Improved robust stability conditions for uncertain neutral systems with discrete and distributed delays

Yonggang Chena,b,*, Wei Qianb, Shumin Feic

aSchool of Mathematical Sciences, Henan Institute of Science and Technology, Xinxiang 453003, PR China
bKey Laboratory of Control Engineering of Henan Province, Henan Polytechnic University, Jiaozuo 454000, PR China
cKey Laboratory of Measurement and Control of CSE (Ministry of Education), School of Automation, Southeast University, Nanjing 210096, PR China

Received 20 August 2014; received in revised form 15 February 2015; accepted 31 March 2015
Available online 9 April 2015

Abstract

This paper considers the robust stability problem for uncertain neutral systems with discrete and distributed delays. Using new augmented Lyapunov–Krasovskii (L–K) functional and some integral inequalities, the less conservative stability and robust stability conditions are well established in terms of linear matrix inequalities (LMIs). Different from the existing L–K functionals, the constructed L–K functional in this paper contains some interconnected terms, which reflect the relationships between discrete delay, neutral delay and distributed delay, and contribute to reduce the possible conservatism. As a special case, stability and robust stability conditions are also proposed in this paper for uncertain linear systems with discrete and distributed delays. Finally, numerical examples illustrate the effectiveness and reduced conservatism of the proposed conditions in this paper.

© 2015 The Franklin Institute. Published by Elsevier Ltd. All rights reserved.

1. Introduction

Time-delay systems have received considerable attention over the past decades due to its wide applications [1,2]. Generally speaking, there are three kinds of time-delays, i.e., discrete delay, neutral delay and distributed delay. For the analysis of time-delay systems, it is widely recognized that the LMI-based conditions are highly appreciated, since such kind of conditions are more convenient for the
synthesis of controllers and filters. During the past two decades, much effort has been made to obtain the less conservative delay-dependent LMI-based analysis and synthesis conditions by introducing some important techniques, such as the bounding inequalities for the cross term [3,4], model transformations [5,6], integral inequalities [7,8], free-weighting matrices [9–11], augmented Lyapunov–Krasovskii (L–K) functionals [12,13], delay decomposition [14], and discretized L–K functionals [15,16]. For more specific introduction of time-delay systems by LMI techniques, refer to [17] and the references therein.

It should be pointed out that the references in [3–17] are mainly concerned with the discrete delay and neutral delay. However, many practical applications can be modelled by systems with distributed delay [1,2,18–20]. During the past years, stability analysis and controller synthesis have also been widely investigated in [19–31] for the systems with distributed delay by using recent developed LMI-based techniques. However, for linear neutral systems with discrete and distributed delays, it is observed that the conditions in [23–31] employ the information of discrete delay, neutral delay, and distributed delay independently, and the relationships between discrete delay, neutral delay and distributed delay are completely neglected. Therefore, the proposed stability and synthesis conditions in [23–31] remain conservative to some extent, and some further improvements are quite necessary.

For linear systems with multiple discrete delays, He et al. pointed out in [32] that the conservatism of the proposed stability conditions can be reduced greatly by utilizing the relationships between multiple discrete delays. Based on such an idea, we recently proposed the improved mixed-delay-dependent stability conditions in [33] for linear neutral systems by incorporating the relationship between discrete delay and neutral delay sufficiently. In this paper, the main objective is to obtain the less conservative stability and robust stability conditions for uncertain neutral systems with discrete and distributed delays. Similar to the ideas in [32,33], we attempt to incorporate the relationships between discrete delay, neutral delay and distributed delay to reduce the conservatism. However, different from the techniques in [32,33], where the relationships between two discrete delays, and discrete and neutral delays are reflected by the terms \(x(t-h)\) and \(x(t-\tau)\), this paper attempts to reflect the relationship between discrete (neutral) delay and distributed delay precisely by the terms \(x(t-h)(x(t-\tau))\) and \(\int_{t-\tau}^{t} x(s) \, ds\).

Motivated by the above discussions, this paper constructs a new augmented L–K functional to analyze the robust stability for uncertain neutral systems with discrete and distributed delays. Different from the L–K functionals used in [23–31], the L–K functional proposed in this paper contains some interconnected terms, which reflect the relationships between discrete delay, neutral delay and distributed delay. Using the proposed L–K functional and some integral inequalities, the improved stability and robust conditions are obtained in terms of LMIs. As a special case, stability and robust stability conditions for uncertain linear systems with discrete and distributed delays are also proposed. Finally, the reduced conservatism of the obtained conditions is well shown by numerical examples. Compared with the conditions in [23–31], the main contributions of this paper can be summarized as follows. Firstly, this paper considers the robust stability of uncertain neutral systems with discrete and distributed delays by incorporating the relationships between discrete delay, neutral delay and distributed delay. Secondly, the proposed L–K functionals in this paper contain some augmented terms, and with the aids of such augmented terms, the relationships between discrete delay and distributed delay, and neutral delay and distributed delay are precisely reflected. Thirdly, by using the proposed L–K functionals in this paper and some integral inequalities, the less conservative stability and robust stability conditions are obtained in terms of LMIs.

Notation: Throughout this paper, the superscript “\(^T\)” stands for matrix transposition. \(\mathbb{R}^n\) and \(\mathbb{R}^{n \times n}\) denote the \(n\)-dimensional Euclidean space and the set of all \(n \times n\) real matrices,
respectively. A real symmetric matrix $P > 0$ ($\geq 0$) denotes $P$ being a positive definite (positive semi-definite) matrix. The symmetric terms in a symmetric matrix are denoted by $\ast$. $I$ denotes an identity matrix with proper dimension. $\| \cdot \|$ refers to the induced matrix 2-norm. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. Problem formulation

Consider the following uncertain neutral system with discrete and distributed delays:
\[
\dot{x}(t) = (A + \Delta A(t))x(t) + (B + \Delta B(t))x(t-h) + (C + \Delta C(t))\dot{x}(t-\tau) + (D + \Delta D(t)) \int_{t-\gamma}^{t} x(s) \, ds, \tag{1}
\]
\[
x(t) = \phi(t), \quad \forall t \in [-\max\{\tau, h, r\}, 0], \tag{2}
\]
where $x(t) \in \mathbb{R}^n$ is the state vector, $h$, $\tau$ and $r$ denote the discrete delay, neutral delay and distributed delay, respectively, $\phi(t) \in \mathbb{R}^n$ is a continuous vector valued initial function. $A, B, C$ and $D$ are known real constant matrices, $\Delta A(t), \Delta B(t), \Delta C(t)$ and $\Delta D(t)$ denote the time-varying parameter uncertainties, and are assumed to be of the following form:
\[
[\Delta A(t) \Delta B(t) \Delta C(t) \Delta D(t)] = MF(t)[N_a \quad N_b \quad N_c \quad N_d], \tag{3}
\]
where $M, N_a, N_b, N_c$ and $N_d$ are known real constant matrices with appropriate dimensions, and $F(t)$ is an unknown continuous time-varying matrix function satisfying $F^T(t)F(t) \leq I$. For system (1), it is also assumed that the condition $\| C + \Delta C(t) \| \leq 1$ holds, which is necessary for guaranteeing the asymptotic stability.

The main purpose of this paper is to propose some less conservative stability and robust stability conditions for system (1). In obtaining the main results of this paper, the lemma introduced below is important.

Lemma 1 (Gu [7], Sun et al. [8]). For any constant symmetric matrix $M \in \mathbb{R}^{n \times n}$, scalars $a$ and $b$ satisfying $0 \leq a < b$, and vector function $\omega(s) \in \mathbb{R}^n$ such that the integrations concerned are well defined, then
\[
\begin{align*}
(i) & \quad \left( \int_{t-b}^{t-a} \omega(s) \, ds \right)^T M \left( \int_{t-b}^{t-a} \omega(s) \, ds \right) \leq (b-a) \int_{t-b}^{t-a} \omega^T(s)M\omega(s) \, ds, \\
(ii) & \quad \left( \int_{t-b}^{t-a} \omega(s) \, ds \right)^T M \left( \int_{t-b}^{t-a} \omega(s) \, ds \right) \\
& \leq \frac{(b^2-a^2)}{2} \int_{t-b}^{t-a} \omega^T(s)M\omega(s) \, ds.
\end{align*}
\]

3. Main results

In this section, we first propose the asymptotic stability condition for the following nominal system:
\[
\dot{x}(t) = Ax(t) + Bx(t-h) + C\dot{x}(t-\tau) + D \int_{t-\gamma}^{t} x(s) \, ds. \tag{4}
\]

Theorem 1. For given scalars $h, \tau$ and $r$, the nominal neutral system (4) is asymptotically stable, if there exist matrices $P = (P_{ij})_{5 \times 5} > 0$, $Q_a > 0 \left[ \begin{array}{c} Q_{21} \\ Q_{22} \\ Q_{23} \end{array} \right]$, $R_u = \left[ \begin{array}{c} R_{11} \\ R_{12} \\ R_{13} \end{array} \right] > 0$,
\( Z_v > 0, T_w, u = 1, 2, \ldots, 6, v = 1, 2, 3, w = 1, 2, 3, 4, \) such that the following LMI holds:

\[
\Omega = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16} & \Omega_{17} & \Omega_{18} & \Omega_{19} \\
* & \Omega_{22} & \Omega_{23} & R_{63} & \Omega_{25} & \Omega_{26} & \Omega_{27} & \Omega_{28} & \Omega_{29} \\
* & * & \Omega_{33} & \Omega_{34} & \Omega_{35} & \Omega_{36} & \Omega_{37} & \Omega_{38} & P^T_{12} \\
* & * & * & \Omega_{44} & 0 & \Omega_{46} & \Omega_{47} & \Omega_{48} & 0 \\
* & * & * & * & \Omega_{55} & P_{23} & P_{24} & \Omega_{58} & \Omega_{59} \\
* & * & * & * & * & \Omega_{66} & R_{41} & R_{61} & P^T_{13} \\
* & * & * & * & * & * & \Omega_{77} & R_{51} & P^T_{14} \\
* & * & * & * & * & * & * & \Omega_{88} & \Omega_{89} \\
* & * & * & * & * & * & * & * & \Omega_{99}
\end{bmatrix} < 0,
\]

where

\[
\begin{align*}
\Omega_{11} &= T_1 A + A^T T_1^T + Q_1 + Q_{21} + Q_3 + h^2 R_{11} + \tau^2 R_{21} + r^2 R_{31} + P_{13} + P_{13}^T + P_{14} + P_{14}^T \\
&\quad + P_{15} + P_{15}^T - R_{13} - R_{23} - R_{33} - h^2 Z_1 - \tau^2 Z_2 - r^2 Z_3 + (\tau - h)^2 R_{41} \\
&\quad + (r - \tau)^2 R_{51} + (h - r)^2 R_{61}, \\
\Omega_{12} &= -P_{13} + T_1 B + A^T T_2^T + R_{13}, \quad \Omega_{13} = -P_{14} + R_{23} + P_{23}^T + P_{24}^T + P_{25}^T, \\
\Omega_{14} &= -P_{15} + R_{33}, \\
\Omega_{15} &= T_1 C + A^T T_3^T + P_{12}, \quad \Omega_{16} = P_{33} + P_{34} + P_{35} + h Z_1 - R_{12}, \\
\Omega_{17} &= P_{34} + P_{44} + P_{45} + \tau Z_2 - R_{22}^T, \quad \Omega_{18} = P_{35} + P_{45} + P_{55} + T_1 D + r Z_3 - R_{32}, \\
\Omega_{19} &= -T_1 + A^T T_4^T + P_{11} + Q_{22} + h^2 R_{12} + \tau^2 R_{22} + r^2 R_{32} + (\tau - h)^2 R_{42} + (r - \tau)^2 R_{52} \\
&\quad + (h - r)^2 R_{62}, \\
\Omega_{22} &= -Q_1 + T_2 B + B^T T_2^T - R_{13} - R_{43} - R_{63} + (\tau - h) Q_4 + (r - h) Q_6, \\
\Omega_{23} &= R_{43} - P_{23}^T, \\
\Omega_{25} &= T_2 C + B^T T_3^T, \quad \Omega_{26} = -P_{33} + R_{12}^T + R_{42}^T + R_{62}, \quad \Omega_{27} = -P_{34} - R_{42}^T, \\
\Omega_{28} &= -P_{35} + T_2 D - R_{62}^T, \quad \Omega_{29} = -T_2 + B^T T_4^T, \\
\Omega_{33} &= -Q_{21} - P_{24} - P_{24}^T - R_{23} - R_{43} - R_{53} + (h - \tau) Q_4 + (r - \tau) Q_5, \\
\Omega_{34} &= -P_{25} + P_{33}, \quad \Omega_{35} = -Q_{22} + P_{22}, \\
\Omega_{36} &= -P_{34} - R_{42}^T, \quad \Omega_{37} = R_{22}^T - P_{44} + R_{42}^T + R_{52}^T, \quad \Omega_{38} = -P_{45} - R_{52}^T, \\
\Omega_{44} &= -Q_3 - R_{33} - R_{53} - R_{63} + (\tau - r) Q_5 + (h - r) Q_6, \quad \Omega_{46} = -P_{35}^T - R_{62}, \\
\Omega_{47} &= -P_{45} - R_{52}^T, \\
\Omega_{48} &= -P_{55} + R_{32}^T + R_{52}^T + R_{62}^T, \quad \Omega_{53} = T_3 C + C^T T_3^T - Q_{23}, \quad \Omega_{58} = P_{25} + T_3 D, \\
\Omega_{59} &= -T_3 + C^T T_4^T, \quad \Omega_{66} = -Z_1 - R_{11} - R_{41} - R_{61}, \quad \Omega_{77} = -Z_2 - R_{21} - R_{41} - R_{51}, \\
\Omega_{88} &= -Z_3 - R_{31} - R_{51} - R_{61}, \quad \Omega_{89} = D^T T_4^T + P_{15}^T, \\
\Omega_{99} &= -T_4 - T_4^T + Q_{23} + (h^2 Z_1 + \tau^2 Z_2 + r^2 Z_3)/4 + h^2 R_{13} + \tau^2 R_{23} + r^2 R_{33} + (\tau - h)^2 R_{43} \\
&\quad + (r - \tau)^2 R_{53} + (h - r)^2 R_{63}.
\end{align*}
\]

**Proof.** Construct the following augmented L–K functional:

\[
V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t),
\]

where

\[
V_1(t) = a^T(t) P a(t) + \int_{t-h}^t x^T(s) Q_1(s) \, ds + \int_{t-\tau}^t p^T(s) Q_2 \beta(s) \, ds + \int_{t-r}^t x^T(s) Q_3 x(s) \, ds,
\]
\[ V_2(t) = h \int_{-h}^{0} \int_{t+\theta}^{t} \beta^T(s)R_1 \beta(s) \, ds \, d\theta + \tau \int_{-\tau}^{0} \int_{t+\theta}^{t} \beta^T(s)R_2 \beta(s) \, ds \, d\theta \]
\[ + r \int_{-r}^{0} \int_{t+\theta}^{t} \beta^T(s)R_3 \beta(s) \, ds \, d\theta, \]
\[ V_3(t) = \frac{h^2}{2} \int_{-h}^{0} \int_{0}^{\tau} \int_{t+\theta}^{t} \dot{x}^T(s)Z_1 \dot{x}(s) \, ds \, d\lambda \, d\theta + \frac{\tau^2}{2} \int_{-\tau}^{0} \int_{0}^{\tau} \int_{t+\theta}^{t} \dot{x}^T(s)Z_2 \dot{x}(s) \, ds \, d\lambda \, d\theta \]
\[ + \frac{r^2}{2} \int_{-r}^{0} \int_{0}^{\tau} \int_{t+\theta}^{t} \dot{x}^T(s)Z_3 \dot{x}(s) \, ds \, d\lambda \, d\theta, \]
\[ V_4(t) = (\tau - h) \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)Q_4 \dot{x}(s) \, ds + (r - \tau) \int_{-r}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)Q_5 \dot{x}(s) \, ds \]
\[ + (h - r) \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)Q_6 \dot{x}(s) \, ds, \]
\[ V_5(t) = (\tau - h) \int_{-\tau}^{0} \int_{t+\theta}^{t} \beta^T(s)R_4 \beta(s) \, ds \, d\theta + (r - \tau) \int_{-r}^{0} \int_{t+\theta}^{t} \beta^T(s)R_5 \beta(s) \, ds \, d\theta \]
\[ + (h - r) \int_{-h}^{0} \int_{t+\theta}^{t} \beta^T(s)R_6 \beta(s) \, ds \, d\theta, \]

and \( \alpha(t) = [\dot{x}^T(t) \dot{x}^T(t - \tau) (\int_{t-\tau}^{t} x(s) \, ds)^T (\int_{t-\tau}^{t} x(s) \, ds)^T (\int_{t-\tau}^{t} x(s) \, ds)^T]^T, \beta(s) = [x^T(s) \dot{x}^T(s)]^T. \)

Differentiating \( V(t) \) with respect to \( t \) along the trajectory of nominal system (4) gives that
\[ \dot{V}_1(t) = 2 \alpha(t)P \dot{x}(t) + \dot{x}^T(t)(Q_1 + Q_3)x(t) + \beta^T(t)Q_2 \beta(t) - x^T(t-h)Q_1 x(t-h) \]
\[ - \beta^T(t)(Q_2 \beta(t) - x^T(t-h)Q_3 x(t-h), \quad (7) \]
\[ \dot{V}_2(t) = \beta^T(t)(h^2 R_1 + \tau^2 R_2 + r^2 R_3) \beta(t) - h \int_{t-h}^{t} \beta^T(s)R_1 \beta(s) \, ds \]
\[ - \tau \int_{t-\tau}^{t} \beta^T(s)R_2 \beta(s) \, ds - r \int_{t-r}^{t} \beta^T(s)R_3 \beta(s) \, ds, \quad (8) \]
\[ \dot{V}_3(t) = \frac{1}{4} \dot{x}^T(t)(h^4 Z_1 + \tau^4 Z_2 + r^4 Z_3) \dot{x}(t) - \frac{h^2}{2} \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)Z_1 \dot{x}(s) \, ds \, d\theta \]
\[ - \frac{\tau^2}{2} \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)Z_2 \dot{x}(s) \, ds \, d\theta - \frac{r^2}{2} \int_{-r}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)Z_3 \dot{x}(s) \, ds \, d\theta, \quad (9) \]
\[ \dot{V}_4(t) = x^T(t-h)[(\tau - h)Q_4 + (r - \tau)Q_6]x(t-h) + \dot{x}^T(t-\tau)[(\tau - \tau)Q_4 + (\tau - \tau)Q_5 + (h - r)Q_6]x(t-h), \quad (10) \]
\[ \dot{V}_5(t) = \beta^T(t)[(\tau - h)^2 R_4 + (\tau - \tau)^2 R_5 + (h - r)^2 R_6] \beta(t) - (\tau - h) \int_{t-h}^{t} \beta^T(s)R_4 \beta(s) \, ds \]
\[ - (r - \tau) \int_{t-r}^{t} \beta^T(s)R_5 \beta(s) \, ds - (h - r) \int_{t-h}^{t} \beta^T(s)R_6 \beta(s) \, ds, \quad (11) \]

Using Lemma 1, then one can obtain that the following inequalities hold:
\[ -h \int_{t-h}^{t} \beta^T(s)R_1 \beta(s) \, ds \leq - \left( \int_{t-h}^{t} \beta(s) \, ds \right)^T R_1 \left( \int_{t-h}^{t} \beta(s) \, ds \right) \]
\[
= - \left[ \int_{t-h}^{t} x(s) \, ds \right] R_1 \left[ \int_{t-h}^{t} x(s) \, ds \right]^{T},
\]
(12)

\[
- \tau \int_{t-\tau}^{t} \beta^T(s) R_2 \beta(s) \, ds \leq - \left( \int_{t-\tau}^{t} \beta(s) \, ds \right)^T R_2 \left( \int_{t-\tau}^{t} \beta(s) \, ds \right),
\]
(13)

\[
- r \int_{t-r}^{t} \beta^T(s) R_3 \beta(s) \, ds \leq - \left( \int_{t-r}^{t} \beta(s) \, ds \right)^T R_3 \left( \int_{t-r}^{t} \beta(s) \, ds \right),
\]
(14)

\[
- (\tau - h) \int_{t-\tau}^{t-h} \beta^T(s) R_4 \beta(s) \, ds \leq - \left( \int_{t-\tau}^{t-h} \beta(s) \, ds \right)^T R_4 \left( \int_{t-\tau}^{t-h} \beta(s) \, ds \right),
\]
(15)

\[
- (r - \tau) \int_{t-r}^{t-r} \beta^T(s) R_5 \beta(s) \, ds \leq - \left( \int_{t-r}^{t-r} \beta(s) \, ds \right)^T R_5 \left( \int_{t-r}^{t-r} \beta(s) \, ds \right),
\]
(16)

\[
- (h - r) \int_{t-h}^{t-r} \beta^T(s) R_6 \beta(s) \, ds \leq - \left( \int_{t-h}^{t-r} \beta(s) \, ds \right)^T R_6 \left( \int_{t-h}^{t-r} \beta(s) \, ds \right),
\]
(17)

\[
- \frac{h^2}{2} \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}(s) \, ds \, d\theta \leq - \left( \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}(s) \, ds \, d\theta \right)^T Z_1 \left( \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}(s) \, ds \, d\theta \right),
\]
(18)

\[
- \frac{\tau^2}{2} \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}(s) \, ds \, d\theta \leq - \left( \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}(s) \, ds \, d\theta \right)^T Z_2 \left( \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}(s) \, ds \, d\theta \right),
\]
(19)

\[
- \frac{r^2}{2} \int_{-r}^{0} \int_{t+\theta}^{t} \dot{x}(s) \, ds \, d\theta \leq - \left( \int_{-r}^{0} \int_{t+\theta}^{t} \dot{x}(s) \, ds \, d\theta \right)^T Z_3 \left( \int_{-r}^{0} \int_{t+\theta}^{t} \dot{x}(s) \, ds \, d\theta \right).
\]
For any matrices \( T_1, T_2, T_3 \) and \( T_4 \), it follows from system equation (4) that
\[
2\rho(t) T \left[ Ax(t) + Bx(t-h) + Cx(t-\tau) + D \int_{t-r}^{t} x(s) \, ds - \dot{x}(t) \right] = 0,
\]
where \( \rho(t) = \left[ x^T(t) x^T(t-h) x^T(t-\tau) x^T(t-r) \right] \dot{x}^T(t) \) and \( T = \begin{bmatrix} T_1^T & T_2^T & T_3^T & T_4^T \end{bmatrix}^T \). Adding the left side of Eq. (21) to \( \dot{V}(t) \), and combining with Eqs. (7)–(20), then one can easily obtain that
\[
\dot{V}(t) \leq \xi^T(t) \Omega \xi(t),
\]
where \( \xi(t) = \left[ x^T(t) x^T(t-h) x^T(t-\tau) x^T(t-r) \right] \dot{x}^T(t-\tau) (\int_{t-h}^{t} x(s) \, ds)^T (\int_{t-r}^{t} x(s) \, ds)^T \dot{x}^T(t) \right]^T \), and \( \Omega \) is defined in Theorem 1. Thus, if LMI \( \Omega < 0 \) in Eq. (5) holds, then it can be concluded that \( \dot{V}(t) < 0 \), which implies that the nominal neutral system (4) is asymptotically stable. This completes the proof. \( \square \)

Remark 1. Compared with the L–K functionals employed in [23–31], the main advantage of the functional (6) proposed in this paper is that the interconnected relationships between discrete delay, neutral delay and distributed delay are utilized, and the terms \( V_4(t) \) and \( V_5(t) \) are further introduced. With the aids of the terms \( V_4(t) \) and \( V_5(t) \), the less conservative asymptotic stability condition can be obtained, which can be explained by the following two aspects. Firstly, one can see from Eq. (11) that the derivative of \( V_5(t) \) can be negative for some positive definite matrices \( R_4, R_5 \) and \( R_6 \) when \( |\tau-h|, |r-\tau| \) and \( |h-r| \) are small, which means that the resulting LMI (5) is less conservative when the differences between \( h, \tau, r \) and \( r \) are small. Secondly, the adjusting parameters \( Q_4, Q_5 \) and \( Q_6 \) are additionally introduced in Theorem 1 by using the term \( V_4(t) \), which results in the LMI (5) more slack.

Remark 2. Different from the L–K functionals proposed in [32,33], where interconnected information between multiple delays is reflected by non-augmented term, our proposed L–K functional (6) contains the augmented term \( V_5(t) \). Due to the augmented term \( V_5(t) \), then the relationship between discrete (neutral) delay and distributed delay is not only reflected by the terms \( x(t-h)(x(t-\tau)) \) and \( x(t-r) \) (see Eqs. (16)–(17)), but also by the terms \( x(t-h)(x(t-\tau)) \) and \( \int_{t-r}^{t} x(s) \, ds \). In fact, from the forms of discrete (neutral) delay and distributed delay appearing in systems (1) and (4), it is expected that the relationship between discrete (neutral) delay and distributed delay can be precisely reflected by \( x(t-h)(x(t-\tau)) \) and \( \int_{t-r}^{t} x(s) \, ds \).

For the uncertain neutral system (1), using the similar techniques as in [25–28] to deal with the uncertainties, then the sufficient condition for guaranteeing the robust asymptotic stability can be described as follows.

Theorem 2. For given scalars \( h, \tau \) and \( r \), the uncertain neutral system (1) is robustly asymptotically stable, if there exist matrices \( P = (P_{ij})_{5 \times 5} > 0 \), \( Q_4 > 0 \), \( Q_2 = \begin{bmatrix} Q_{21} & Q_{22} \\ Q_{22} & Q_{23} \end{bmatrix} \), \( R_u = \begin{bmatrix} r_{u1} & r_{u2} & r_{u3} \end{bmatrix}^T > 0 \), \( Z_v > 0 \), \( T_w, u = 1, 2, \ldots, 6, v = 1, 2, 3, w = 1, 2, 3, 4 \), and a scalar \( \mu > 0 \),
such that the following LMI holds:
\[
\begin{bmatrix}
\Omega & \bar{M} & \mu \bar{N}^T \\
* & -\mu I & 0 \\
* & * & -\mu I
\end{bmatrix} < 0,
\]
where \(\Omega\) is defined in Eq. (5) and
\[
\bar{M} = [M^T T_1^T M^T T_2^T 0 0 M^T T_3^T 0 0 0 M^T T_4^T]^T, \\
\bar{N} = [N_a \ N_b \ 0 \ 0 \ N_c \ 0 \ 0 \ N_d \ 0].
\]
In the case that the neutral terms do not occur in Eqs. (1) and (4), using the following L–K functional:
\[
\hat{V}(t) = \gamma^T(t)P\gamma(t) + \int_{t-h}^t x^T(s)Q_1(s) ds + \int_{t-r}^t x^T(s)Q_2x(s) ds
\]
\[
+ h \int_{t-h}^t \int_{t+\theta}^t \beta^T(s)R_1\beta(s) ds \, d\theta + r \int_{t-r}^t \int_{t+\theta}^t \beta^T(s)R_2\beta(s) ds \, d\theta
\]
\[
+ \frac{h^2}{2} \int_{t-h}^t \int_{t+\theta}^t \hat{x}^T(s)Z_1\hat{x} ds \, d\lambda + \frac{r^2}{2} \int_{t-r}^t \int_{t+\theta}^t \hat{x}^T(s)Z_2\hat{x} ds \, d\lambda \, d\theta
\]
\[
+ (h-r) \int_{t-h}^t \int_{t-r}^t x^T(s)Q_3(s) ds + (h-r) \int_{t-h}^t \int_{t-r}^t \beta^T(s)R_3\beta(s) ds \, d\theta,
\]
where \(\gamma(t) = \begin{bmatrix} x^T(t) \left( \int_{t-h}^t x(s) ds \right) \end{bmatrix}^T \begin{bmatrix} \left( \int_{t-r}^t x(s) ds \right)^T \end{bmatrix} \), then the following stability conditions can be easily obtained.

**Corollary 1.** For given scalars \(h\) and \(r\), the nominal system (4) with \(C=0\) is asymptotically stable, if there exist matrices \(P = (P_0)_{3 \times 3} > 0, Q_u > 0, R_u = \begin{bmatrix} R_{u1} & R_{u2} \\ R_{u2} & R_{u3} \end{bmatrix} > 0, Z_r > 0, T_u, u = 1, 2, 3, v = 1, 2,\) such that the following LMI holds:
\[
\Xi = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\
* & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & \Xi_{26} \\
* & * & \Xi_{33} & \Xi_{34} & \Xi_{35} & 0 \\
* & * & * & \Xi_{44} & \Xi_{45} & \Xi_{46} \\
* & * & * & * & \Xi_{55} & \Xi_{56} \\
* & * & * & * & * & \Xi_{66}
\end{bmatrix} < 0,
\]
where
\[
\Xi_{11} = T_1A + A^T T_1^T + Q_1 + Q_2 + h^2 R_{11} + r^2 R_{21} + P_{12} + P_{13}^T + P_{13}
\]
\[
+ P_{13}^T - R_{13} - R_{23} - h^2 Z_1 - r^2 Z_2 + (h-r)^2 R_{31}, \quad \Xi_{12} = T_1B + A^T T_2^T + R_{13} - P_{12},
\]
\[
\Xi_{13} = R_{23} - P_{13},
\]
\[
\Xi_{14} = P_{22} + P_{23}^T + h Z_1 - R_{12}, \quad \Xi_{15} = T_1D + P_{23} + P_{33} + r Z_2 - R_{22},
\]
\[
\Xi_{16} = - T_1 + A^T T_3^T + P_{11} + h^2 R_{12} + r^2 R_{22} + (h-r)^2 R_{32},
\]
\[
\Xi_{22} = T_2B + B^T T_2^T - Q_1 - R_{13} - R_{33} + (h-r)Q_3, \quad \Xi_{24} = - P_{22} + P_{12}^T + R_{32},
\]
\[
\Xi_{25} = T_2D - P_{23} - R_{32}, \quad \Xi_{26} = - T_2 + B^T T_3^T, \quad \Xi_{33} = - Q_2 - R_{23} - R_{33} + (h-r)Q_3,
\]
\[
\Xi_{34} = - P_{23}^T - R_{32}, \quad \Xi_{35} = - P_{33} + R_{22}^T + R_{32}, \quad \Xi_{44} = - Z_1 - R_{11} - R_{31},
\]
Case A

Theorem 2.6.7.6.67 6.67 5.52 1.95 1.55 1.46 34

Example 1

Maximum allowable delay bounds on $r$ for $\tau = 0.1$ and different $h$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>0.1</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>1.6</th>
<th>1.7</th>
<th>Number of variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>[27]</td>
<td>6.64</td>
<td>5.55</td>
<td>1.62</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>27$n^2$ + 5$n$ + 1</td>
</tr>
<tr>
<td>[28]</td>
<td>6.67</td>
<td>6.12</td>
<td>2.75</td>
<td>1.31</td>
<td>0.93</td>
<td>0.42</td>
<td>21$n^2$ + 9$n$ + 1</td>
</tr>
<tr>
<td>[29]</td>
<td>6.65</td>
<td>6.02</td>
<td>2.68</td>
<td>0.88</td>
<td>–</td>
<td>–</td>
<td>14$n^2$ + 8$n$ + 1</td>
</tr>
<tr>
<td>[30]</td>
<td>6.67</td>
<td>5.83</td>
<td>2.97</td>
<td>1.53</td>
<td>1.33</td>
<td>1.14</td>
<td>21$n^2$ + 7$n$ + 1</td>
</tr>
</tbody>
</table>

Theorem 2

Case A

4. Numerical examples

Example 1 (Li and Zhu [27], Sun et al. [28], Hu et al. [29], Chen et al. [30], Hui et al. [31]). Consider the uncertain neutral system (1) with the following parameters:

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}, \quad C = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix},$$

$$D = \begin{bmatrix} -0.12 & -0.12 \\ -0.12 & 0.12 \end{bmatrix}, \quad M = I, \quad N_a = N_b = N_c = N_d = 0.1I.$$
For this example, by solving Theorem 2 with \( \tau = 0.1 \), one can obtain the maximum allowable upper bounds on \( r \) and \( h \) for different \( h \) and \( r \), which are listed in Tables 1 and 2 (– denotes infeasible), respectively, where Case A corresponds to Theorem 2 with \( Q_j = R_j = 0, j = 4, 5, 6 \), and the delay bounds in [30] are some corrected values. It is clear from Tables 1 and 2 that Theorem 2 proposed in this paper can provide the larger delay bounds than the conditions in [27–30]. Meanwhile, it can be seen that Case A provides the smaller delay bounds than Theorem 2, which implies that the conservatism increases when the relationships between mixed delays are abandoned.

Using the delay decomposition approach, Hui et al. provide a smaller delay bound 1.70 (\( N = 10, M = 2, r = 0.1 \)) on \( h \) and a larger delay bound 6.94 (\( N = 2, M = 10, h = 0.1 \)) on \( r \) in [31], which means that Theorem 2 proposed in this paper is not always less conservative than the condition in [31]. However, by incorporating the delay decomposition approach in this paper, it is expected that some more effective conditions can be obtained.

**Example 2** (Yue et al. [24], Chen and Zheng [25]). Consider the nominal system (4) with the following parameters:

\[
A = \begin{bmatrix} -0.9 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -0.12 \\ 0.12 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} -0.12 & -0.12 \\ -0.12 & 0.12 \end{bmatrix}, \quad C = 0.
\]

For this example, by the conditions in [24,25] with \( r = 1 \), the allowable delay bounds on \( h \) for guaranteeing the stability are 1.8302 and 2.8011, respectively, while by Corollary 1 in this paper, one can obtain a larger delay bound 3.5823. In addition, Tables 3 and 4 list some delay bounds obtained by Corollary 1, Case B and Case C, where Case B corresponds to Corollary 1 with \( Q_3 = R_3 = 0 \), and Case C corresponds to Corollary 1 with \( R_{31} = R_{32} = 0 \). From Tables 3 and 4, it

<table>
<thead>
<tr>
<th>Table 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum allowable upper bounds on ( r ) for different ( h ).</td>
</tr>
<tr>
<td>( h )</td>
</tr>
<tr>
<td>Corollary 1</td>
</tr>
<tr>
<td>Case B</td>
</tr>
<tr>
<td>Case C</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum allowable upper bounds on ( h ) for different ( r ).</td>
</tr>
<tr>
<td>( r )</td>
</tr>
<tr>
<td>Corollary 1</td>
</tr>
<tr>
<td>Case B</td>
</tr>
<tr>
<td>Case C</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numbers of variables involved in [24,25], Corollary 1, Case A and Case B.</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>( 3.5n^2 + 3.5n )</td>
</tr>
</tbody>
</table>
can be seen that Case B provides the smaller delay bounds than Corollary 1, which shows the importance of using the relationship between discrete delay and distributed delay. From Tables 3 and 4, one can also see that Case C provides the smaller delay bounds than Corollary 1, which implies that the conservatism increases when the augmented term is abandoned in the proposed L–K functional (24).

**Remark 3.** From Examples 1 and 2, it can be seen that our proposed conditions can provide the less conservative results. However, it should be pointed out from Tables 1 and 5 that more variables are involved in our conditions, and thus testing the conditions is more time-consuming. For further research, it is expected that some more effective conditions are obtained, and meanwhile the less variables are contained in the proposed conditions.

5. Conclusion

Based on the new augmented L–K functional and integral inequalities, the new stability and robust stability conditions have been proposed for uncertain neutral systems with discrete and distributed delays in terms of LMIs. The proposed stability conditions in this paper are less conservative due to the constructed augmented L–K functional, which utilizes the relationships between discrete delay, neutral delay and distributed delay. Also, the robust stability problem for uncertain systems without neutral term is well considered. Finally, we have presented numerical examples to show the reduced conservatism of the proposed conditions and techniques in this paper.

Acknowledgment

This work was supported by the National Natural Science Foundations of PR China under Grants 61304061, 61273119 and 61174076. This work was also partly supported by the Open Foundation of Key Laboratory of Control Engineering of Henan Province under Grant KG 2014-02.

References
