if } i \in C_1, \\
    = f_1(x_i(t)) - f_1(s_i(t)) + c \sum_{j=1}^{N} a_{ij} \Gamma e_j(t) & \text{if } i \notin C_1,
    \end{cases}
\end{equation}

Recall the definition (12), we have the following useful lemma.
For the case of adaptive coupling strength $c(t)$, with the same notations as those in Theorem 1, for $1 \leq i \leq N$, we have

$$
\dot{e}_i(t) = \left\{ \begin{array}{ll}
\dot{e}_i(t) = f_i(x_i(t)) - f_i(s(t)) + c(t) \left( \sum_{j=1}^{N} a_{ij} \Gamma e_j(t) - c(t) \Gamma \dot{c}(x_i(t) - s(t)) \right) \\
- c(t) \alpha_1 \Gamma \dot{s}(t) - c(t) \left( \sum_{k=2}^{m} \alpha_{1k} \Gamma \dot{x}_k(t) \right) & \text{if } i \in C_1 \\
\dot{e}_i(t) = f_i(x_i(t)) - f_i(s(t)) + c(t) \left( \sum_{j=1}^{N} a_{ij} \Gamma e_j(t) + c(t) \left( \sum_{k=1}^{r_i} \alpha_{1k} \Gamma \dot{x}_k(t) \right) \right) \\
+ c(t) \left( \sum_{j=r_i+1}^{N} a_{ij} \Gamma \dot{x}_j(t) + f_i(\dot{x}_i(t)) - \dot{x}_i(t) \right) & \text{if } i \in C_k, k \neq 1.
\end{array} \right.
$$

Theorem 2. If function $f_k(\cdot)$ satisfies the QUAD condition, then coupled network (29) can realize cluster synchronization with the following adaptive rule:

$$
\dot{c}(t) = - \frac{\rho}{2} \left( \sum_{i=1}^{N} e_i(t)^\top \right) \Gamma \left( \sum_{j=1}^{N} b_{ij} \Gamma e_j(t) + \sum_{i=1}^{r_i} q_i e_i(t)^\top e_i(t) \right),
$$

where $\rho > 0, c(0) = 0, B = \left[ b_{ij} \right]_{i,j=1}^{N}$ can be any matrix belonging to $A2$, $q_i > 0, i = 1, \ldots, r_i$, and matrix $\Gamma^*$ can be any diagonally positive definite matrix.

The proof can be found in Appendix B.

Remark 5. Because of the arbitrariness of matrix $B$ and $Q = \text{diag} \{ q_1, \ldots, q_N \}$, the matrix is easy to be found out and the adaptive rule can apply a variety of situations, even the matrix $A$ is unknown. A simple example is that $B$ has the full coupling mode as the following:

$$
B = 1 \cdot 1^\top - N \cdot I_N,
$$

where $1 = [1, \ldots, 1_N]^\top$. While a simple example for matrix $Q$ can be

$$
Q = \text{diag} \{ q_1, \ldots, q_N \}.
$$

We will also use this form of matrix $B$ and $Q$ in later numerical simulations.

If there is only one cluster in the system, then

$$
x_i(t) = f_i(x_i(t)) + c(t) \left( \sum_{j=1}^{N} a_{ij} \Gamma x_j(t) + u_i(t) \right), \quad i = 1, \ldots, N,
$$

where $u_i, i = 1, \ldots, N$, are defined by (18). By Theorem 2, it is easy to obtain the following result concerning the complete synchronization.

Corollary 3. If function $f(\cdot)$ satisfies the QUAD condition, then coupled system (33) with the adaptive rule

$$
\dot{c}(t) = - \frac{\rho}{2} \left( \sum_{i=1}^{N} e_i(t)^\top \right) \Gamma \left( \sum_{j=1}^{N} b_{ij} \Gamma e_j(t) + \sum_{i=1}^{r_i} q_i e_i(t)^\top e_i(t) \right),
$$

or

$$
\dot{c}(t) = - \frac{\rho}{2} \sum_{i=1}^{N} q_i e_i(t)^\top e_i(t),
$$

can realize complete synchronization, where $e_i(t) = x_i(t) - s(t)$, $s(t)$ and $B = \left[ b_{ij} \right]_{i,j=1}^{N}$ are defined in Theorem 2, $q_i > 0, i = 1, \ldots, N$, and matrix $\Gamma^*$ can be any diagonally positive definite matrix.

5. Numerical examples

In this section, we give some numerical examples to demonstrate the effectiveness of the obtained theoretical results for cluster synchronization.

5.1. Cluster synchronization in a network with 5 nodes

Consider a linearly coupled network with 5 nodes, and suppose the network can be divided into two clusters: $C_1 = \{1, 2\}$ and $C_2 = \{3, 4, 5\}$. Then the network can be described as follows:

$$
\begin{align*}
\dot{x}_1(t) &= f_1(x_1(t)) + c \sum_{j=1}^{5} a_{1j} x_j(t), \quad i = 1, 2, \\
\dot{x}_3(t) &= f_3(x_3(t)) + c \sum_{j=1}^{5} a_{3j} x_j(t), \quad i = 3, 4, 5
\end{align*}
$$

where $f_i(\cdot)$ are nonidentical Chua’s circuit

$$
f_1(x(t)) = L_1 x(t) + \text{diag}(h(x^3(t)), 0, 0);
$$

where

$$
L_k = \begin{pmatrix}
0 & 0 & 0 \\
1 & -1 & 1 \\
0 & w_k & 0
\end{pmatrix},
$$

and

$$
h(v) = \begin{cases}
0.16 v + (m_0 - m_1) & \text{if } v \geq 1 \\
0.04 - 0.18 & 0.07 \\
0.01 v - (m_1 - m_0) & \text{if } v \leq -1
\end{cases}
$$

For all $k=1,2$, we take $m_0 = -1/7$ and $m_1 = 2/7$. The parameter pairs $(o_x, w_k), k = 1, 2$, distinguish the clusters and are picked as $(9, 100/7), (10, -110/7)$ respectively. The coupling matrix $A$ has the following form:

$$
A = \begin{pmatrix}
-0.16 & 0.02 & 0.03 & 0.05 & 0.06 \\
0.04 & -0.18 & 0.07 & 0 & 0.07 \\
0.1 & 0 & 2.5 & -2.6 & 0 \\
0.06 & 0.04 & 1 & 0 & -1.1
\end{pmatrix}
$$

Then we will calculate the value of $\alpha$ for the QUAD condition. Simple calculations lead to the following fact:

$$
(x - y)^\top [f_1(x) - f_2(y)] \leq \max(\max_{i=1}^{5} (L_k x(t)) - y)^\top (x - y),
$$

where $L_k = (L_k + L_k^\top)/2 + \text{diag}(\max(m_0, m_1), 0, 0)$, so we can have $\alpha = 8.7546$.

In this example, the initial values are taken as

$$
x_1(0) = (-0.0405, -0.1472, 0.0719)^\top, \\
x_2(0) = (0.0163, -0.0377, -0.0785)^\top, \\
x_3(0) = (-0.0856, -0.0051, -0.0121)^\top, \\
x_4(0) = (0.0160, 0.0156, 0.0432)^\top, \\
x_5(0) = (-0.0015, -0.0082, -0.0314)^\top.
$$

Moreover, we assume the inner coupling matrix $\Gamma = I$ and coupling strength $c=7.7$. We will take the fourth-order Runge-Kutta scheme to solve all the ordinary differential equations in numerical simulations with step-size 0.01.

At first, we consider the network (36) without external control. The errors in clusters are defined as follows:

$$
\text{ER}_1(t) = \sqrt{\sum_{i=1}^{5} \| \dot{x}_i(t) - \dot{x}_i(t) \|^2 / 2},
$$

$$
\text{ER}_2(t) = \sqrt{\sum_{i=1}^{5} \| \dot{x}_i(t) - \dot{x}_i(t) \|^2 / 2}.
$$
where

where ER1(t) = \sum_{i=1}^{5} \|x_i(t) - \mathbf{x}_1(t)\|^2 / 3,

where In Fig. 1 show that the cluster synchronization cannot be realized.

where \( \mathbf{A} = [a_{ij}] \) is the coupling matrix and satisfy the condition (26), \( \epsilon = 1, s(t) = f_1(s(t)) \), where \( s(0) = (0.0547, 0.0555, -0.0432)^T \), and \( a_{kk'} \) is defined in (6) and \( \mathbf{x}_k \) is defined in (12).

Choose the matrix \( P = I \), obviously \( \|P(c\mathbf{A} + c\mathbf{I})\|_F \leq 0 \). Fig. 2 shows that cluster synchronization can be realized and different clusters have different synchronized trajectories.

5.2. Cluster synchronization with an adaptive coupling strength

Simulation 1: Cluster synchronization in a simple network. The cluster synchronization can be realized if the condition (26) is satisfied. But in practice, it is difficult to find the qualified parameters. If the coupling strength \( c = 0.01 \) in the previous simulation, the trajectories of each node can be seen in Fig. 3, which shows that the cluster synchronization cannot be realized.

Instead, by adapting the coupling strength

with the following adaptive rules:

where

Notations \( E_1(t) \) and \( E_2(t) \) are used to denote the synchronization errors in clusters \( c_1 \) and \( c_2 \) as follows:

Simulations show that cluster synchronization can be realized, and \( c(t) \rightarrow 0.8871, t \rightarrow +\infty \), see Fig. 4.

Simulation 2: Cluster synchronization in a small-world network. We consider cluster synchronization in a small-world network.
with 200 nodes, and suppose its nodes can be divided into four clusters:

\[ C_1 = \{1, \ldots, 50\}, \quad C_2 = \{51, \ldots, 100\}, \quad C_3 = \{101, \ldots, 150\}, \quad C_4 = \{151, \ldots, 200\} \]

(44)

The original dynamic function of each node is defined by (37) and the adaptive rule we adopt is as follows:

\[
\dot{c}(t) = -0.05 \left( \sum_{i=1}^{200} e_i(t) - \sum_{i=1}^{200} b_i e_i(t) + \sum_{i=1}^{50} e_i(t)^T e_i(t) \right).
\]

(45)

The corresponding definitions of error systems \( E_1(t), E_2(t), E_3(t), E_4(t) \) are given by follows:

\[
E_1(t) = \sum_{i=1}^{50} \| x_i(t) - s(t) \|^2 / 50, \quad E_2(t) = \sum_{i=51}^{100} \| x_i(t) - \overline{x}_2(t) \|^2 / 50,
\]

\[
E_3(t) = \sum_{i=101}^{150} \| x_i(t) - \overline{x}_3(t) \|^2 / 50,
\]

\[
E_4(t) = \sum_{i=151}^{200} \| x_i(t) - \overline{x}_4(t) \|^2 / 50
\]

(46)

The simulation result is depicted in Fig. 5, which shows that cluster synchronization is finally realized.

**Simulation 3:** Cluster synchronization in a scale-free network. Now, we consider the cluster synchronization in a scale-free network with 200 nodes and each cluster has 50 nodes, see the partition in (44). The definitions of adaptive rule and error systems are the same with (45) and (46), respectively. Then, the simulation result is depicted in Fig. 6, therefore, the cluster synchronization can also be realized.

6. Conclusion

In this paper, cluster synchronization under pinning control is investigated. It is assumed that the dynamics on each node is nonidentical, and all nodes are linearly coupled. The coupling scheme assumes that connections between nodes are cooperative only. Firstly, the criteria for cluster synchronization are derived via linear matrix inequality. Secondly, we point out that to realize cluster synchronization, enlarging the couplings of nodes is the key point. Then, we propose an adaptive coupling strength approach to realize cluster synchronization, whose validity is rigorously proved. Simulations are also given to demonstrate the effectiveness of the proposed scheme.

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Appendix A. Proof of Theorem 1

**Proof.** We define a Lyapunov function to measure the distance from $x(t)$ to the cluster synchronization manifold as follows:

$$V(t) = \frac{1}{2} \sum_{i=1}^{N} p_i e_i(t)^{\top} e_i(t).$$

Differentiating $V(t)$, we have

$$\dot{V}(t) = \sum_{i=1}^{N} p_i e_i(t)^{\top} \dot{e}_i(t)$$

$$= \sum_{i=1}^{N} p_i e_i(t)^{\top} (f_i(x_i(t)) - f_i(x(t)) + c \sum_{j=1}^{N} \delta_{ij} G_j e_j(t))$$

$$+ \sum_{i=1}^{N} p_i e_i(t)^{\top} \left( f_i(x_i(t)) - f_i(x(t)) + c \sum_{j=1}^{N} a_{ij} G_j e_j(t) ight)$$

$$+ c \sum_{i=1}^{N} a_{ii} G_i t(t) + \sum_{i=1}^{N} p_i e_i(t)^{\top} (f_i(x_i(t)) - f_i(x_i(t)))$$

By Lemma 2, for $k = 2, \ldots, m$, we have

$$\prod_{k=1}^{m} p_i(x_i(t) - x_i(t))^\top x_i(t) = 0,$$

$$\prod_{k=1}^{m} p_i(x_i(t) - x_i(t))^\top x_i(t) = 0,$$

$$\prod_{k=1}^{m} p_i(x_i(t) - x_i(t))^\top x_i(t) = 0.$$

Thus,

$$\dot{V}(t) = -\sum_{i=1}^{N} p_i e_i(t)^{\top} \left( f_i(x_i(t)) - f_i(x_i(t)) + c \sum_{j=1}^{N} \delta_{ij} G_j e_j(t) ight)$$

$$+ \sum_{i=1}^{N} p_i e_i(t)^{\top} \left( f_i(x_i(t)) - f_i(x_i(t)) + c \sum_{j=1}^{N} a_{ij} G_j e_j(t) ight)$$

$$- c \sum_{i=1}^{N} a_{ii} G_i t(t) + c \sum_{i=1}^{N} p_i e_i(t)^{\top} (f_i(x_i(t)) - f_i(x_i(t)))$$

$$= -\sum_{i=1}^{N} p_i e_i(t)^{\top} \left( f_i(x_i(t)) - f_i(x_i(t)) + c \sum_{j=1}^{N} \delta_{ij} G_j e_j(t) ight)$$

$$+ c \sum_{i=1}^{N} p_i e_i(t)^{\top} (f_i(x_i(t)) - f_i(x_i(t)))$$

$$\leq -\eta \sum_{i=1}^{N} p_i e_i(t)^{\top} e_i(t) + \sum_{i=1}^{N} p_i e_i(t)^{\top} (c \sum_{j=1}^{N} \delta_{ij} G_j e_j(t) + c \sum_{i=1}^{N} a_{ii} G_i t(t) + c \sum_{i=1}^{N} p_i e_i(t)^{\top} (f_i(x_i(t)) - f_i(x_i(t))))$$

$$= -\eta \sum_{i=1}^{N} p_i e_i(t)^{\top} e_i(t) - \sum_{i=1}^{N} p_i e_i(t)^{\top} (c \sum_{j=1}^{N} \delta_{ij} G_j e_j(t) + c \sum_{i=1}^{N} a_{ii} G_i t(t) + c \sum_{i=1}^{N} p_i e_i(t)^{\top} (f_i(x_i(t)) - f_i(x_i(t))))$$

It is clear that $|\prod_{i=1}^{N} p_i(x_i(t) - x_i(t))| \leq 0$ implies $|\prod_{i=1}^{N} p_i(x_i(t) - x_i(t))| \otimes G \leq 0$. Therefore,

$$E(t)^{\top} [\prod_{i=1}^{N} p_i(x_i(t) - x_i(t))] \otimes G E(t) \leq 0.$$

Hence, we have

$$V(t) \leq -\eta E(t)^{\top} (P \otimes I_M) E(t) = -2\eta V(t).$$

This implies that $V(t) \leq \exp(-2\eta t) V(0)$. Therefore, $\lim_{t \to \infty} V(t) = 0$, namely, $\lim_{t \to \infty} e_i(t) = 0$ holds. According to the assumption that $f_i(x_i(t))$ are different, so final synchronization trajectories in clusters are also different. The proof is completed.\[\square\]

Appendix B. Proof of Theorem 2

**Proof.** By the QUAD condition, we have

$$\sum_{i=1}^{N} p_i e_i(t)^{\top} (f_i(x_i(t)) - f_i(x_i(t)) - a t e_i(t)) \leq -\eta \sum_{i=1}^{N} p_i e_i(t)^{\top} e_i(t).$$

(47)

Denote $\Gamma = diag[y_1, \ldots, y_n]$ and $\Gamma = diag[y_1^*, \ldots, y_n^*]$, then

$$\chi_i \Gamma \leq \chi_i \Gamma \leq \chi_i \Gamma,$$

where positive scalars $\chi_i = \min y_i / \max y_i$ and $\chi_i = \max y_i / \min y_i$.

For $j = 1, \ldots, n$, then by Lemma 1, we can pick a sufficiently small constant $\zeta > 0$ such that

$$\tilde{y}_j (P \tilde{A}^\top - \zeta \tilde{X} \tilde{B}) \leq 0,$$

(48)

where $\tilde{B} = -\tilde{A} - diag(q_1, \ldots, q_n, 0, \ldots, 0)$. For this chosen $\zeta$, pick a sufficiently large constant $c^* > 0$ such that

$$\zeta \tilde{c} \bigl(\chi_i \tilde{X} \tilde{B} + a \tilde{P}\bigr) \leq 0.$$

(49)

With these parameters, define a candidate function

$$V(t) = \frac{1}{2} E(t)^{\top} (P \otimes I_M) E(t) + \frac{\zeta}{\rho} (c^* - c(t))^2,$$

$$= \frac{1}{2} \sum_{i=1}^{N} \sum_{i=1}^{N} p_i e_i(t)^{\top} e_i(t) + \frac{\zeta}{\rho} (c^* - c(t))^2,$$

where $\zeta$ and $c^*$ are defined such that inequalities (48) and (49) hold.

Differentiating it and combining with the inequality (47), we have

$$\dot{V}(t) = \sum_{i=1}^{N} \sum_{i=1}^{N} p_i e_i(t)^{\top} \left( f_i(x_i(t)) - f_i(x_i(t)) + c \sum_{j=1}^{N} \delta_{ij} G_j e_j(t) ight)$$

$$+ \sum_{i=1}^{N} \sum_{i=1}^{N} p_i e_i(t)^{\top} \left( f_i(x_i(t)) - f_i(x_i(t)) + c \sum_{j=1}^{N} a_{ij} G_j e_j(t) ight)$$

$$+ \zeta (c^* - c(t)) \sum_{i=1}^{N} \sum_{i=1}^{N} p_i e_i(t)^{\top} e_i(t)$$

$$= -\eta E(t)^{\top} (P \otimes I_M) E(t) + c(t) \sum_{i=1}^{N} \sum_{i=1}^{N} p_i e_i(t)^{\top} e_i(t)$$

$$+ \zeta (c^* - c(t)) \sum_{i=1}^{N} \sum_{i=1}^{N} p_i e_i(t)^{\top} e_i(t)$$

$$\leq -\eta E(t)^{\top} (P \otimes I_M) E(t) + \zeta (c^* - c(t)) \sum_{i=1}^{N} \sum_{i=1}^{N} p_i e_i(t)^{\top} e_i(t)$$

$$\leq -\eta E(t)^{\top} (P \otimes I_M) E(t) \leq 0.$$

By the LaSalle–Yoshizawa theorem [43], we have $\lim_{t \to \infty} e_i(t) = 0$, $i = 1, \ldots, N$, which means that the cluster synchronization is finally realized. The proof is completed.\[\square\]

References


