Distributivity and conditional distributivity of a uninorm with continuous underlying operators over a continuous t-conorm

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Abstract

This paper deals with the distributivity and conditional distributivity of uninorms with continuous underlying operators over continuous triangular conorms. In particular, the involved triangular conorm is either maximum operator or an ordinal sum with only one summand in which the corresponding triangular conorm is strict. From the obtained results, it is deduced that distributivity and conditional distributivity are equivalent. Moreover, we obtain the full characterization of some cases of this class of uninorms of which either the underlying triangular norm or triangular conorm is strict.

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1. Introduction

The problem of distributivity has been posed many years ago [1]. In the literature, there are many directions of research related to distributivity of different operators used in fuzzy set theory, e.g. triangular norms and triangular conorms [5], uninorms and nullnorms [10,20,23,30], increasing binary operators [2,7,24], implications over triangular conorms [3,4], implications over uninorms [5,26].

This paper is devoted to the distributivity and conditional distributivity of a uninorm over a continuous triangular conorm (see Definitions 6 and 7 in Section 4). Let us note that this problem is related to pseudo-analysis in measure theory [6,17,29] and nonlinear PDE [21]. Distributivity and conditional distributivity of a uninorm over a continuous triangular conorm have been solved in [25] for four well known classes of uninorms. The results in [25] show that nontrivial solutions only appear for representable uninorms [12] and for uninorms continuous in ]0, 1[ [14] and that in both cases, solutions involve strict triangular conorms. This paper is focused on the class of uninorms (see Definition 5 in Section 2) of which the underlying triangular norm and triangular conorm are both continuous. This paper will deal with the distributivity and conditional distributivity of this class of uninorms over a continuous t-conorm. It is

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obvious that uninorms continuous in \([0, 1]^2\) are properly included in this class of uninorms with continuous underlying operators. In this sense, the results presented here involving distributivity or conditional distributivity can be viewed as complementary to those proved in [25].

2. Preliminaries

In this section, we summarize some of the essential results about triangular norms, triangular conorms and uninorms.

**Definition 1.** (See [16].) A triangular conorm (t-conorm for short) is a commutative, associative, non-decreasing function \(S : [0, 1]^2 \rightarrow [0, 1]\) such that \(S(0, x) = x\) for all \(x \in [0, 1]\).

A t-conorm \(S\) satisfies \(S(x, y) \geq S_M(x, y) = \max(x, y)\) for all \(x, y \in [0, 1]\). If a continuous t-conorm \(S\) satisfies \(S(x, x) > x\) for all \(x \in [0, 1]\), then it is called a continuous Archimedean t-conorm. As it is well-known, each continuous Archimedean t-conorm \(S\) can be represented by means of a continuous additive generator, i.e., a strictly increasing continuous function \(s : [0, 1] \rightarrow [0, \infty]\) with \(s(0) = 0\) such that

\[
S(x, y) = s^{(-1)}(s(x) + s(y)),
\]

where \(s^{(-1)} : [0, \infty) \rightarrow [0, 1]\) is the pseudo-inverse of \(s\), and it is given by

\[
s^{(-1)}(u) = s^{-1}(\min(u, s(1)))\).
\]

Moreover, if \(S\) is continuous Archimedean and for all \(x \in [0, 1]\), \(0 < y < z < 1\) implies \(S(x, y) < S(x, z)\), then \(S\) is called strict. If \(S\) is continuous Archimedean and for all \(x \in [0, 1]\), there exists \(y \in [0, 1]\) such that \(S(x, y) = 1\), then \(S\) is called nilpotent. A nilpotent t-conorm \(S\) has a continuous additive generator \(s\) such that \(s(1) < +\infty\). This implies that \(S\) is strictly increasing on that part of the unit square where it is not equal to 1. We will use this fact in some proofs later on.

Each continuous t-conorm can be represented as an ordinal sum of continuous Archimedean t-conorms, i.e., there exist a uniquely determined index set \(K\), and a family of uniquely determined continuous Archimedean t-conorms \(S_k, k \in K\) such that \(S = (\langle a_k, b_k, S_k \rangle)_{k \in K}\), where \(a_k < b_k\), each \(\langle a_k, b_k, S_k \rangle\) is called a summand and \(S_k\) is called the corresponding t-conorm in summand \(\langle a_k, b_k, S_k \rangle\) [16].

Dually, we have the similar concepts and results about triangular norm.

**Definition 2.** (See [16].) A triangular norm (t-norm for short) is a commutative, associative, non-decreasing function \(T : [0, 1]^2 \rightarrow [0, 1]\) such that \(T(1, x) = x\) for all \(x \in [0, 1]\).

A t-norm \(T\) satisfies \(T(x, y) \leq T_M(x, y) = \min(x, y)\) for all \(x, y \in [0, 1]\). If a continuous t-norm \(T\) satisfies \(T(x, x) < x\) for all \(x \in [0, 1]\), then it is called a continuous Archimedean t-norm. As it is well-known, each continuous Archimedean t-norm \(T\) can be represented by means of a continuous additive generator, i.e., a strictly decreasing continuous function \(t : [0, 1] \rightarrow [0, \infty]\) with \(t(1) = 0\) such that

\[
T(x, y) = t^{(-1)}(t(x) + t(y)),
\]

where \(t^{(-1)} : [0, \infty) \rightarrow [0, 1]\) is the pseudo-inverse of \(t\), and it is given by

\[
t^{(-1)}(u) = t^{-1}(\min(u, t(0)))\).
\]

Moreover, if \(T\) is continuous Archimedean and for all \(x \in [0, 1]\), \(0 < y < z < 1\) implies \(T(x, y) < T(x, z)\), then \(T\) is called strict. If \(T\) is continuous Archimedean and for all \(x \in [0, 1]\), there exists \(y \in [0, 1]\) such that \(T(x, y) = 0\), then \(T\) is called nilpotent. A nilpotent t-norm \(T\) has a continuous additive generator \(t\) such that \(t(0) < +\infty\). This implies that \(T\) is strictly increasing on that part of the unit square where it is positive. We will use this fact in some proofs later on.

Each continuous t-norm can be represented as an ordinal sum of continuous Archimedean t-norms, i.e., there exist a uniquely determined index set \(I\), and a family of uniquely determined continuous Archimedean t-norms \(T_i, i \in I\) such that \(T = (\langle a_i, b_i, T_i \rangle)_{i \in I}\) [16].

More information concerning t-norms and t-conorms can be found in [16].
**Definition 3.** (See [31].) A uninorm is a two-place function: \( U : [0, 1]^2 \to [0, 1] \) which is associative, commutative, increasing in each place and there exists some element \( e \in [0, 1] \), called neutral element, such that \( U(e, x) = x \) for all \( x \in [0, 1] \).

We summarize some fundamental results from [12].

It is clear that the function \( U \) becomes a t-norm when \( e = 1 \) and a t-conorm when \( e = 0 \). For any uninorm we have \( U(0, 1) \in [0, 1] \). Throughout this paper, we exclusively consider uninorms with a neutral element \( e \) strictly between 0 and 1. With any uninorm \( U \) with neutral element \( e \in ]0, 1[ \), we can associate two binary operations \( T_U, S_U : [0, 1]^2 \to [0, 1] \) defined by

\[
T_U(x, y) = \frac{1}{e} \cdot U(ex, ey)
\]

and

\[
S_U(x, y) = \frac{1}{1-e}(U(e + (1-e)x, e + (1-e)y) - e).
\]

It is easy to see that \( T_U \) is a t-norm and that \( S_U \) is a t-conorm. In other words, on \([0, e] \) any uninorm \( U \) is determined by a t-norm \( T_U \), and on \([e, 1] \) any uninorm \( U \) is determined by a t-conorm \( S_U \); \( T_U \) is called the underlying t-norm, and \( S_U \) is called the underlying t-conorm. Let us denote the remaining part of the unit square by \( E \), i.e., \( E = [0, 1]^2 \setminus ([0, e] \cup [e, 1]^2) \). On the set \( E \), any uninorm \( U \) is bounded by the minimum and maximum of its arguments, i.e. for any \((x, y) \in E \) it holds that

\[
\min(x, y) \leq U(x, y) \leq \max(x, y).
\tag{1}
\]

Now, we recall the characterizations of several classes of uninorms.

**Theorem 1.** (See [12].) Suppose that \( U \) is a uninorm with neutral element \( e \in ]0, 1[ \) and both functions \( x \mapsto U(x, 1) \) and \( x \mapsto U(x, 0) \ (x \in [0, 1]) \) are continuous except perhaps at the point \( x = e \). Then \( U \) is given by one of the following forms.

(i) If \( U(0, 1) = 0 \), then

\[
U(x, y) = \begin{cases} 
  eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\
  e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x, y) \in [e, 1]^2, \\
  \min(x, y) & \text{otherwise.}
\end{cases}
\tag{2}
\]

(ii) If \( U(0, 1) = 1 \), then

\[
U(x, y) = \begin{cases} 
  eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\
  e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x, y) \in [e, 1]^2, \\
  \max(x, y) & \text{otherwise.}
\end{cases}
\tag{3}
\]

Denote \( U_{\min} \) the class of uninorms having form (2) and \( U_{\max} \) the class of uninorms with form (3).

**Proposition 1.** (See [10,21].) Consider \( e \in ]0, 1[ \) and a strictly increasing continuous function \( h : [0, 1] \to [0, +\infty] \) with \( h(0) = 0, h(e) = 1 \) and \( h(1) = +\infty \). The binary operator \( U \) defined by

\[
U(x, y) = h^{-1}(h(x) \cdot h(y))
\]

for all \((x, y) \in [0, 1]^2 \setminus \{(0, 0), (1, 0)\}\) and either \( U(0, 1) = U(1, 0) = 0 \) or \( U(0, 1) = U(1, 0) = 1 \), is a uninorm which is continuous in \( [0, 1]^2 \setminus \{(0, 1), (1, 0)\} \).

Uninorms defined in **Proposition 1** are called \( \text{representable uninorms} \) and function \( h \) is called a \( \text{multiplicative generator of } U \).
Remark 1. For a representable uninorm $U$, the underlying t-norm $T_U$ and t-conorm $S_U$ are both strict.

Uninorms continuous in $]0,1[$ form another class of uninorms that contains the class of representable uninorms. They were characterized in [14,26] as follows.

Theorem 2. Suppose that $U$ is a uninorm continuous in $]0,1[$ with neutral element $e \in ]0,1[$. Then one of the following cases is satisfied:

(i) There exist $u \in [0,e], \lambda \in [0,u]$, two continuous t-norms $T_1$, $T_2$ and a representable uninorm $R$ such that $U$ can be represented as

$$U(x,y) = \begin{cases} 
\lambda T_1(\frac{x}{\lambda}, \frac{y}{\lambda}) & \text{if } x, y \in [0,\lambda], \\
\lambda + (u - \lambda)T_2(\frac{x-u}{u-\lambda}, \frac{y-u}{u-\lambda}) & \text{if } x, y \in [\lambda, u], \\
u + (1-u)R(\frac{x-u}{u}, \frac{y-u}{u}) & \text{if } x, y \in [u,1], \\
1 & \text{if } \min(x,y) \in [\lambda,1], \max(x,y) = 1,
\end{cases}$$

(ii) There exist $\gamma \in [e,1], \delta \in [\gamma,1]$, two continuous t-conorms $S_1$, $S_2$ and a representable uninorm $R$ such that $U$ can be represented as

$$U(x,y) = \begin{cases} 
\gamma R(\frac{x}{\gamma}, \frac{y}{\gamma}) & \text{if } x, y \in [0,\gamma], \\
\gamma + (\delta - \gamma)S_1(\frac{x-\gamma}{\delta-\gamma}, \frac{y-\gamma}{\delta-\gamma}) & \text{if } x, y \in [\gamma, \delta], \\
\delta + (1-\delta)S_2(\frac{x-\delta}{1-\delta}, \frac{y-\delta}{1-\delta}) & \text{if } x, y \in [\delta,1], \\
0 & \text{if } \max(x,y) \in [0,\delta], \min(x,y) = 0,
\end{cases}$$

Denote $CU_{\min}$ the class of uninorms with form (4) and $CU_{\max}$ the class of uninorms with form (5). A uninorm $U$ in $CU_{\min}$ (or in $CU_{\max}$) will be denoted as $U = (e, u, \lambda, T_1, T_2, R)$ (or $U = (e, \gamma, \delta, R, S_1, S_2)$) to represent its parameters.

Remark 2. Any uninorm $U$ in $CU_{\min}$ with $u = 0$ or $U$ in $CU_{\max}$ with $\gamma = 1$ is a representable uninorm. Both the underlying t-norm $T_U$ and underlying t-conorm $S_U$ of $U$ in $CU_{\min}$ (or in $CU_{\max}$) are continuous.

Definition 4. A uninorm $U$ is called idempotent whenever $U(x,x) = x$ for all $x \in [0,1]$. A complete characterization of idempotent uninorms can be found in [8,27].

Now, we recall the definition of uninorms with continuous underlying operators which were introduced in [9,28].

Definition 5. A uninorm $U$ is called with continuous underlying operators if the underlying t-norm $T_U$ and t-conorm $S_U$ are both continuous.

Denote $COU$ the class of uninorms with continuous underlying operators. It is obvious that any uninorm $U \in CU_{\min}$ (or $U \in CU_{\max}$) belongs to the class $COU$. There exist some uninorms $U \in COU$ which are not continuous in $]0,1[^2$ (see examples in Section 4).

3. The structure of some cases of uninorms with continuous underlying operators

In this section, we will focus on the structure of uninorms with continuous underlying operators and obtain the full characterization of uninorms of which either the underlying t-norm or t-conorm is strict.

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Recently, a characterization of the class of uninorms with strict underlying \( t \)-norm and \( t \)-conorm was proved in [13].

**Theorem 3.** Let \( U \) be a uninorm with neutral element \( e \in [0, 1] \) such that \( T_U \) is strict and \( S_U \) is strict. Then one of the following seven statements holds:

\[
\begin{align*}
\text{(i)} & \quad U \in U_{\text{min}}, \\
\text{(ii)} & \quad U(x, y) = \begin{cases} 
  eT_U\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x, y) \in [0, e]^2, \\
  e + (1 - e)S_U\left(\frac{x - e}{1 - e}, \frac{y - e}{1 - e}\right) & \text{if } (x, y) \in [e, 1]^2, \\
  1 & \text{if } x = 1 \text{ or } y = 1, \\
  \min(x, y) & \text{otherwise.}
\end{cases} \\
\text{(iii)} & \quad U(x, y) = \begin{cases} 
  eT_U\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x, y) \in [0, e]^2, \\
  e + (1 - e)S_U\left(\frac{x - e}{1 - e}, \frac{y - e}{1 - e}\right) & \text{if } (x, y) \in [e, 1]^2, \\
  1 & \text{if } x = 1, y \neq 0 \text{ or } x \neq 0, y = 1, \\
  \min(x, y) & \text{otherwise.}
\end{cases} \\
\text{(iv)} & \quad U \in U_{\text{max}}, \\
\text{(v)} & \quad U(x, y) = \begin{cases} 
  eT_U\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x, y) \in [0, e]^2, \\
  e + (1 - e)S_U\left(\frac{x - e}{1 - e}, \frac{y - e}{1 - e}\right) & \text{if } (x, y) \in [e, 1]^2, \\
  0 & \text{if } x = 0 \text{ or } y = 0, \\
  \max(x, y) & \text{otherwise.}
\end{cases} \\
\text{(vi)} & \quad U(x, y) = \begin{cases} 
  eT_U\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x, y) \in [0, e]^2, \\
  e + (1 - e)S_U\left(\frac{x - e}{1 - e}, \frac{y - e}{1 - e}\right) & \text{if } (x, y) \in [e, 1]^2, \\
  0 & \text{if } x = 0, y \neq 1 \text{ or } x \neq 1, y = 0, \\
  \max(x, y) & \text{otherwise.}
\end{cases} \\
\text{(vii)} & \quad U \text{ is representable.}
\end{align*}
\]

**Proof.** To prove the result, Lemma 2 and Corollary 1 in [13] imply that we need to distinguish three cases: \( \min(x, y) < U(x, y) < \max(x, y) \) for all \( x \in [0, e[ \text{ and } y \in ]e, 1[; U(x, y) = \min(x, y) \) for all \( x \in [0, e[ \text{ and } y \in ]e, 1[; U(x, y) = \max(x, y) \) for all \( x \in ]0, e[ \text{ and } y \in ]e, 1[. Theorem 4 in [13] implies that \( U \) is a representable uninorm.

Case 1: \( \min(x, y) < U(x, y) < \max(x, y) \) for all \( x \in [0, e[ \text{ and } y \in ]e, 1[. First, we prove that \( U(1, x) = 1 \text{ or } x \text{ for all } x \in ]0, e[. \) On the contrary, suppose that there exists \( t \in ]0, e[ \text{ such that } t < U(1, t) < 1. \) By the associativity of \( U \), we have

\[
U(1, t) = U(U(1, 1), t) = U(1, U(1, t)).
\]

Hence, by the structure of \( U \), we have \( U(1, t) < e. \) Denote \( c = U(1, t). \) We have \( t = \min(1, t) \leq c < e. \)

By induction, Eq. (13) can be rewritten as
\[
c = U(1, c) = U(1, U(c, \ldots, c))
\]
for any positive integer \( n \). Since \( c < e \), we have \( \lim_{n \to \infty} U(c, \ldots, c) = 0 \) by the structure of \( T_U \). Hence, by the monotonicity of \( U \), we have \( U(z, 1) = c = U(t, 1) \) for any \( z \in [0, c] \). The strict monotonicity of \( T_U \) implies that \( 0 < U(t, t) < t \leq c \). By the associativity of \( U \), we have
\[
U(U(1, t), t) = U(1, U(t, t)) = c = U(1, t) = U(U(1, t), e),
\]
a contradiction with the strict monotonicity of \( T_U \). Hence \( U(1, x) = x \) or 1 for all \( x \in ]0, e[ \). Suppose that there exist \( x_1, x_2 \in ]0, e[ \) such that \( x_1 \neq x_2 \), \( U(1, x_1) = x_1 \) and \( U(1, x_2) = 1 \). Then the monotonicity of \( U \) implies that \( U(1, x) = x \) for all \( x \in ]0, 1[ \). By the strict monotonicity of \( T_U \), we have \( 0 < U(x_1, x_2) < U(x_1, e) = x_1 \). The associativity and commutativity of \( U \) imply that
\[
U(x_1, x_2) = U(U(1, x_1), x_2) = U(U(1, x_2), x_1) = U(1, x_1) = x_1,
\]
a contradiction. Hence \( U(1, x) = 1 \) for all \( x \in ]0, e[ \) or \( U(1, x) = x \) for all \( x \in ]0, e[ \).

If \( U(1, x) = x \) for all \( x \in ]0, e[ \), then \( U(1, 0) = 0 \) by the monotonicity of \( U \). Hence \( U(x, y) = \min(x, y) \) for all \( (x, y) \in ]0, 1[^2 \setminus (0, e]^2 \cup [e, 1]^2 \), i.e., \( U \in U_{\min} \).

If \( U(1, x) = 1 \) for all \( x \in ]0, e[ \), then \( U(1, 0) = 0 \) or \( U(1, 0) = 1 \). Hence, \( U \) is with form (9) or with form (10).

**Remark 3.** Theorem 5 in [13] points that the uninorm with strict underlying t-norm and t-conorm belongs to \( U_{\min} \) or \( U_{\max} \) or the class of representable uninorms. However, Theorem 3 implies that such a uninorm may either be with form (6), or be with form (7), or it is with form (8), or it is with form (9).

Now, we discuss the structure of uninorm \( U \in CQU \) with strict underlying t-norm and nilpotent underlying t-conorm.

**Lemma 1.** Let \( U \) be a uninorm with neutral element \( e \in ]0, 1[ \) such that \( T_U \) is strict and \( S_U \) is nilpotent.

(i) If \( U(x_0, y_0) = \min(x_0, y_0) \) for some \( x_0 \in ]0, e[ \) and some \( y_0 \in ]e, 1[ \), then \( U(x, y_0) = \min(x, y_0) \) for all \( x \in ]0, e[ \).

(ii) If \( U(x_0, y_0) = \max(x_0, y_0) \) for some \( x_0 \in ]0, e[ \) and some \( y_0 \in ]e, 1[ \), then \( U(x_0, y) = \max(x_0, y) \) for all \( y \in ]e, 1[ \).

**Proof.** We prove statement (ii) only. Suppose that \( U(x_0, y_0) = \max(x_0, y_0) \) for some \( x_0 \in ]0, e[ \) and \( y_0 \in ]e, 1[ \), and hence \( U(x_0, y_0) = y_0 \). Consider \( y_1 \) such that \( y_0 < y_1 < 1 \). Due to the continuity of \( S_U \), there exists \( t_1 \in ]e, 1[ \) such that \( y_1 = U(t_1, y_0) \). The associativity of \( U \) now allows to write
\[
U(x_0, y_1) = U(x_0, U(t_1, y_0)) = U(t_1, U(x_0, y_0)) = U(t_1, y_0) = y_1 = \max(x_0, y_1).
\]

The monotonicity of \( U \) implies that \( U(x_0, 1) = \max(x_0, 1) = 1 \). Hence, the statement holds for any \( y \in ]y_0, 1[ \).

Next, consider \( y_2 \) such that \( e < y_2 < y_0 \). Again due to the continuity of \( S_U \), there exists \( t_2 \in ]e, 1[ \) such that \( y_0 = U(t_2, y_2) \). Contrary to the statement, suppose that \( U(x_0, y_2) \neq y_2 \). We need to distinguish two cases: \( U(x_0, y_2) \leq e \) and \( e < U(x_0, y_2) < y_2 \).

Case 1: \( U(x_0, y_2) \leq e \). Then the associativity and monotonicity of \( U \) and the structure of \( S_U \) imply that
\[
y_0 = U(x_0, y_0) = U(x_0, U(t_2, y_2)) = U(t_2, U(x_0, y_2)) \leq U(t_2, e) < U(t_2, y_2) = y_0 < 1,
\]
a contradiction.

Case 2: \( e < U(x_0, y_2) < y_2 \). Then the associativity of \( U \) and the structure of \( S_U \) imply that
\[
y_0 = U(x_0, y_0) = U(x_0, U(t_2, y_2)) = U(t_2, U(x_0, y_2)) < U(t_2, y_2) = y_0 < 1,
\]
a contradiction.

Hence, \( U(x_0, y_2) = y_2 \). Therefore, the statement holds for any \( y \in ]e, y_0[ \).

**Lemma 2.** Let \( U \) be a uninorm with neutral element \( e \in ]0, 1[ \) such that \( T_U \) is strict and \( S_U \) is nilpotent.
(i) If \( U(x_0, y_0) = \min(x_0, y_0) \) for some \( x_0 \in [0, 1] \) and some \( y_0 \in [0, 1] \), then \( U(x, y) = \min(x, y) \) for all \( x \in [0, e] \) and \( y \in [e, 1] \).

(ii) If \( U(x_0, y_0) = \max(x_0, y_0) \) for some \( x_0 \in [0, 1] \) and some \( y_0 \in [0, 1] \), then \( U(x, y) = \max(x, y) \) for all \( x \in [0, e] \) and \( y \in [e, 1] \).

Proof. We prove statement (ii) only. Suppose that \( U(x_0, y_0) = \max(x_0, y_0) \) for some \( x_0 \in [0, e] \) and \( y_0 \in [e, 1] \). Due to Lemma 1, it follows that \( U(x_0, y) = \max(x_0, y) = y \) for all \( y \in [e, 1] \). Since \( U(e, y) = y \), it then holds that \( U(x, y) = y \) for all \( x \in [x_0, e] \) and \( y \in [e, 1] \). Hence, there exists \( d \in [0, x_0] \) such that \( U(x, y) = \max(x, y) = y \) for all \( x \in [d, e] \) and \( y \in [e, 1] \).

Contrary to the statement, suppose that there exists \( a_0 \in [0, e] \) such that \( U(a_0, y_0) < y_0 \), i.e., \( d > 0 \). We need to distinguish two cases: \( U(a_0, y_0) \leq e \) and \( e < U(a_0, y_0) < y_0 \). Note that the associativity and commutativity of \( U \) imply that

\[
U(a_0, y_0) = U\left(a_0, U(x_0, y_0)\right) = U\left(x_0, U(a_0, y_0)\right). \tag{11}
\]

Case 1: \( U(a_0, y_0) \leq e \). The structure of \( T_U \) implies that \( U(a_0, y_0) = 0 \). However, by (1), it also holds that \( U(a_0, y_0) \geq \min(a_0, y_0) = a_0 > 0 \), a contradiction.

Case 2: \( e < U(a_0, y_0) < y_0 \). Denote \( t_0 = U(a_0, y_0) \). Eq. (11) can be rewritten as \( t_0 = U(x_0, t_0) \). By induction, we have

\[
t_0 = U\left(U\left(x_0, \ldots, x_0, t_0\right)\right)
\]

for any positive integer \( n \). Since \( x_0 < e \), the strict monotonicity of \( T_U \) implies that the sequence

\[
u_n := U\left(x_0, \ldots, x_0\right)
\]

is strictly decreasing and tends to 0. This implies that \( U(z, t_0) = t_0 = \max(z, t_0) \) for all \( z \in [0, e] \). From Lemma 1, it then follows that \( U(z, y) = \max(z, y) \) for all \( z \in [0, e] \) and \( y \in [e, 1] \), a contradiction with the assumption \( U(a_0, y_0) < y_0 \). \( \square \)

Theorem 4. Let \( U \) be a uninorm with neutral element \( e \in [0, 1] \) such that \( T_U \) is strict and \( S_U \) is nilpotent. Then one of the following three statements holds:

(i) \( U \in U_{\text{min}} \).

(ii) \( U \in U_{\text{max}} \).

(iii)

\[
U(x, y) = \begin{cases} 
eq T_U \left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x, y) \in [0, e]^2, \\ e + (1 - e)S_U \left(\frac{x}{1-e}, \frac{y}{1-e}\right) & \text{if } (x, y) \in [e, 1]^2, \\ \max(x, y) & \text{if } x = 0 \text{ or } y = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{12}
\]

Proof. For arbitrary \( x_0 \in [0, e] \), we distinguish two cases: \( U(1, x_0) = 1 \) and \( U(1, x_0) < 1 \). Denote \( d = \inf\{z \in [e, 1] : U(z, z) = 1\} \). Then \( e < d < 1 \).

Case 1: \( U(1, x_0) = 1 \). By the associativity of \( U \), we have

\[
1 = U(1, x_0) = U\left(U(d, d), x_0\right) = U\left(U(x_0, d), d\right).
\]

Hence, \( U(d, x_0) = U(x_0, d) \geq d \). Since \( U(d, x_0) \leq \max(x_0, d) = d \), we have \( U(d, x_0) = d \). By Lemma 2, we have \( U(x, y) = \max(x, y) \) for all \( x \in [0, e] \) and \( y \in [e, 1] \).

First, we prove that \( U(0, x) = 0 \) or \( x \) for all \( x \in [e, 1] \). On the contrary, suppose that there exists \( t \in [e, 1] \) such that \( 0 = \min(0, t) < U(0, t) < \max(0, t) = t \). Due to the associativity of \( U \), we have


\[ U(0, t) = U(U(0, 0), t) = U(0, U(0, t)). \]

Hence \( U(0, t) > e \). Indeed, on the contrary, if \( U(0, t) \leq e \) then \( U(0, U(0, t)) = 0 \), a contradiction. Denote \( c = U(0, t) \). Then \( e < c < t \). By induction, we have

\[ c = U(0, c) = U(0, U(c, \ldots, c)) \]

for any positive integer \( n \). Since \( c > e \), there exists a positive integer \( n_0 \) such that \( U(c, \ldots, c) = 1 \). Hence \( U(0, 1) = c \in [e, t[ \), a contradiction with the fact that \( U(0, 1) \in \{0, 1\} \). Hence \( U(0, x) = 0 \) or \( x \) for all \( x \in ]e, 1[ \). Suppose that there exist \( x_1, x_2 \in ]e, 1[ \) such that \( x_1 < x_2 \), \( U(0, x_1) = 0 \) and \( U(0, x_2) = x_2 \). The strict monotonicity of \( S_U \) implies that \( U(x_2, x_1) > U(x_2, e) = x_2 \). By the commutativity and associativity of \( U \), we have

\[ x_2 = U(0, x_2) = U(U(0, x_1), x_2) = U(U(0, x_2), x_1) = U(x_2, x_1) > x_2, \]

a contradiction. Hence \( U(0, x) = 0 \) for all \( x \in ]e, 1[ \) or \( U(0, x) = x \) for all \( x \in ]e, 1[ \).

If \( U(0, x) = x \) for all \( x \in ]e, 1[ \), then \( U(0, 1) = 1 \) by the monotonicity of \( U \). Hence \( U(x, y) = \max(x, y) \) for all \( (x, y) \in [0, 1]^2 \setminus ([0, e]^2 \cup [e, 1]^2) \), i.e., \( U \in U_{\text{max}} \).

If \( U(0, x) = 0 \) for all \( x \in ]e, 1[ \), then we only need to prove that \( U(0, 1) = 0 \). On the contrary, suppose that \( U(0, 1) = 1 \). By the associativity of \( U \), we have

\[ 1 = U(0, 1) = U(0, U(d, d)) = U(U(0, d), d) = U(0, d) = 0, \]

a contradiction. Hence, \( U(x, y) = \max(x, y) \) for all \( x \in ]0, e[ \) and \( y \in ]e, 1[ \), and \( U(0, x) = 0 \) for all \( x \in ]e, 1[ \), i.e., \( U \) is with form (12).

Case 2: \( U(1, x_0) < 1 \). By the associativity of \( U \), we have

\[ 1 > U(1, x_0) = U(U(1, d), x_0) = U(U(1, x_0), d). \]

Hence \( 0 < \min(1, x_0) \leq U(1, x_0) < e \). Indeed, on the contrary, if \( U(1, x_0) \geq e \) then \( U(U(1, x_0), d) > U(1, x_0) \) by the structure of \( S_U \). So \( U(U(1, x_0), d) = U(1, x_0) = \min(U(1, x_0), d) \). By Lemma 2, we have that \( U(x, y) = \min(x, y) \) for all \( x \in ]0, e[ \) and \( y \in ]e, 1[ \).

Now, we prove that \( U(1, x) = x \) for all \( x \in ]0, e[ \). On the contrary, suppose that there exists \( t \in ]0, e[ \) such that \( U(1, t) > t \). The associativity of \( U \) and strict monotonicity of \( T_U \) imply that

\[ U(U(d, t), U(d, t)) = U(U(d, d), U(t, t)) = U(U(1, t), t) = U(1, U(t, t)) > U(t, t). \]

Hence \( U(t, d) = U(d, t) > t \), a contradiction. Therefore, \( U(1, x) = x \) for all \( x \in ]0, e[ \). By the monotonicity of \( U \), we have \( U(0, 1) = U(1, 0) = 0 \). So, \( U(x, y) = \min(x, y) \) for all \( (x, y) \in [0, 1]^2 \setminus ([0, e]^2 \cup [e, 1]^2) \), i.e., \( U \in U_{\text{min}} \).

Similarly, we have the following result.

Theorem 5. Let \( U \) be a uninorm with neutral element \( e \in ]0, 1[ \) such that \( T_U \) is nilpotent and \( S_U \) is strict. Then one of the following three statements holds:

(i) \( U \in U_{\text{max}} \).
(ii) \( U \in U_{\min} \).
(iii) \[
U(x, y) = \begin{cases} 
\frac{x}{e}T_U\left(\frac{1}{e}, y\right) & \text{if } (x, y) \in [0, e]^2, \\
\frac{y}{e}T_U\left(\frac{x}{e}, \frac{1}{e}\right) & \text{if } (x, y) \in [e, 1]^2, \\
1 & \text{if } x = 1 \text{ or } y = 1, \\
\min(x, y) & \text{otherwise.}
\end{cases}
\]

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Remark 4. Theorem 3 in [19] points out that the following operators are uninorms.

\[
U(x, y) = \begin{cases} 
  eT(x, y) & \text{if } (x, y) \in [0, e)^2, \\
  e + (1 - e)S\left( \frac{x - e}{1 - e}, \frac{y - e}{1 - e} \right) & \text{if } (x, y) \in [e, 1)^2, \\
  \min(x, y) & \text{otherwise,} 
\end{cases}
\]

and

\[
U(x, y) = \begin{cases} 
  eT(x, y) & \text{if } (x, y) \in [0, e)^2, \\
  e + (1 - e)S\left( \frac{x - e}{1 - e}, \frac{y - e}{1 - e} \right) & \text{if } (x, y) \in [e, 1)^2, \\
  0 & \text{if } x = 0 \text{ or } y = 0, \\
  \max(x, y) & \text{otherwise,} 
\end{cases}
\]  

(13)  

(14)

where \( T \) is any t-norm and \( S \) is any t-conorm. However, Theorem 4 implies that the operator having form (13) is not a uninorm when \( T \) is a strict t-norm and \( S \) is a nilpotent t-conorm and Theorem 5 implies that the operator with form (14) is not a uninorm when \( T \) is a nilpotent t-norm and \( S \) is a strict t-conorm.  

4. Distributivity and conditional distributivity of a uninorm with continuous underlying operators over a continuous t-conorm

Definition 6. (See [25] ) Let \( U \) be a uninorm with neutral element \( e \in [0, 1] \) and \( S \) be a t-conorm. Then we say that \( U \) is conditionally distributive over \( S \) if, for all \( x, y, z \in [0, 1] \)

\[
U(x, S(y, z)) = S(U(x, y), U(x, z)) \quad \text{whenever } S(y, z) < 1. 
\]

(15)

Definition 7. (See [25] ) Let \( U \) be a uninorm with neutral element \( e \in [0, 1] \) and \( S \) be a t-conorm. Then we say that \( U \) is distributive over \( S \) if for all \( x, y, z \in [0, 1] \)

\[
U(x, S(y, z)) = S(U(x, y), U(x, z)). 
\]

(16)

We summarize some fundamental results from [25].

Proposition 2. (See [25] ) Let \( U \) be a uninorm with neutral element \( e \in ]0, 1[ \) and \( S \) be a t-conorm. Then \( U \) is conditionally distributive (or distributive) over \( S \) fulfilling \( S(e, e) = e \) if and only if \( S = S_M \).

Theorem 6. (See [25] ) A uninorm \( U \) with neutral element \( e \in ]0, 1[ \) which either is idempotent, or is in \( U_{\max} \) or is in \( U_{\min} \), or it is representable, or it is continuous in \( ]0, 1[^2 \), is distributive over a continuous t-conorm \( S \) if and only if \( U \) is conditionally distributive over \( S \).

Theorem 7. (See [25] ) Let \( U \) be a uninorm with neutral element \( e \in ]0, 1[ \) which either is idempotent, or is in \( U_{\max} \) or is in \( U_{\min} \), or it is representable, or it is continuous in \( ]0, 1[^2 \), and \( S \) be a continuous t-conorm. The following statements are equivalent.

(i) \( U \) is conditionally distributive (or distributive) over \( S \).
(ii) We have either one of the following cases.
   a) \( S = S_M \)
   b) \( S \) is strict and \( U \) is representable, and if \( s \) is the additive generator of \( S \) satisfying \( s(e) = 1 \), then \( s \) is also a multiplicative generator of \( U \).
   c) \( U = (e, u, \lambda, T_1, T_2, R) \) is in \( C U_{\min} \) and there exists a strict t-conorm \( S^* \) such that its additive generator \( s \) with \( s\left( \frac{\lambda - u}{\lambda} \right) = 1 \) is also a multiplicative generator of \( R \), and \( S = (1, u, S^*) \).
   d) \( U = (e, y, \delta, R, S_1, S_2) \) is in \( C U_{\max} \) and there exists a strict t-conorm \( S^* \) such that its additive generator \( s \) with \( s\left( \frac{\gamma}{\gamma} \right) = 1 \) is also a multiplicative generator of \( R \), and \( S = (0, y, S^*) \).
Now, we discuss the conditional distributivity of a uninorm $U \in \mathcal{COU}$ over a continuous t-conorm $S$. From Proposition 2, we will focus on the case: the neutral element $e$ of $U$ is not an idempotent element of t-conorm $S$.

**Lemma 3.** Let $U \in \mathcal{COU}$ be a uninorm with neutral element $e \in ]0, 1]$ and $S$ be a continuous t-conorm. If $U$ is conditionally distributive over $S$ fulfilling $S(e, e) > e$, then $S$ has only one ordinal summand.

**Proof.** Suppose $S = \{(a_k, b_k, S_k)\}_{k \in K}$. Since $e$ is not the idempotent element of $S$, there exists $k \in K$ such that $e \not\in \{a_k, b_k\}$. To prove the result, we need to distinguish two cases: $b_k < 1$ and $b_k = 1$.

Case 1: $b_k < 1$. Then $a_k = S(a_k, a_k) < b_k < 1$, $S(b_k, b_k) = b_k < 1$.

Applying $y = z = a_k$ in (15), we have

$$U(x, a_k) = U(x, S(a_k, a_k)) = S(U(x, a_k), U(x, a_k))$$

for all $x \in [0, 1]$. So, $U(x, a_k)$ is an idempotent element of $S$ for all $x \in [0, 1]$. As $T_U$ is continuous and $U(0, a_k) = 0$, $U(e, a_k) = a_k$, every element in $[0, a_k]$ is an idempotent element of $S$.

Applying $y = z = b_k$ in (15), we have

$$U(x, b_k) = U(x, S(b_k, b_k)) = S(U(x, b_k), U(x, b_k))$$

for all $x \in [0, 1]$. So, $U(x, b_k)$ is an idempotent element of $S$ for all $x \in [0, 1]$. As $S_U$ is continuous and $U(e, b_k) = b_k$, $U(1, b_k) = 1$, every element in $[b_k, 1]$ is an idempotent element of $S$. Hence, $S = \{(a_k, b_k, S_k)\}$.

Case 2: $b_k = 1$. Then $S = \{(a_k, 1, S_k)\}$ from above discussion.  

According to the proof of Lemma 3, we have the following corollary.

**Corollary 1.** Let $U \in \mathcal{COU}$ be a uninorm with neutral element $e \in ]0, 1]$ and $S$ be a continuous t-conorm. If $U$ is conditionally distributive over $S$ and $a$ is an idempotent element of $S$, then one of the following statements holds.

(i) If $a \leq e$ then every element of $[0, a]$ is an idempotent element of $S$.

(ii) If $a \geq e$ then every element of $[a, 1]$ is an idempotent element of $S$.

According to Lemma 3, we only need to discuss continuous t-conorm $S$ with only one ordinal summand. We have the following result which also appears in [15]. For the completeness of the paper, we present the proof here.

**Theorem 8.** Let $U \in \mathcal{COU}$ be a uninorm with neutral element $e \in ]0, 1]$ and $S = \{(a, b, S^*)\}$ be a continuous t-conorm. If $U$ is conditionally distributive over $S$, then the set $[a, b]^2$ is closed under $U$.

**Proof.** If $a = 0$ and $b = 1$ then the result holds. If $a > 0$ or $b < 1$ then we only need prove $U(a, a) = a$, $U(b, b) = b$ by the monotonicity of $U$. Notice that $a < e < b$ by Lemma 3. Now, we distinguish two cases: $S^*$ is either strict or nilpotent.

Case 1: $S^*$ is strict. Applying $x = b$, $y = z = e$ in (15), as $S(e, e) < S(b, b) \leq 1$, we have

$$U(b, S(e, e)) = S(U(b, e), U(b, e)) = U(b, e) = b$$

because $U(b, e) = b > e$ and Corollary 1.

By induction, we get

$$b = U(b, e) = U(b, e^2) = U(b, e^{2n})$$

for any positive integer $n$, where $e^{2n} = S(e, \ldots, e)$. Since $\lim_{n \to \infty} e^{2n} = b$ and $S_U$ is continuous, $b = U(b, b)$.

Applying $x = a$, $y = z = e$ in (15), we have

$$U(a, S(e, e)) = S(U(a, e), U(a, e)) = U(a, e) = a$$

because $U(a, a) = e < e$ and Corollary 1.
By induction, we get
\[ a = U(a, e) = U((e_S^{1/2})_S^{(2)}) = U(a, e_S^{1/2}) = U(a, e_S^{(2-n)}) \]
for any positive integer \( n \), where \( e_S^{(2-n)} = \inf\{z \in [0, 1] : z^{(2^n)} > e\} \). Since \( \lim_{n \to \infty} e_S^{(2-n)} = a \) and \( T_U \) is continuous, \( a = U(a,a) \).

Case 2: \( S^* \) is nilpotent. Denote \( c = \inf\{z \in [0, 1] : S(z, z) = 1\} \).

- If \( c < 1 \) then \( b = 1 \) and \( U(b, b) = b = 1 \).
- If \( c = 1 \) then \( b < 1 \). Applying \( x = b, y = z = e \) in (15), we have
  \[ U(b, S(e, e)) = U(U(b, e), U(b, e)) = S(b, b) = b. \]
  - If \( S(e, e) = b \) then \( U(b, b) = b \).
  - If \( S(e, e) < b \) then by induction, we have
    \[ b = U(b, S(e, e)) = U(b, U(e_S^{(2)}, e_S^{(2)})) = U(b, U(e_S^{(2)}, \ldots, e_S^{(2)})) \]
    for any positive integer \( n \). Denote \( d = \inf\{z \in [e, 1] : U(z, z) = z\} \). Since \( S_U \) is continuous, we have \( d = \min\{z \in [e, 1] : U(z, z) = z\} \). If \( b \leq d \) then there exists a positive integer \( n_0 \) such that \( U(e_S^{(2)}, \ldots, e_S^{(2)}) = d \)
    by the structure of \( S_U \). Hence, \( b = U(b, S(e, e)) = U(b, d) \). By the continuity of \( S_U \), we have \( b = U(b, b) \).

Now, we prove \( U(a, a) = a \).

- If \( e < c \) then \( S(e, e) < 1 \). We can get \( U(a, a) = a \) as in Case 1.
- If \( c \leq e \) then \( S(x, x) < 1 \) for all \( x < c \). Applying \( x = a, y = z = d \) in (15), where \( a < d < c \leq e \), we have
  \[ U(a, S(d, d)) = S(U(a, d), U(a, d)) = U(a, d) \]
  because \( S(d, d) < S(c, c) = 1, U(a, d) \leq a \) and Corollary 1.

By induction, we get
\[ U(a, S(d, d)) = U(a, d) = U(a, (d_S^{1/2})_S^{(2)}) = U(a, d_S^{1/2}) = U(a, d_S^{(2-n)}) \]
for any positive integer \( n \). Since \( \lim_{n \to \infty} d_S^{(2-n)} = a \) and \( T_U \) is continuous, we have
\[ U(a, S(d, d)) = U(a, d) = U(a, a). \quad (17) \]
Since \( d_S^{(2)} = a, e_S^{(2)} = 1, e_S^{(1/2)}_S \leq e < 1 \), by the monotonicity of \( S \), we have \( a < e_S^{(1/2)} < c \). Taking \( d = e_S^{(1/2)} \) in (17), we have \( U(a, S(e_S^{(1/2)}, e_S^{(1/2)})) = U(a, a) \). Hence \( U(a, a) = U(a, e) = a \). \( \square \)

Now, we characterize the uninorm \( U \in COU \) which is conditionally distributive over a strict t-conorm \( S \).

**Lemma 4.** Let \( U \in COU \) be a uninorm with neutral element \( e \in [0, 1] \) and \( S \) be a continuous t-conorm. If \( U \) is conditionally distributive over \( S \) and \( U \) has an idempotent element \( a \in [0, e[ \), then every element of \( [0, a] \) is an idempotent element of \( S \).
Proof. If $S(e, e) = e$, then $S = S_M$ by Proposition 2, and then the result is true. Otherwise, $S(e, e) > e$ and as $S$ is continuous, there exists $z \in (0, e]$ such that $S(z, z) = e$. Suppose $\alpha \in (0, e]$ such that $U(\alpha, \alpha) = \alpha$. To prove the result, we need to distinguish two cases: $S(\alpha, \alpha) \geq e$ and $S(\alpha, \alpha) < e$.

Case 1: $S(\alpha, \alpha) \geq e$. Then $z \leq \alpha < e$. Applying $x = y = z$ in (15), we have

$$\alpha = U(\alpha, S(z, z)) = S(U(\alpha, z), U(\alpha, z)) = S(\min(\alpha, z), \min(\alpha, z)) = S(z, z) = e,$$

a contradiction.

Case 2: $S(\alpha, \alpha) < e$. Then $\alpha < z < e$. Applying $x = \alpha, y = z$ in (15), we have

$$\alpha = U(\alpha, S(z, z)) = S(U(\alpha, z), U(\alpha, z)) = S(\min(\alpha, z), \min(\alpha, z)) = S(\alpha, \alpha).$$

$\alpha$ is an idempotent element of $S$. Due to Corollary 1, the result holds. \qed

**Theorem 9.** Let $U \in \mathcal{C}(\mathcal{U})$ be a uninorm with neutral element $e \in (0, 1]$ and $S$ be a strict $t$-conorm. Then $U$ is conditionally distributive over $S$ if and only if $U$ is a representable uninorm which multiplicative generator is also an additive generators of $S$.

Proof. In view of Theorem 7, the sufficiency is obvious.

Conversely, if $U$ is conditionally distributive over $S$ then there exists no idempotent element of $U$ in $(0, e]$ by Lemma 4. Hence, $T_U$ is strict or nilpotent.

First, we prove that $T_U$ is strict. On the contrary, suppose that $T_U$ is nilpotent. Denote $\beta = \sup\{z \in [0, e] : U(z, z) = 0\}$. Then $0 < \beta < e$ and $\beta < S(\beta, \beta) < 1$ by the strict monotonicity of $S$. Applying $x = y = z = \beta$ in (15), we have

$$0 < U(\beta, S(\beta, \beta)) = S(U(\beta, \beta), U(\beta, \beta)) = S(0, 0) = 0,$$

a contradiction.

Second, we prove that $S_U$ is strict. On the contrary, suppose that $S_U$ is nilpotent or an ordinal sum. In the following, we distinguish two cases: $S_U$ is nilpotent or $S_U$ is an ordinal sum.

Case 1: $S_U$ is nilpotent. Denote $\alpha = \inf\{z \in [e, 1] : U(z, z) = 1\}$. Then $e < \alpha < 1$. Since $S$ is strict, there exists $\gamma \in (0, e]$ such that $\gamma < \alpha < S(\gamma, \gamma) < 1$ and $U(\alpha, \gamma) < 1$. Applying $x = \alpha, y = z = \gamma$ in (15), we have

$$1 = U(\alpha, S(\gamma, \gamma)) = S(U(\alpha, \gamma), U(\alpha, \gamma)) < 1,$$

a contradiction.

Case 2: $S_U$ is an ordinal sum. Then there exists an idempotent element $a \in (e, 1]$ of $U$. Applying $x = y = a, z = e$ in (15), we have

$$S(a, e) = \max(a, S(a, e)) = U(a, S(a, e)) = S(U(a, a), U(a, e)) = S(a, a).$$

This is possible only if $S(a, e) = S(a, a) = 1$, which leads to a contradiction.

Hence, both $T_U$ and $S_U$ are strict from above discussion. By Theorem 3, we can divide our proof in some cases.

- $U \in \mathcal{U}_{\min}$ or $U \in \mathcal{U}_{\max}$. Then Theorem 7 implies that $U$ is only conditionally distributive over $S = S_M$.
- $U$ is with form (6) or with form (7). Then $U$ is only conditionally distributive over $S = S_M$. Indeed, on the contrary, if $S(e, e) > e$, then there exists $0 < t < e$ such that $S(t, t) = e < 1$ since $S$ is continuous. Applying $y = z = t$ in (15), we have

$$x = U(x, S(t, t)) = S(U(x, t), U(x, t))$$

for all $x \in [0, 1]$. Taking $x$ verifying $t < e < x < 1$, we have

$$x = S(U(x, t), U(x, t)) = S(\min(x, t), \min(x, t)) = S(t, t) = e.$$

But this is a contradiction. Hence, $(e, e) = e$, and by Proposition 2 one has $S = S_M$. 

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U has form (8) or form (9). Then U is only conditionally distributive over \( S = S_M \). Indeed, as S is strict, \( S(e, e) < 1 \) and there exists \( t \in [0, 1] \) such that \( 0 < t < e < S(t, t) \). Applying \( x = t, y = z = e \) in (15), we have

\[
U(t, S(e, e)) = S(U(t, e), U(t, e)) = S(t, t).
\]

If \( S(e, e) > e \) then \( S(e, e) = \max(t, S(e, e)) = U(t, S(e, e)) = S(t, t) \). This is possible only if \( t = e \) which leads to a contradiction. So, \( S(e, e) = e \). By Proposition 2, we have \( S = S_M \).

U is a representable uninorm. The result holds by Theorem 7.

By Theorem 7 and Theorem 9, we have the following corollary.

**Corollary 2.** Let \( U \in COU \) be a uninorm with neutral element \( e \in [0, 1] \) and \( S \) be a strict t-conorm. Then \( U \) is conditionally distributive over \( S \) if and only if \( U \) is distributive over \( S \).

Now, we characterize the uninorms \( U \in COU \) which are conditionally distributive over a nilpotent t-conorm \( S \).

**Theorem 10.** Let \( S \) be a nilpotent t-conorm. There exists no uninorm \( U \in COU \) with neutral element \( e \in [0, 1] \) such that \( U \) is conditionally distributive over \( S \).

**Proof.** In order to prove the result, we divide our proof in four steps.

Step 1: By Lemma 4, we know that there exists no idempotent element of \( U \) in \( [0, e[ \). Hence, \( T_U \) is strict or nilpotent. We prove that \( T_U \) is strict. On the contrary, suppose that \( T_U \) is nilpotent. Denote \( \beta = \sup[z \in [0, e]: U(z, z) = 0] \). Then \( e > \beta > 0 \) and there exists \( z \in [0, 1] \) such that \( z < \beta < S(z, z) < 1 \) since \( S \) is nilpotent. Applying \( x = \beta, y = z \) in (15), we have

\[
0 < U(\beta, S(z, z)) = S(U(\beta, z), U(\beta, z)) = S(0, 0) = 0,
\]

which is a contradiction.

Step 2: We prove that \( S_U \) is not strict. On the contrary, suppose that \( S_U \) is strict. Then, we have the uninorm \( U \) of which the underlying t-norm \( T_U \) and t-conorm \( S_U \) are both strict. By the proof of Theorem 9, \( U \) is conditionally distributive over a continuous t-conorm \( S \) only when \( S = S_M \) or \( S \) is strict. This is a contradiction.

Step 3: We prove that \( S_U \) is not an ordinal sum. On the contrary, suppose that there exists an idempotent element \( b \in [e, 1[ \) of \( U \). From the continuity of \( S \), it then follows that there exists \( 0 < \alpha < e \) such that \( S(\alpha, \alpha) = e \). Applying \( x = \beta, y = \alpha \) in (15), we have

\[
b = U(b, S(\alpha, \alpha)) = S(U(b, \alpha), U(b, \alpha)).
\]

We divide our proof in two cases.

Case 1: \( U(b, \alpha) \geq e \). Then \( U(b, \alpha) = b \) by Lemma 5.5 in [11]. For the completeness of this paper, we present the proof here. Indeed, we have

\[
b = U(b, b) \geq U(b, t) \geq U(b, e) = b
\]

for all \( t \in [e, b] \). Thus, by the associativity of \( U \), we have

\[
U(b, \alpha) = U(U(b, b)), \alpha = U(b, U(b, \alpha)) \geq U(b, e) = b
\]

and

\[
U(b, \alpha) \leq \max(b, \alpha) = b.
\]

Hence, \( U(b, \alpha) = b \). Now, Eq. (18) can be rewritten as

\[
1 > b = U(b, S(\alpha, \alpha)) = S(U(b, \alpha), U(b, \alpha)) = S(b, b),
\]

a contradiction.

Case 2: \( U(b, \alpha) < e \). Then \( U(b, \alpha) = \alpha \). Indeed, suppose that \( e > U(b, \alpha) > \alpha \). Then there exists \( z \in [0, e[ \) such that \( U(U(b, \alpha), z) = \alpha \) by the continuity of \( T_U \). We divide our proof in two subcases.
• $U(b, z) \geq e$. By Lemma 5.5 in [11], we have $U(b, z) = b$ from above discussion. From the commutativity and associativity of $U$, we have

$$U(b, \alpha) = U(U(b, z), \alpha) = U(U(b, \alpha), z) = \alpha,$$

a contradiction.

• $U(b, z) < e$. By the commutativity and associativity of $U$, we have

$$U(U(b, z), \alpha) = U(U(b, \alpha), z) = \alpha.$$

This is possible only if $\alpha = 0$ or $U(b, z) = e$ which leads to a contradiction.

Hence, $U(b, \alpha) = \alpha$ whenever $U(b, \alpha) < e$. Eq. (18) can be read as

$$e < b = U(b, S(\alpha, \alpha)) = S(U(b, \alpha), U(b, \alpha)) = S(\alpha, \alpha) = e,$$

a contradiction.

Step 4: We prove that $SU$ is not nilpotent. On the contrary, suppose that $SU$ is nilpotent. Denote $c = \inf\{z \in [0, 1]: S(z, z) = 1\}$ and $d = \inf\{z \in [e, 1]: U(z, z) = 1\}$. Then $0 < c < 1$ and $e < d < 1$. Due to the continuity of $S$, there exists $0 < \alpha < e$ such that $S(\alpha, \alpha) = e$. Applying $\gamma = z = \alpha$ in (15), we have

$$x = U(x, S(\alpha, \alpha)) = S(U(x, \alpha), U(x, \alpha))$$

(19)

for all $x \in [0, 1]$. If there exists $\gamma \in [0, 1]$ such that $U(\gamma, \alpha) \geq c$, then applying $x = \gamma$ in (19), we have $\gamma = S(U(x, \alpha), U(x, \alpha)) \geq \gamma(c, c) = 1$, a contradiction. On the contrary, suppose that $U(x, \alpha) < c$ for all $x \in [0, 1]$. Then there exists $\delta \in [0, 1]$ such that $1 - \delta > c$. Hence $U(1 - \delta, \alpha) < c < 1 - \delta = \max(1 - \delta, \alpha)$. Theorem 4 implies that $U \in U_{\min}$. Theorem 7 implies that $U$ is only conditionally distributive over $S_M$.

Hence, there exists no uninorm $U \in COU$ such that $U$ is conditionally distributive over a nilpotent t-conorm $S$. □

By Theorem 10, we have the following corollary.

**Corollary 3.** Let $U \in COU$ be a uninorm with neutral element $e \in [0, 1]$ and $S$ be a nilpotent t-conorm. Then $U$ is conditionally distributive over $S$ if and only if $U$ is distributive over $S$.

Now, we characterize the uninorm $U \in COU$ which is conditionally distributive over a continuous t-conorm $S = (\langle a, b, S^a \rangle), a > 0 \text{ or } b < 1$.

**Proposition 3.** Let $U \in COU$ be a uninorm with neutral element $e \in [0, 1]$ and $S = (\langle a, b, S^a \rangle)$ be a continuous t-conorm, $a > 0 \text{ or } b < 1$. If $U$ is conditionally distributive over $S$, then the following statements hold.

(i) $a = \max\{z \in [0, e]: U(z, z) = z\}$.

(ii) $b = \min\{z \in [e, 1]: U(z, z) = z\}$.

**Proof.** Notice that $e \in [a, b]$ by Lemma 3.

(i) Denote $k_1 = \sup\{z \in [0, e]: U(z, z) = z\}$. Since $T_U$ is continuous, we have $k_1 = \max\{z \in [0, e]: U(z, z) = z\}$. In order to prove that $k_1 = a$, we need to distinguish two cases: $a = 0$ and $a > 0$.

Case 1: $a = 0$. Notice that Lemma 4 implies $k_1 \leq a$. Hence $a = k_1 = 0$.

Case 2: $a > 0$. On the contrary, suppose $k_1 < a$. Proposition 2 implies that $S(e, e) > e$. Lemma 3 implies that $a < e < b$. As $S$ is continuous, there exists $t \in [a, e]$ such that $S(t, t) = e$. Applying $x = a, y = z = t$ in (15), we have

$$a = U(a, e) = U(A, S(t, t)) = S(U(a, t), U(a, t)) = U(a, t) < e$$

since $U(a, t) \leq \min(a, t) = a$ and Corollary 1. This is possible only if $a$ is an idempotent element of $U$ which leads to a contradiction. Hence $a \leq k_1$. Moreover, Lemma 4 implies that $k_1 \leq a$. The result holds.
(ii) Denote $k_2 = \inf \{ z \in [e, 1] : U(z, z) = z \}$. Since $S_U$ is continuous, we have $k_2 = \min \{ z \in [e, 1] : U(z, z) = z \}$. Hence, $k_2$ is an idempotent element of $U$. In order to prove $k_2 = b$, we need to distinguish two cases: $b = 1$ and $b < 1$.

Case 1: $b = 1$. On the contrary, suppose $k_2 < b$. As $S$ is continuous, there exists $t \in [a, e]$ such that $S(t, t) = e$. Applying $x = k_2$, $y = z = t$ in (15), we have
\[
k_2 = U(k_2, e) = U(k_2, S(t, t)) = S(U(k_2, t), U(k_2, t)).
\]

We divide our proof in two subcases.

- **$U(k_2, t) < e$.** Then $U(k_2, t) = t$ whose proof is similar to Case 2 in the proof of Theorem 10. Eq. (20) can be rewritten as
\[
k_2 = S(U(k_2, t), U(k_2, t)) = S(t, t) = e,
\]
a contradiction.

- **$U(k_2, t) \geq e$.** Then $U(k_2, t) = k_2$ by Lemma 5.5 in [11]. Eq. (20) can be rewritten as
\[
k_2 = S(U(k_2, t), U(k_2, t)) = S(k_2, k_2).
\]

This is possible only if $k_2$ is an idempotent element of $S$ which leads to a contradiction with the supposition $k_2 < b$.

Case 2: $b < 1$. As in Case 1, we can prove that $k_2 < b$ is impossible. On the contrary, suppose $k_2 > b$. Notice that $e < S(e, e) \leq b < 1$. Applying $x = b$, $y = z = e$ in (15), by the ordinal sum structure of $S_U$, we have
\[
b = U(b, e) < U(b, S(e, e)) = S(U(b, e), U(b, e)) = S(b, b) = b < k_2,
\]
a contradiction. Hence $k_2 = b$. \(\square\)

**Proposition 4.** Let $U \in COU$ be a uninorm with neutral element $e \in [0, 1]$ and $S = ([a, b], S^e)$ be a continuous $t$-conorm, $a > 0$ or $b < 1$. If $U$ is conditionally distributive over $S$, then the following statements hold.

(i) There exists a representable uninorm $R$ such that $U(x, y) = a + (b - a)R(\frac{x-a}{b-a}, \frac{y-a}{b-a})$ for all $(x, y) \in [a, b]^2$ and $S^e$ is strict.

(ii) $U(x, y) = \min(x, y)$ for all $(x, y) \in [0, a] \times [a, b]$. Moreover, if $U(a, b) = a$ then $U(x, y) = \min(x, y)$ for all $(x, y) \in [0, a] \times [a, b]$.

(iii) $U(x, y) = \max(x, y)$ for all $(x, y) \in [a, b] \times [b, 1]$. Moreover, if $U(a, b) = b$ then $U(x, y) = \max(x, y)$ for all $(x, y) \in [a, b] \times [b, 1]$.

**Proof.**

(i) Theorem 9, Theorem 10 and Proposition 3 imply that statement (i) holds.

(ii) By the proof of Proposition 3, we have $U(a, a) = a$. By Lemma 3, we have $e \in [a, b]$. From the structure of continuous $T_U$, we have $U(x, y) = \min(x, y)$ for all $(x, y) \in [0, a] \times [a, e]$. By the statement (i) and Proposition 1, we have $U(a, y) = \min(a, y) = a$ for all $y \in [a, b]$. Hence, $U(x, y) = x = \min(x, y)$ for $(x, y) \in [0, a] \times \{e, b\}$ by Lemma 3.11 in [22].

Moreover, if $U(a, b) = a$ then $U(x, y) = x = \min(x, y)$ for all $(x, y) \in [0, a] \times \{e, b\}$ by Lemma 3.11 in [22]. Hence, the result holds.

(iii) Notice that Proposition 3 implies that $b$ is an idempotent element of $S_U$. The structure of continuous $S_U$ implies that $U(x, y) = \max(x, y)$ for all $(x, y) \in [e, b] \times [b, 1]$. By the statement (i) and Proposition 1, we have $U(x, b) = \max(x, b) = b$ for all $x \in [a, e]$. Hence, $U(x, y) = \max(x, y)$ for $(x, y) \in [a, e] \times [b, 1]$ by Corollary 3.12 in [22].

Moreover, if $U(a, b) = b$ then $U(x, y) = y = \max(x, y)$ for all $(x, y) \in [a, e] \times [b, 1]$ by Corollary 3.12 in [22]. Hence, the result holds. \(\square\)

**Problem.** The full characterization of all uninorms with continuous underlying operators which are conditionally distributive over a continuous $t$-conorm represented as an ordinal sum is still an open problem. From the proofs of
the above theorems, in our opinion, the answer depends on the full characterization of all uninorms of which both the underlying t-norm and underlying t-conorm are ordinal sums. In [11], Drygaś presented some properties of uninorms with underlying t-norm and t-conorm given as ordinal sums. Moreover, Drygaś characterized a class of uninorms of which the underlying operations are basic t-norms and t-conorms and one of them is idempotent. Proposition 4 implies that such uninorm is not conditionally distributive over a continuous t-conorm which is represented as an ordinal sum.

From Proposition 4, one has the following corollary.

**Corollary 4.** Let $U \in COU$ be a uninorm with neutral element $e \in [0, 1]$ and $S = (a, b, S^*)$ $(a > 0$ or $b < 1)$ be a continuous t-conorm. Then $U$ is conditionally distributive over $S$ if and only if $U$ is distributive over $S$.

**Proof.** If $U$ is distributive over $S$ then it is obviously conditionally distributive over $S$. Conversely, we only need to prove

$$U(x, S(y, z)) = S(U(x, y), U(x, z))$$

(21)

for all $x, y, z \in [0, 1]$ such that $S(y, z) = 1$. We divide our proof in two cases.

Case 1: $b = 1$. By the statement (i) in Proposition 4, we know that $S^*$ is strict. Hence, if $S(y, z) = 1$ then $y = 1$ or $z = 1$. Eq. (21) can be easily verified.

Case 2: $b < 1$. By the structure of $S$, we have that $y = 1$ or $z = 1$ if $S(y, z) = 1$. Eq. (21) can be easily verified. □

By Proposition 2 and Corollaries 1–4, we have the following result.

**Theorem 11.** Let $U \in COU$ be a uninorm with neutral element $e \in [0, 1]$ and $S$ be a continuous t-conorm. Then $U$ is conditionally distributive over $S$ if and only if $U$ is distributive over $S$.

Now, we list a few examples of uninorms with continuous underlying operators that are conditionally distributive and distributive over a continuous t-conorm.

**Example 1.** Let $U$ be a uninorm in $COU$ defined as follows

$$U(x, y) = \begin{cases} 
eq (\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\ e + (1 - e)S(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [e, 1]^2, \\ 1 & \text{if } x = 1 \text{ or } y = 1, \\ \min(x, y) & \text{otherwise}, \end{cases}$$

where $T$ and $S$ are both strict. $U \notin U_{\min}$. It is easily verified that $U$ is conditionally distributive and distributive only over $S_M$.

**Example 2.** Let $U = (e, \gamma, \delta, R, S_1, S_2)$ be a uninorm in $CU_{\max}$. $U$ is also a uninorm in $COU$ of which the underlying t-norm is strict and the underlying t-conorm is an ordinal sum. $R$ is defined as follows

$$R(x, y) = \begin{cases} 1 & \text{if } (x, y) = [0, 1], \\ \frac{\alpha xy}{(1-x)(1-y)+\alpha xy} & \text{otherwise}, \end{cases}$$

where $\alpha = \frac{\gamma - \epsilon}{\gamma - \epsilon}$. $S$ is a t-conorm defined by $S = ((0, \delta, S^*))$ where $S^* = S_0^H$ (the Hamacher t-conorm with parameter $0$, see [16]). It is easily verified that $s_\delta(x) = \frac{\alpha x}{1-x}$ is an additive generator of $S$ and a multiplicative generator of the representable uninorm $R$. As $s_\delta(\frac{x}{\gamma}) = 1$, it is easily verified that $U$ is conditionally distributive and distributive over $S$.

**Example 3.** Let us define $U \in COU$ as follows:
\[ U(x, y) = \begin{cases} 
    a + (b - a)R\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}\right) & \text{if } (x, y) \in [a, b]^2, \\
    \min(x, y) & \text{if } (x, y) \in [0, a] \times [0, b] \cup [0, b] \times [0, a], \\
    \max(x, y) & \text{otherwise}, 
\end{cases} \]

where \( R \) is a disjunctive representable uninorm with neutral element \( \frac{1}{1 + a} \) defined in the previous example and \( 0 < a < b < 1 \). \( S \) is a t-conorm defined by \( S = (a, b, S^\alpha) \) where \( S^\alpha = S^H_0 \) (the Hamacher t-conorm with parameter 0, see [16]). The additive generator \( s \) of \( S^\alpha \) satisfying \( s\left(\frac{1}{1 + a}\right) = 1 \) is also a multiplicative generator of representable uninorm \( R \). It is easy to verify that \( U \) is conditionally distributive and distributive only over \( S \) and \( S_M \).

5. Conclusions

In this paper, we have investigated the distributivity and conditional distributivity of a uninorm \( U \) with continuous underlying operators over a continuous t-conorm \( S \). Furthermore, we have characterized all uninorms of which either the underlying t-norm or underlying t-conorm is strict. We have not deal with the case when the underlying t-norm and t-conorm are both nilpotent as this case has already been solved elsewhere [18].

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