A Model for Decision Making with Missing, Imprecise, and Uncertain Evaluations of Multiple Criteria

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In real-life multicriteria decision making (MCDM) problems, the evaluations against some criteria are often missing, inaccurate, and even uncertain, but the existing theories and models cannot handle such evaluations well. To address the issue, this paper extends the Dempster–Shafer (DS)/analytic hierarchy process (DS/AHP) approach of MCDM to handle three types of ambiguous evaluations: missing, interval-valued, and ambiguous lottery evaluations. In our extension, the aggregation of criteria’s evaluation takes the following six steps: (i) calculate the expected evaluation interval and the ambiguity degree of each group of decision alternatives regarding each criterion, (ii) from them to obtain the preference degree of each group of decision alternatives, (iii) apply the DS/AHP method to obtain the mass function distribution of each group of decision alternatives, (iv) use the Dempster’s rule of combination to get the overall mass function of each group of decision alternatives with respect to all criteria, (v) according to the overall mass function to count the belief function and the plausibility function of each decision alternative, and (vi) set the overall preference ordering of decision alternatives by our regret-avoid ambiguous principle and then find the optimal solution. Finally, we give an example of real estate investment to illustrate how our approach is employed to deal with real-life MCDM problems. © 2012 Wiley Periodicals, Inc.

1. INTRODUCTION

Generally speaking, decision making can be regarded as a process that results in the selection of a course of action among several alternatives (Wikipedia). In real-life, often the decisions have to be made in the case that only limited or ambiguous information is available because of time pressure, lack of data, disturbance of unknown factors, randomness outcome of some attributes, and so on.1–3 In particular, for multicriteria decision-making (MCDM) problems, the consequence of
an action chosen with respect to some criteria often is (i) unknown; (ii) interval valued; and (iii) of multiple possibilities, even including a possible loss (i.e., risk).\textsuperscript{4} For example, the problem of real estate investment is such a MCDM problem under uncertainty and risk. In the problem, the investor’s decision is affected by different criteria such as house price, his wealth, the spillover effects of the house, the policy of the government, the economic environment, investment profit, and so on. Moreover, some criteria such as the economic environment are usually uncertain for the decision maker because its growth or recession or depression is all possible in the future. As a result, to make decisions based on ambiguous information provided by several different sources, we need a model that can assist the decision maker to rank decision alternatives regarding different criteria and properly make use of ambiguous information in decision analyzing. To this end, this paper incorporates the Dempster–Shafer (D-S) theory of evidence\textsuperscript{5, 6} with analytic hierarchy process (AHP)\textsuperscript{7} to construct such a model.

In the literature, the AHP is one of the most well-known MCDM techniques. Recently, Beynon et al.\textsuperscript{8–10} extended the AHP model to a DS/AHP method, with which a decision maker can make preference judgments on groups of decision alternatives rather on individual decision alternatives or through pairwise comparisons of decision alternatives. After that, many researchers have used the DS/AHP technique to analyze different problems in their domains. For example, Beynon et al.\textsuperscript{10} analyzed the consumer choice, and Awasthi and Chauhan\textsuperscript{11} evaluated sustainable transport solutions. Nevertheless, although Beynon et al.\textsuperscript{8–10, 12} claimed that ignorance in the judgments is allowed in DS/AHP, they did not provide any method for solving MCDM problems with some kinds of ambiguous information, such as the interval-valued evaluations for a decision alternative according to some criteria and imprecise probabilities over possible outcomes of a decision alternative against some criteria.

To address the issue, this paper extends the DS/AHP method to handle ambiguous evaluations of multiple criteria. More specifically, this paper will discern two kinds of mass function that model the ambiguity evaluations of criteria. The first kind of mass function is about the possibilities of multiple evaluations of the decision alternatives against each criterion under different situations. For example, against the criterion of investment profit, the evaluation of a house compared with the whole set of decision alternatives depends on the economic situation (i.e., growth or recession or depression). The second kind of mass function is about the possibility of the identified group of decision alternatives to be the optimal decision alternative regarding a given criterion. According to the first kind of mass function, this paper will present a method to obtain the preference degree over a group of decision alternatives. After considering the criteria weights, the second kind of mass function will be obtained by using the method of the DS/AHP model. After combining the second kind of mass function by using Dempster’s rule of combination, this paper will apply an ambiguity aversion principle of minimax regret to set the complete preference ordering based on the belief intervals, which is determined by the belief function and plausibility function that are induced by the overall mass function. Thus, the optimal decision alternative is obtained.
This paper advances the state of art in the field of MCDM in the following aspects: (i) We identify three types of ambiguous evaluations for MCDM problems in real life. More specifically, our method can deal with missing, interval-valued, and ambiguous lottery evaluations. (ii) We introduce the concept of preference degree, which can be used to handle the three types of ambiguous evaluations well. (iii) We introduce the concept of ambiguity degree based on the belief interval, which can reflect the decision maker’s attitude of ambiguity aversion in finding the optimal alternative. And, (iv) we propose an ambiguity aversion principle of minimax regret to obtain the complete preference ordering so that the belief interval can be compared properly. In short, we relax the assumption of the precise point-valued evaluations of MCDM to ambiguous ones, which is more feasible in real-life.

The rest of this paper is organized as follows: Section 2 briefly reviews some basics of D-S theory, DS/AHP theory, and the ambiguity aversion principle of minimax regret. Section 3 gives a formal definition of the problems we will solve in this paper. In this section, we also identify three kinds of ambiguity in decision matrix. Section 4 discusses how to obtain the preference degree of each group of decision alternatives against a given criterion. Section 5 proposes a method to obtain the optimal decision alternative according to their belief interval. Section 6 illustrates the applicability of our decision model by a real estate investment problem. Section 7 discusses some related work. Finally, Section 8 summarizes the paper and highlights the further work that will be worth carrying out in the future.

2. PRELIMINARIES

In this section, we recap some basic concepts and notations of D-S theory, DS/AHP method, and the ambiguity aversion principle of minimax regret.

2.1. Basics of D-S Theory

First, we recap the basic definitions of D-S theory.

Definition 1 Let an exhaustive set of mutually disjoint atomics be a frame of discernment and denoted as Θ.

(i) Function \( m : 2^Θ \rightarrow [0, 1] \) is called a basic probability assignment or a mass function if \( m(\phi) = 0 \) and \( \sum_{A \subseteq \Theta} m(A) = 1 \).

(ii) Function Bel : \( 2^Θ \rightarrow [0, 1] \), defined as follows, is a belief function over Θ:

\[
Bel(A) = \sum_{B \subseteq A} m(B). \tag{1}
\]

If \( m(B) > 0 \), then \( B \) is said to be a focal element of Bel.

(iii) Function Pl : \( 2^Θ \rightarrow [0, 1] \), defined as follows, is called a plausibility function over Θ:

\[
Pl(A) = \sum_{B \cap A \neq \phi} m(B). \tag{2}
\]
D-S theory also provides a rule to combine several mass functions that are evidenced from different kinds of independent sources.\textsuperscript{6,14,15}

**Definition 2** (Dempster’s rule of combination). *Let* $m_1$ and $m_2$ *be two basic probability assignment over discernment frame* $\Theta$. *Then function* $m_{12} = m_1 \oplus m_2$ *is given by*

$$
m_{12}(\{x\}) = \begin{cases} 
0 & \text{if } x = \emptyset \\
\frac{\sum_{A_i \cap B_j = \emptyset} m_1(A)m_2(B)}{k} & \text{if } x \neq \emptyset
\end{cases}
$$

*where the normalization factor*

$$
k = 1 - \sum_{A_i \cap B_j = \emptyset} m_1(A)m_2(B),
$$

*which is a measure of the conflict between the pieces of evidence and is called the normalization factor in the combination. Moreover, the Dempster’s rule of combination is commutative and associative,\textsuperscript{6} i.e.,

(i) $m_1 \oplus m_2 = m_2 \oplus m_1$ *commutativity and*

(ii) $(m_1 \oplus m_2) \oplus m_3 = m_1 \oplus (m_2 \oplus m_3)$ *associativity.*

Accordingly, the pieces of evidence can be combined in any order.\textsuperscript{5}

**Definition 3** *Let* $m$ *be a mass function over a discernment frame* $\Theta$, *and* $|A|$ *be the cardinality of set* $A$. *Then the ambiguity degree of* $m$, *denoted as* $\delta$, *is given by*

$$
\delta = \frac{\sum_{A \subseteq \Theta} m(A) \log_2 |A|}{\log_2 |\Theta|}.
$$
Let \( a \) be a decision alternative, which corresponds to a mass function \( m \) over \( \Theta \). Based on the concept of mass function, the point-valued expected utility formula can be extended to the interval-valued expected utility:\(^{20}\)

\[
\text{DEFINITION 4 For decision alternative } a \text{ specified by mass function } m \text{ over } \Theta, \text{ its expected utility interval is } EUI(a) = [E(a), \bar{E}(a)], \text{ where }
\]

\[
E(a) = \sum_{A \subseteq \Theta} \min(A)m(A), \tag{6}
\]

\[
\bar{E}(a) = \sum_{A \subseteq \Theta} \max(A)m(A). \tag{7}
\]

In the above definition, if each \( A \subseteq \Theta \) has only one element, \( m(A) \) degenerates to probability and thus formulas (6) and (7) degenerate to the point-valued expected utility. In other words, the interval value of an expected utility is caused by \( m(A) > 0 \), where \( A \subseteq \Theta \) has at least two elements. That is, the interval value of expected utility is due to the decision maker’s ambiguity about which consequence will be caused.

\[
\text{DEFINITION 5 Given a set of utilities } \Theta, \text{ set } A \subseteq \Theta \text{ is called an ambiguity set if it is a nonsingleton subset of } \Theta.
\]

### 2.2. DS/AHP Method

The DS/AHP method\(^{8,9,12}\) is a method for solving MCDM problems, which integrates D-S theory into AHP\(^{7}\) so that a preference on groups of decision alternatives can be set with respect to different criteria. More specifically, using the DS/AHP method, a decision maker just needs to make judgments on (i) the weights of the criteria and (ii) the preference level of groups of decision alternatives with respect to different criteria. Then, based on these two sets of judgments, the decision maker needs to

- (i) set the goal of the decision problem, the related criteria, and the discernment frame \( \Theta \) for the decision alternatives with respect to the criteria;
- (ii) assign a preference scale value on groups of decision alternatives (focal elements according to the evidence) to discern the preferences of the decision maker with respect to an individual criterion;
- (iii) assign a weight to each criterion based on the judgments on their relative importance;
- (iv) set the mass function for the focal elements with respect to each criterion by

\[
m(s_i) = \frac{a_i \omega}{\sum_{j=1}^{d} a_j \omega + \sqrt{d}}, \quad (i = 1, 2, \ldots, d), \tag{8}
\]

\(^{4}\)If this mass function is a probability function, the specified decision alternative is actually a lottery, defined by Von Neumann and Morgenstern.\(^{19}\)
\[ m(\Theta) = \frac{\sqrt{d}}{\sum_{j=1}^{d} a_j \omega + \sqrt{d}}. \]  

(9)

where \( s_i \) is one of \( d \) focal elements of a criterion that assigns the preference scale value \( a_i \), \( \Theta \) is the frame of discernment, and \( \omega \) is the weight of that criterion (indicating how important it is); and

(v) use Dempster’s rule of combination to combine the mass function of each criterion and thus get the overall evaluation of a decision alternative.

2.3. Ambiguity Aversion Principle of Minimax Regret

In this section, we will recap a principle to set a proper preference ordering over interval-valued expected utility, which we have proposed in Refs. 2, 13.

Our approach is to extend the minimax regret principle,\(^{21}\) which is based on the psychological observation in real life: When people choose a decision alternative, they might feel regret when the utility of another decision alternative turns out to be higher than his decision alternative.\(^{22}\) Furthermore, we consider one more cognitive factor—ambiguity aversion, which has also been observed in both laboratory experiments\(^ {23}\) and real-world problem of health care,\(^ {24}\) to revise the definition of regret under uncertainty. Formally, in Ref. 2 we introduced

**Definition 6 (The ambiguity aversion principle of minimax regret).** Let \( m \) be an initial mass function over a set of utilities \( \Theta = \{x_1, \ldots, x_n\} \), \( EUI(y) = [E(y), \overline{E}(y)] \) be the expected utility interval of decision alternative \( y (y \in \{a, b\}) \), and \( \delta(b) \) be the ambiguity degree of decision alternative \( b \). Then the ambiguity-avoiding maximum regret of the decision alternative \( a \) against decision alternative \( b \) is defined as

\[ \Re^b_a = \epsilon(b) - E(a), \]  

(10)

where \( \epsilon(b) \), called the ambiguity-avoided upper expected utility of decision alternative \( b \), is given by

\[ \epsilon(b) = E(b) + (1 - \delta(b))(\overline{E}(b) - E(b)). \]  

(11)

For any two decision alternatives \( a \) and \( b \), the strict preference ordering \( > \) is defined as follows:

\[ a > b \iff \Re^b_a < \Re^a_b. \]  

(12)

By Definition 6, we have

\[ a > b \iff \Re^b_a < \Re^a_b \iff \epsilon(b) - E(a) < \epsilon(a) - \overline{E}(b) \iff E(a) + \epsilon(a) > \overline{E}(b) + \epsilon(b). \]

Moreover, according to formula (11), we can see \( E(a) \leq \epsilon(a) \leq \overline{E}(a) \) and so ambiguity degree \( \delta \) actually works as a discounting factor: the higher it is, the more the upper utility of a decision alternative will be discounted.
It is interesting that the point-valued evaluation can also be model in D-S theory. In fact, for a decision alternative \( a_i \), whose point-value evaluation is \( x_j \in \Theta = \{x_1, \ldots, x_n\} \), by Definitions 1, 3, and 6, we have

\[
m([x_j]) = 1, \quad m(A) = 0,
\]

where \( A \subseteq \Theta \) and \( A \neq \{x_j\} \). Therefore, the expect utility of decision alternative \( a_i \) is

\[
E(a_i) = x_i, \quad \delta = 0, \quad \varepsilon(a_i) = x_i.
\]

The following theorem\(^2\) confirms the preference ordering \( \succ \) defined in the above enables us to compare any two decision alternatives properly.

**Theorem 1** Let \( A \) be a finite decision alternative set and the interval-valued expected utility of decision alternative \( a_i \in A \) be \( EU I(a_i) = [\underline{E}(a_i), \overline{E}(a_i)] \), and its ambiguity degree be \( \delta(a_i) \). A binary relation \( \succ \) over \( A \), which is defined by formula (12), satisfies

(i) if \( \underline{E}(a_1) > \overline{E}(a_2) \), then \( a_1 \succ a_2 \);
(ii) if \( \underline{E}(a_1) > \overline{E}(a_2) \), \( \overline{E}(a_1) > \overline{E}(a_2) \), and \( \delta(a_1) \leq \delta(a_2) \), then \( a_1 \succ a_2 \);
(iii) if \( \frac{\underline{E}(a_1)+\overline{E}(a_1)}{2} = \frac{\underline{E}(a_2)+\overline{E}(a_2)}{2} \), \( \underline{E}(a_1) \geq \overline{E}(a_2) \), and \( 0 < \delta(a_1) \leq \delta(a_2) \), then \( a_1 \simeq a_2 \) (in particular, \( a_1 \sim a_2 \) when \( \underline{E}(a_1) = \overline{E}(a_2) \) and \( \delta(a_1) = \delta(a_2) \)); and
(iv) if \( \underline{E}(a_1) > \overline{E}(a_2) \), and \( \delta(a_1) = \delta(a_2) = 1 \), then \( a_1 \succ a_2 \).

### 3. Problem Definition

In this section, we give the formal definition of the problems that we will solve in this paper.

In real life, often the decisions have to be made under uncertainty of ambiguity. Actually, we can distinguish the three types of ambiguity based on their performance: (i) There is complete lack of evaluation about the performance of a decision alternative against a criterion. As a result, the values of the decision alternative regarding this criterion in the decision matrix is unknown (denoted as “null”). (ii) The evaluation for a decision alternative against a criterion is not a precise point value but an interval value. (iii) The evaluation for a decision alternative against a criterion is in a set of possible values, but their probabilities are unknown or not even meaningful. That is, in this case, the evaluation is actually a mass distribution on a set of possible evaluation values.

For example, suppose a person wants to buy a house for investment as well as family. There are five alternatives \( a_1, a_2, a_3, a_4, \) and \( a_5 \); and six criteria for evaluating these alternatives: Price, Quality, Service Level, External Environment, Investment Profit (after 5 years), and Style. As shown in Table I, against the criteria of Quality, External Environment, and Style, a decision alternative could be evaluated as one of the following grades: Excellent (E), Good (G), Average (A), and Poor (P).
Table I. Decision matrix of the real estate investment problem.

<table>
<thead>
<tr>
<th></th>
<th>Price ($/m²)</th>
<th>Quality (grade)</th>
<th>Service level (%)</th>
<th>External environment (grade)</th>
<th>Investment Profit ($/m²)</th>
<th>Style (grade)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$, $a_2$</td>
<td>2200–3200</td>
<td>Null</td>
<td>Null</td>
<td>Null</td>
<td>Null</td>
<td>Null</td>
</tr>
<tr>
<td>$a_1$</td>
<td>Null</td>
<td>G</td>
<td>85</td>
<td>$m{E, G} = 0.3$</td>
<td>3800</td>
<td>E,G</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$m{G, A} = 0.5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_2$</td>
<td>Null</td>
<td>E</td>
<td>Null</td>
<td>E,G</td>
<td>3200</td>
<td>G</td>
</tr>
<tr>
<td>$a_3$</td>
<td>2800</td>
<td>Null</td>
<td>81–85</td>
<td>G,A</td>
<td>3000</td>
<td>G,A</td>
</tr>
<tr>
<td>$a_4$</td>
<td>3200</td>
<td>A,E</td>
<td>91</td>
<td>E</td>
<td>4200</td>
<td>E</td>
</tr>
<tr>
<td>$a_5$</td>
<td>2800–3000</td>
<td>A,G</td>
<td>81–95</td>
<td>A</td>
<td>3200</td>
<td>Null</td>
</tr>
</tbody>
</table>

- Some values in the decision matrix of the real estate investment problem are unknown. For example, the quality of house $a_3$ is unknown.
- For the Style criterion, some evaluations of the decision alternatives are uncertain for the buyer because they are presale of uncompleted house and the price of an alternative could be estimated only by an interval value considering some random factors.
- For the criterion of Investment Profit that is determined in the economic context, each decision alternative has one of the three different outcomes of growth, recession, and depression. In particular, according to the market survey result of the trend of economic environment, we can set $m\{\text{growth}\} = 0.4$, $m\{\text{growth, recession}\} = 0.3$, and $m\{\text{recession, depression}\} = 0.3$.
- For alternative $a_1$, probably a noisy manufactory or a beautiful park will be built nearby and so the evaluation against the External Environment criterion for alternative $a_1$ is ambiguous. It means that we cannot give a point-valued probability for each of possible evaluations.
- Also the price of alternatives $a_1$ and $a_2$ cannot be comparable because the decision maker just knows that they both in the interval of 2200–3200$/$m² but does not know which one exactly.

Then, which house should he choose?

In these cases, we might be not able to assign a unique evaluation value to each group of decision alternatives regarding each criterion, but it is relatively easier for us to give an ambiguous evaluation of the following categories:

(i) **Complete ignorance**: The decision maker lacks judgments on the consequence the decision alternatives with respect to some criteria.

(ii) **Interval value**: The decision maker only knows a range of the evaluation values for a decision alternative against a criterion, but has no idea exactly where it is. For example, when people buy a second-hand house, the real estate agency might tell him the house owner can accept a price between $300,000 and $350,000, but normally they would not tell the buyer how this price is calculated.

(iii) **Ambiguous lottery**: The decision makers only know that there is a mass (belief or plausibility) function over the set of all possibilities (see Definition 5 for the concept of mass function over the ambiguous sets).

To apply the AHP approach to conduct decision analysis for MCDM problems with ambiguous evaluations, all the values of the evaluations need to be in a unified
range. This is because of the following four reasons: (i) The criteria in MCDM problems might be classified into different categories. For example, against the *benefit* criterion, the higher a group of house alternative is evaluated, the more the decision maker *prefers* this group of house alternatives; but against the *cost* criterion, the higher a group decision alternatives is evaluated, the less the decision maker *prefers* this group of alternatives. (ii) Different groups of decision alternatives might be evaluated in different ways against different criteria, and so the evaluations cannot be combined directly. For example, the price of a house is measured by the price per square meter, whereas the service level is measured by the satisfaction degree. (iii) The criteria might be measured in different levels. The cost of building a house will be measured by million dollars, whereas the number of new houses built by a company per year might not be more than 100. (iv) The decision maker cannot combine one criterion’s qualitative evaluations with another’s quantitative evaluation directly.

To unify ambiguous evaluations for a decision alternative against criteria, we just need a mutually exclusive and collectively exhaustive set of numeral assessment grade $H_n = \{x_i \mid x_i \in \mathbb{R}, i = 1, \ldots, n, 0 = x_1 < \cdots < x_n\}$, which presents a scale set of $n$-scale unit preference on groups of decision alternatives (focal elements according to the evidence) compared with the whole set decision alternatives. Now, we can give the formal definition of a MCDM problem with ambiguous evaluations of criteria as follows:

**Definition 7** A MCDM problem under uncertainty of ambiguity, or call an ambiguous MCDM problem, is a 5-tuple of $(\Lambda, C, S, H, M)$, where

(i) $\Lambda = \{a_1, a_2, \ldots, a_l\}$ is a nonempty finite decision alternatives set;

(ii) $C = \{c_1, c_2, \ldots, c_m\}$ is a nonempty finite criteria set;

(iii) $S = \{s_{A,c} \mid A \subseteq \Lambda, \text{and } c \in C\}$ is the set of all consequence states of group of decision alternatives $A$ regarding criterion $c$.

(iv) $H = \{x_1, x_2, \ldots, x_n\}$ (where $0 = x_1 < x_2 < \cdots < x_n$) is a nonempty finite of the numeral assessment grades so that $\forall A \subseteq \Lambda, c \in C, \text{and } s_{A,c} \in S, H_{A,c}(s) \in H \subseteq \mathbb{R}$, i.e., $H_{A,c}(s)$ is the assessment grades of consequence $s_{A,c} \in S$; and

(v) $M = \{m_{A,c} \mid A \subseteq \Lambda, \text{and } c \in C\}$ where mass function $m_{A,c}$ represents the decision makers’ ambiguous judgment about the consequence states that group of decision alternatives $A$ could cause regarding criterion $c$.

And the evaluation of a group of decision alternatives against a criterion could be one of the five cases as follows:

(i) **point-valued evaluation**: $\exists x \in H, m(\{x\}) = 1$

(ii) **risk lottery evaluation**: $\forall T \subseteq H, \text{if } m_{A,c}(T) > 0, \text{then } |T| = 1$;

(iii) **missing evaluation**: $m(\emptyset) = 1, \emptyset = H$;

(iv) **interval-valued evaluation**: $H_{A,c} = [x_i, x_j], \text{where } x_i, x_j \in H$; and

(v) **ambiguous lottery evaluation**: $\exists T \subseteq H, |T| > 1, m_{A,c}(T) > 0$.

The missing, interval-valued, and ambiguous lottery case are called ambiguous evaluations.

Although the DS/AHP method\(^{8–10,12,25}\) does not mention its applicability to $(\Lambda, C, S, H, M)$ with missing evaluation, we find that in Ref. 25 Beynon has applied
this method to deal example with missing values. So, this paper focuses on the other two cases of ambiguity.

To solve such an ambiguous MCDM problem, we have to answer the following three questions: (i) How to calculate the interval-valued evaluation for each group of decision alternatives over \( \Theta \) against each criterion in these three types of ambiguous evaluations?; (ii) to apply the DS/AHP method, how to obtain a unique preference degree of each identified group of decision alternatives against each criterion?; and (iii) how to apply the belief function and plausibility function obtained by the DS/AHP method to find the optimal decision alternative?

Now we will discuss the first questions. We consider the case of missing evaluation first. As the decision maker is completely unknown in this situation, it means that

\[ m(\Theta) = 1, \Theta = H = \{x_1, x_2, \ldots, x_n\}, \text{where } x_1 = 0, x_i < x_{i+1}. \]

By formulas (6) and (7), the decision maker can obtain:

\[
\begin{align*}
E(A, c) &= \sum_{B \subseteq \Theta} \min(B)m(B) = 0 \times m(\Theta) = 0, \\
\overline{E}(A, c) &= \sum_{B \subseteq \Theta} \max(B)m(B) = x_n \times m(\Theta) = x_n.
\end{align*}
\]

Now, we turn to consider the case of the interval-valued evaluation of a group of decision alternatives against a criterion. Suppose the discernment frame of the criterion is \( \{x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\} (x_1 = 0 \text{ and } x_i < x_{i+1}) \). Then the complete ignorance of the decision maker for the interval-valued evaluation \([x_i, x_j]\) of a group of decision alternatives against a criterion can be represented as follows:

\[ m_{A,c}(\{x_i, \ldots, x_j\}) = 1. \]

It means that the decision maker only knows that the evaluation value that he obtains could be any between \( x_i \) and \( x_j \), but which one cannot be sure. Then, by formulas (6) and (7), we can obtain the expected evaluation interval:

\[ \text{EUI}(A, c) = [x_i \times 1, x_j \times 1] = [x_i, x_j]. \]

That is, the expected evaluation interval is the same as the interval-valued evaluation itself. This is consistent with our intuitions of the interval-valued evaluation.

In the case of ambiguous lottery evaluation, it can also be represented by a mass function. We can regard the evaluation of each possible outcome of a group of decision alternatives against a criterion as their utilities and accumulate the lower and upper bounds of their expected evaluation by formulas (6) and (7) as well.

In addition, in our method, we will adopt another kind of mass functions to represent the possibility for decision alternatives to be optimal decision alternative regarding a given criterion. In this case, the discernment frame of the second type is a set of given decision alternatives in the MCDM problem: \( \Theta' = \Lambda = \{a_1, a_2, \ldots, a_m\} \).
In the step of calculating the expected evaluation interval of a group of decision alternatives against a given criterion, we first adopt a mass function to represent the multiple possibilities of the group of decision alternatives against the criterion and calculate the expected evaluation interval based on this kind of mass function. However, as the decision maker has to consider the influence of the criteria weights for selecting the optimal decision alternative against all criteria, the decision maker cannot apply the Dempster combine’s rule to combine this kind of mass function of each criterion and obtain the optimal decision alternative directly. As a result, after the decision maker uses this kind of mass functions to obtain the expected evaluation interval regarding each criterion, we need to give a method to set a point-valued preference degree. Then, according to the weight of criterion and the preference degree of the group of decision alternatives regarding this criterion, we can apply the DS/AHP method to obtain the second kind of mass functions, which expresses the possibility of a given decision alternative to be the optimal decision alternative regarding the given criterion. Now, as the weight of each criterion is compatible with the preference degree in the second kind of mass function by the DS/AHP method, the decision maker can apply the Dempster’s combine rule to obtain the final mass function of each identified groups of decision alternatives regarding the whole decision alternative sets. Therefore, the decision maker can find the optimal decision alternative by the belief interval. In the following sections, we will discuss how to formally introduce the concept of preference degree and then how to select the optimal decision alternatives according to the belief intervals of the alternatives.

4. PREFERENCE DEGREES OF GROUP OF DECISION ALTERNATIVES

After obtaining the expected evaluation interval, to apply the DS/AHP method the decision maker has to face the second problem: How to obtain a preference degree of each group of decision alternatives? This section will answer this question.

How to assign preference degrees is very important for the decision makers in, for example, the problem of real estate investment, which will cost most of money for a normal person in China; the policy decision problem of a government, which might results in the depression of economic environment; and the banker decision problem about whether or not to loan money to an investor. In all of these situations, the decision maker should be more cautious than daily decision situations to avoid the loss that he cannot afford. As a result, we suggest that the decision maker should follow the ambiguous aversion principle of minimax regret (see Definition 6).

To apply the DS/AHP model, we need to give a point-valued preference degree over a group of identified decision alternatives with the expected utility interval, which can induce a preference ordering equivalent to that in Definition 6, i.e.,

**Definition 8** Let $EI(I(A, c) = [E(A, c), E(A, c)]$ be an interval-valued expected utility of the group of decision alternatives $A$ against criterion $c$, and $\delta(A, c)$ be the
ambiguuous degree of $EUI(A, c)$, then its preference degree is given by

$$
\rho_c(A) = \frac{2E(A, c) + (1 - \delta(A, c))(\bar{E}(A, c) - E(A, c))}{2}.
$$

(15)

Moreover, the preference degree $\rho(A, c)$ of a group of decision alternatives $A$ satisfies the following theorem:

**Theorem 2.** Let $EUI(X, c) = [E(X, c), \bar{E}(X, c)](X \in \{A, B\})$ be an interval-valued expected utility of decision alternatives set $X$ against criterion $c$, and $\delta(X, c)$ be the ambiguous degree of $EUI(X, c)$, then its preference degree $\rho(X, c)$ of the group of decision alternatives $X$ satisfies

(i) $A \succ B \Leftrightarrow \rho(A, c) > \rho(B, c)$ and
(ii) $E(X, c) < \rho(X, c) < \bar{E}(X, c)$.

**Proof.** By

$$
A \succ B \Leftrightarrow \Re_A^B < \Re_B^A,
$$

$$
\Re_A^B = \varepsilon(B, c) - \bar{E}(A, c),
$$

we can obtain that

$$
A \succ B \Leftrightarrow \frac{\varepsilon(A, c) + E(A, c)}{2} > \frac{\varepsilon(B, c) + E(B, c)}{2}.
$$

Moreover, we have

$$
2E(A, c) < \varepsilon(A, c) + E(A, c) < 2\varepsilon(A, c)
$$

$$
\Leftrightarrow E(A, c) < \frac{\varepsilon(A, c) + E(A, c)}{2} < \varepsilon(A, c).
$$

Thus, for missing evaluation, we have the following theorem:

**Theorem 3** For a group of decision alternatives $A$ regarding a given criterion $c$ in an ambiguous MCDM problem $(\Lambda, C, S, H, M)$, which satisfies that $m(\Theta) = 1$, $\Theta = \{x_1, x_2, \ldots, x_n\}$ ($0 = x_1 < x_2 < \ldots < x_n$), the following holds:

$$
m_c([A]) = 0.
$$

(16)

**Proof.** By formula (5), we can obtain the ambiguous degree as follows:

$$
\delta(A, c) = \frac{m_{A,c}([x_1, x_2, \ldots, x_n])\log_2(n)}{\log_2(n)} = 1.
$$
Thus, by formula (11), we have

\[ \varepsilon(A, c) = \mathbf{E}(A, c) + (1 - \delta(A, c))(\mathbf{\bar{E}}(A, c) - \mathbf{E}(A, c)) = 0 + (1 - 1)(x_n - 0) = 0. \]

Moreover, according to Definition 8, we have

\[ \rho_c(A) = \frac{2\mathbf{E}(A, c) + (1 - \delta(A, c))(\mathbf{\bar{E}}(A, c) - \mathbf{E}(A, c))}{2} = 0. \]

As a result, by formula (8), we have

\[ m_c([A]) = \frac{\rho_c(A) \omega}{\sum_{j=1}^{d} a_j \omega + \sqrt{d}} = \frac{0 \times \omega}{\sum_{j=1}^{d} a_j \omega + \sqrt{d}} = 0. \]

This theorem means that in the condition of missing evaluations of criterion, the decision maker is completely ignorant about the possibility that a group of decision alternative \( A \) would contain the optimal decision alternative regarding given criterion \( c \).

For the second type of ambiguous evaluation of interval values, we have the following theorem:

**Theorem 4.** In an ambiguous MCDM problem \((\Lambda, C, S, H, M)\), for a group of decision alternatives \( A \) regarding criterion \( c \), if \( H_{A,c} = [x_i, x_j] \), where \( x_i, x_j \in H \), then

\[ \rho_c(A) = x_i + \frac{(1 - \log_2(j - i + 1))(x_j - x_i)}{2}. \]  

(17)

**Proof.** In this condition, we just know the interval evaluation of selecting the decision alternatives regarding a criterion, but have no ideas where it is exactly. As a result, we can suppose the discernment frame of the criterion as \( \{x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\} \) \((x_1 = 0 \text{ and } x_i < x_{i+1})\). Thus, the ignorance of the decision maker for the interval-valued evaluation \([x_i, x_j]\) can be represented as follows:

\[ m_{A,C}([x_i, \ldots, x_j]) = 1. \]

Then, by formulas (6) and (7), from the above mass function the expected evaluation interval can be obtained as follows:

\[ EUI(A, c) = [x_i \times 1, x_j \times 1] = [x_i, x_j]. \]
And by formula (5), we have the ambiguous degree as follows:

\[ \delta(A, c) = m_i([x_i, \ldots, x_j]) \frac{\log_2(j - i + 1)}{\log_2 n} = \log_n(j - i + 1), \]

where \( n \) is the cardinality of the discernment frame. Thus, by formula (11) we have

\[ \varepsilon(A, c) = E(A, c) + (1 - \delta(A, c))(\overline{E}(A, c) - E(A, c)) = x_i + (1 - \log_n(j - i + 1))(x_j - x_i). \]

Finally, according to Definition 8, we have formula (17) holds. □

At last, for the third type of ambiguous evaluation (i.e., ambiguous lottery evaluation), we can obtain its preference degree by using formulas (5) and (11) directly.

After applying the ambiguous principle of minimax regret to obtain the unique preference degree over alternatives with different expected utility intervals, a decision maker can apply the DS/AHP method\(^5,10\) to obtain the optimal decision alternative.

## 5 PREFERENCE ORDERING OVER BELIEF INTERVALS

In this section, we present a method to obtain the optimal decision alternative by the idea behind the ambiguous aversion principle of minimax regret.

As the belief (Bel) function of a decision alternative represents the confidence of the decision maker toward all evidences and the plausibility (Pl) function of a decision alternative represents the extent to which the decision maker fails to disbelieve the decision alternative is the best,\(^6\) the decision maker’s Bel function of a decision alternative can be taken as the lower bound of a belief interval and its Pl function can be taken as the upper bound of a belief interval. Moreover, as we find that the more elements of the subset has the less confident decision maker believes that the Pl function to be the probability of the decision alternative. So, we can consider the impact of the ambiguity of the belief interval for the decision alternative in finding the optimal one. For example, suppose decision alternatives \( A \) and \( B \) have the same Bel function, but the nonzero value of their Pl functions is only over subset \( \{a_1, a_2\} \) and \( \{a_2, a_3, a_5\} \), respectively. Although the Bel and Pl functions are both the same for these two decision alternatives, a decision alternative \( a_1 \) still is better than a decision alternative \( a_2 \) because the elements in subset \( \{a_1, a_4\} \) are less than that in \( \{a_2, a_3, a_5\} \). Formally, we can give the definition of ambiguity degree of such a belief interval and the degree of preferring one decision alternative to another as follows:

**Definition 9** Let \( m(\{a_i\}) \) is a mass function of decision alternative \( a_i \), which induces the belief function \( \text{Bel}(\{a_i\}) \) and plausibility function \( \text{Pl}(\{a_i\}) \) over \( \Theta = \{a_1, a_2, \ldots, a_n\} \), then the ambiguity degree for uncertain interval \([\text{Bel}(\{a_i\}), \text{Pl}(\{a_i\})]\)
of decision alternative $a_i$, denoted as $\delta_U(a_i)$, is given by

$$\delta_U(a_i) = \frac{\sum_{\{a_i\} \cap B \neq \emptyset} m(B) \log_2 |B|}{\log_2 |\Theta|}. \quad (18)$$

Definition 9 is different from Definition 3 in the following two aspects: (i) The ambiguity degree of a belief interval concentrates on the singleton element of the discernment frame rather than the whole set of the discernment frame as Definition 3; and (ii) the ambiguity degree of $a_i$ in belief interval considers the subset $B$ that satisfies $\{a_i\} \cap B \neq \emptyset$, but in Definition 3 we consider all subsets of the discernment frame. Actually, the definition reflects two intuitions: (i) the decision maker just needs to consider the mass function of set $x$ ($x \subseteq \Theta$ and $x \cap A \neq \emptyset$) to confirm the ambiguity degree of decision alternative $A$ in belief interval; and (ii) the more elements are contained in set $B$ ($A \cap B \neq \emptyset$), the more ambiguous the belief interval of decision alternative $A$.

**Definition 10** The degree of preference of $a_1$ over $a_2$, denoted by $P_{a_2}^{a_1} \in [0, 1]$, is given by

$$P_{a_2}^{a_1} = (\text{Bel}(\{a_1\}) + \mu(\{a_1\})) - (\text{Bel}(\{a_2\}) + \mu(\{a_2\})), \quad (19)$$

where

$$\mu(\{a_i\}) = (1 - \delta_U(a_i))(\text{Pl}(\{a_i\}) - \text{Bel}(\{a_i\})), \quad i = 1, 2. \quad (20)$$

By the above definition, the preference relation between two decision alternatives can be defined as follows:

**Definition 11** The preference relation between two decision alternatives $a$ and $b$ is defined as follows: (i) $a > b$ if $P_b^a > 0$, (ii) $a < b$ if $P_b^a < 0$, and (iii) $a \sim b$ if $P_b^a = 0$.

The following theorem implies that our preference relation defined as above captures some intuitions well.

**Theorem 5** Let $\Lambda$ be a finite set of decision alternatives, $\text{Bel}([a])$ and $\text{Pl}([a])$ be the belief function and the plausibility function of decision alternative $a \in \Lambda$, and the ambiguity degree for belief interval $[\text{Bel}([a]), \text{Pl}([a])]$ over decision alternative $a$ be $\delta_U(a)$, then the preference ordering $\succ$ over $\Lambda$ satisfies

(i) if $\text{Bel}([a_1]) > \text{Pl}([a_2])$, then $a_1 \succ a_2$;
(ii) if $\text{Bel}([a_1]) > \text{Bel}([a_2])$, $\text{Pl}([a_1]) > \text{Pl}([a_2])$, and $\delta_U(a_1) \leq \delta_U(a_2)$, then $a_1 \succ a_2$;
(iii) if $\frac{\text{Bel}([a_1]) + \text{Pl}([a_1])}{2} = \frac{\text{Bel}([a_2]) + \text{Pl}([a_2])}{2}$, $\text{Bel}([a_1]) \geq \text{Bel}([a_2])$, and $0 < \delta_U(a_1) \leq \delta_U(a_2)$, then $a_1 \geq a_2$ (in particular, $a_1 \sim a_2$) when $\text{Bel}([a_1]) = \text{Bel}([a_2])$ and $\delta_U(a_1) = \delta_U(a_2)$; and
(iv) if $\text{Bel}([a_1]) > \text{Bel}([a_2])$ and $\delta_U(a_1) = \delta_U(a_2) = 1$, then $a_1 \succ a_2$. 

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Thus, since

\[ \delta(\{a_1\}) = Bel(\{a_1\}) > Pl(\{a_2\}) \]

we have

\[ \mu(\{a_1\}) \geq Bel(\{a_1\}) > Pl(\{a_2\}) \geq \mu(\{a_2\}) \geq Bel(\{a_2\}) \]

\[ \Leftrightarrow \mu(\{a_1\}) \geq Bel(\{a_1\}) > Pl(\{a_2\}) \geq \mu(\{a_2\}) \]

\[ \Leftrightarrow Bel(\{a_1\}) + \mu(\{a_1\}) > Bel(\{a_2\}) + \mu(\{a_2\}) \]

\[ \Leftrightarrow P_{a_1}^{a_2} > 0 \]

\[ \Leftrightarrow a_1 > a_2. \]

(ii) When \( Bel(\{a_1\}) > Bel(\{a_2\}) \) and \( 0 \leq \delta(\{a_1\}) \leq \delta(\{a_2\}) \leq 1 \), we have

\[ P_{a_1}^{a_2} = \left( Bel(\{a_1\}) + \mu(\{a_1\}) - Bel(\{a_2\}) + \mu(\{a_2\}) \right) \]

\[ = \left( 2 Bel(\{a_1\}) + (1 - \delta(\{a_1\})) Pl(\{a_1\}) - Bel(\{a_1\}) \right) - (2 Bel(\{a_2\}) \]

\[ + (1 - \delta(\{a_2\})) Pl(\{a_2\}) \]

\[ \geq (2 Bel(\{a_1\}) + (1 - \delta(\{a_1\})) Pl(\{a_1\}) - Bel(\{a_1\})) - (2 Bel(\{a_2\}) \]

\[ + (1 - \delta(\{a_2\})) Pl(\{a_2\}) \]

\[ = (1 - \delta(\{a_2\})) Pl(\{a_1\}) + Pl(\{a_2\}) \]

\[ + (1 + \delta(\{a_2\})) Bel(\{a_1\}) - Bel(\{a_2\}) \]

\[ > 0. \]

That is, \( a_1 > a_2 \).

(iii) When \( Bel(\{a_1\}) \geq Bel(\{a_2\}) \), \( Bel(\{a_1\}) + Pl(\{a_1\}) = Bel(\{a_2\}) + Pl(\{a_2\}) \), and \( 0 < \delta(\{a_1\}) = \delta(\{a_2\}) \), we have

\[ P_{a_1}^{a_2} = \left( Bel(\{a_1\}) + \mu(\{a_1\}) - Bel(\{a_2\}) + \mu(\{a_2\}) \right) \]

\[ = \left( 2 Bel(\{a_1\}) + (1 - \delta(\{a_1\})) Pl(\{a_1\}) - Bel(\{a_1\}) \right) - (2 Bel(\{a_2\}) \]

\[ + (1 - \delta(\{a_2\})) Pl(\{a_2\}) - Bel(\{a_2\}) \]

\[ = Bel(\{a_1\}) + Pl(\{a_1\}) - Bel(\{a_2\}) + Pl(\{a_2\}) - \delta(\{a_1\}) Pl(\{a_1\}) \]

\[ - Bel(\{a_1\}) + \delta(\{a_2\}) Pl(\{a_2\}) - Bel(\{a_2\}) \]

\[ \geq -\delta(\{a_2\}) Pl(\{a_1\}) - Pl(\{a_2\}) + \delta(\{a_2\}) Bel(\{a_1\}) - Bel(\{a_2\}). \]

Also \( Bel(\{a_1\}) \geq Bel(\{a_2\}) \) and \( Bel(\{a_1\}) + Pl(\{a_1\}) = Bel(\{a_2\}) + Pl(\{a_2\}) \) imply that \( Pl(\{a_1\}) \leq Pl(\{a_2\}) \). Thus, since \( 0 < \delta(\{a_1\}) \leq \delta(\{a_2\}) \), we have \( P_{a_1}^{a_2} \geq 0 \) and, in particular, when \( Bel(\{a_1\}) = Bel(\{a_2\}) \), \( \delta(\{a_1\}) = \delta(\{a_2\}) \), we have \( P_{a_1}^{a_2} = 0 \). Thus \( a_1 \geq a_2 \).

(iv) When \( \delta(\{a_1\}) = \delta(\{a_2\}) = 1 \), we have

\[ \mu(\{a_1\}) = Bel(\{a_1\}) + (1 - \delta(\{a_1\})) Pl(\{a_1\}) - Bel(\{a_1\}) = Bel(\{a_1\}), \]

\[ \mu(\{a_2\}) = Bel(\{a_2\}) + (1 - \delta(\{a_2\})) Pl(\{a_2\}) - Bel(\{a_2\}) = Bel(\{a_2\}). \]

Thus, since \( Bel(\{a_1\}) > Bel(\{a_2\}) \), we have \( a_1 > a_2 \).
According to Definition 11 and Theorem 5, we can set the preference ordering over the decision alternatives regarding all the criteria and obtain the optimal decision alternative.

6. A SCENARIO OF THE REAL ESTATE INVESTMENT

This section will illustrate how our method is used to solve a real estate investment problem of multicriteria.

Similarly to the DS/AHP, using our regret-avoided ambiguous DS/AHP method, a decision maker also needs to make two sets of judgments: (i) the judgment on the criteria weights and (ii) the judgment on the preference degree of groups of decision alternatives against different criteria. As shown in Figure 1, the procedure of our regret-avoided ambiguous DS/AHP method is as follows:

(i) Determine the criteria and set the goal of the decision problem.\(^4\)
(ii) Construct the decision matrix based on the ambiguous evaluations against criteria.

\(^4\)The criteria weights of buying house for family only or for investment only might be different. The family-only buyer might think \textit{quality} is the most important, but the investment-only one might think that \textit{investment profit} is the most important. Therefore, different goals of the decision problem might cause different preference ordering of decision alternatives.
Table II. Decision matrix of the real estate investment problem.

<table>
<thead>
<tr>
<th></th>
<th>Price</th>
<th>Quality</th>
<th>Service level</th>
<th>External environment</th>
<th>Investment profit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Growth</td>
<td>Recession</td>
</tr>
<tr>
<td>(a_1, a_2)</td>
<td>2–8</td>
<td>Null</td>
<td>Null</td>
<td>Null</td>
<td>Null</td>
</tr>
<tr>
<td>(a_1)</td>
<td>Null</td>
<td>6</td>
<td>4</td>
<td>(m((6, 8)) = 0.3,)</td>
<td>7</td>
</tr>
<tr>
<td>(a_2)</td>
<td>Null</td>
<td>8</td>
<td>Null</td>
<td>6–8</td>
<td>6</td>
</tr>
<tr>
<td>(a_3)</td>
<td>6</td>
<td>Null</td>
<td>2–4</td>
<td>4–6</td>
<td>5</td>
</tr>
<tr>
<td>(a_4)</td>
<td>2</td>
<td>4–8</td>
<td>6</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>(a_5)</td>
<td>4–6</td>
<td>4–6</td>
<td>2–8</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

(iii) Identify three types of ambiguous evaluations and calculate the expected evaluation intervals by formulas (6) and (7).
(iv) Obtain the preference degrees of decision alternatives with expected evaluation intervals by formula (15).
(v) Assign a weight to each criterion based on the judgment on their relative importance.
(vi) Determine the mass function for the focal elements of each criterion by formulas (8) and (9).
(vii) Combine the mass function of each group of decision alternatives regarding each criterion by Dempster’s combine rule and then obtain the overall belief function and the plausibility function of each decision alternative.
(viii) Set the preference ordering on belief interval of each decision alternative by formula (19) and Definition (11).
(ix) Take the optimal decision alternative.

Now, we will illustrate the applicability of our model by the real estate investment problem mentioned in Section 3. In this real estate investment problem, the goal of the decision maker is to select the best house according to six criteria: price \(p\), quality \(q\), service level \(sl\), external environment \(ee\), investment profit \(ip\), and style \(s\). According to the ambiguous evaluation of a decision alternative against each criterion, without loss of generality, we can give the decision matrix as shown in Table I. Moreover, suppose the decision maker’s evaluations take the values in \(H = \{0, 1, 2, 3, \ldots , 8\}\), meaning from “extremely dislike” (scale value 0) to “extremely preferred” (scale value 8). More specifically, without loss of generality we assume that for the external environment criterion and the style criterion, the evaluations of the group of decision alternatives over \(\Theta\) is the four assessment grades: \(u(Poor) = 2, u(Average) = 4, u(Good) = 6,\) and \(u(Excellent) = 8\). And taking the external environment criterion as an example, the evaluation of decision alternative \(B\) over \(\Theta\) for excellence should be different from the evaluation that for good. Therefore, the decision maker should use the interval-valued evaluation to express his ambiguity. Moreover, as the preference scale value of the decision alternative \(A\) is determined by the mass function, the decision matrix should also contain this situation. As a result, we can use a table about the evaluations for each criterion as shown in Table II to express decision matrix of each criterion. The null in Table II means that the corresponding evaluations are unknown.
From Table II, we can observe that the three types of ambiguity evaluation in the real estate investment problem are: (i) missing evaluation (i.e., null); (ii) interval-valued evaluation (e.g., 4–8, 4–6, 2–4, 2–8); and (iii) ambiguous lottery evaluation (i.e., \(m((6, 8)) = 0.3, m((4, 6)) = 0.5\)).

After evaluating on groups of decision alternatives against each criterion, formulas (6) and (7) will be used to calculate their expected evaluation interval with respect to each criterion. Take the *investment profit (ip)* criterion of the real estate problem as an example. First, the frame of discernment is set as

\[
\Theta = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}.
\]

Thus, according to a survey of real estate, mass function \(m\) over \(\Theta\) can be set as follows:

\[
m_{\Theta, ip}(\{x_8\}) = m_{\Theta, ip}(\{x_7\}) = m_{\Theta, ip}(\{x_6\}) = m_{\Theta, ip}(\{x_9\}) = 0.4, \\
m_{\Theta, ip}(\{x_8, x_5\}) = m_{\Theta, ip}(\{x_7, x_6\}) = m_{\Theta, ip}(\{x_6, x_5\}) = m_{\Theta, ip}(\{x_9, x_7\}) = 0.3, \\
m_{\Theta, ip}(\{x_5, x_3\}) = m_{\Theta, ip}(\{x_3, x_4\}) = m_{\Theta, ip}(\{x_4, x_3\}) = m_{\Theta, ip}(\{x_7, x_3\}) = 0.3.
\]

By formulas (6) and (7), we can obtain the lower and upper bounds of expected utilities as follows:

\[
E(\{a_1\}, ip) = \sum_{A \subseteq \Theta} \min(A)m(A) \\
= \sum_{i=1}^{8} m(\{x_i\})x_i + m(\{x_7, x_4\})x_4 + m(\{x_4, x_2\})x_2 \\
= 4.6,
\]

\[
\bar{E}(\{a_1\}, ip) = \sum_{A \subseteq \Theta} \max(A)m(A) \\
= \sum_{i=1}^{8} m(\{x_i\})x_i + m(\{x_7, x_4\})x_7 + m(\{x_4, x_2\})x_4 \\
= 6.1.
\]
Similarly, we can obtain

\[ EUI(\{a_2\}, ip) = [5.1, 5.7], EUI(\{a_3\}, ip) = [4.1, 4.7], \]
\[ EUI(\{a_4\}, ip) = [5.9, 7.4], EUI(\{a_5\}, ip) = [4.8, 6]. \]

Also, the expected evaluation interval of decision alternative \( a_1 \) regarding the external environment (ee) criterion can be obtained by formulas (6) and (7) as follows:

\[
E(\{a_1\}, ee) = \sum_{A \subseteq \Theta} \min(A)m(A) \\
= m(\{x_7, x_9\})x_6 + m(\{x_5, x_7\})x_4 + m(\{x_1, \ldots, x_9\})x_1 \\
= 0.4 \times 6 + 0.45 \times 4 + 0.15 \times 0 \\
= 4.2,
\]

\[
\bar{E}(\{a_1\}, ee) = \sum_{A \subseteq \Theta} \max(A)m(A) \\
= m(\{x_7, x_9\})x_9 + m(\{x_5, x_7\})x_7 + m(\{x_1, \ldots, x_9\})x_9 \\
= 0.4 \times 8 + 0.45 \times 6 + 0.15 \times 8 \\
= 7.1.
\]

From the expected evaluation (interval) of the groups of decision alternatives, by formula (14), the decision maker can calculate the preference degrees of the groups of decision alternative regarding each criterion. There are the three different cases: (i) point-valued expected evaluation, (ii) interval-valued expected evaluation based on interval-valued evaluation, and (iii) interval-valued expected evaluation based on ambiguity lottery evaluation. Clearly, the evaluation of decision alternative \( a_1 \) regarding the service-level criterion, the style criterion, and the investment profit criterion belongs to the first, second, and third cases, respectively. Therefore, the decision maker can obtain:

- For the decision alternative \( a_1 \) regarding the service-level (sl) criterion, by formula (14) and Definition 3, the preference degree
  \[
  \rho_{sl}(\{a_1\}) = \frac{E(\{a_1\}, sl) + \varepsilon(\{a_1\}, sl)}{2} = \frac{E(\{a_1\}, sl) + \bar{E}(\{a_1\}, sl)}{2} = \frac{4 + 4}{2} = 4.
  \]

- For the decision alternative \( a_1 \) regarding the style (s) criterion, the decision maker can express his complete ignorance of the interval-valued evaluation as follows:
  \[
  m_{\{a_1\}, s}(\{x_7, \ldots, x_9\}) = 1, E(\{a_1\}, s) = x_7 = 6, \bar{E}(\{a_1\}, s) = x_9 = 8,
  \]
  which means that the decision maker only knows that the evaluation value that he can obtain and could be any between \( x_7 \) and \( x_9 \), but which one is unknown. Then, by formulas
those groups are not the focal elements with respect to each criterion): groups of decision alternatives if its preference degrees is 0 for the reason that all of decision alternatives regarding each criterion as follow (we will not mention the ambiguity degree 4.1): 

\[ \rho_{\text{ip}}([a_1]) = \frac{E([a_1], ip) + \varepsilon([a_1], ip)}{2} \]

\[ = 6 + 6 + (1 - \log_9(9 - 7 + 1)) \times (8 - 6) \]

\[ = 6.5. \]

- For the decision alternative \( a_1 \) regarding the investment profit criterion, by Definition 3, ambiguity degree

\[ \delta([a_1], ip) = \frac{\sum_{A \subseteq \Theta} m(A) \log_2 |A|}{\log_2 |\Theta|} \]

\[ = 0.4 \log_2 1 + 0.3 \log_2 2 + 0.3 \log_2 2 \]

\[ = 0.189. \]

Hence, by Definition 8, preference degree

\[ \rho_{\text{ip}}([a_1]) = \frac{E([a_1], ip) + \varepsilon([a_1], ip)}{2} \]

\[ = 4.6 + 4.6 + (1 - 0.189) \times (6.1 - 4.6) \]

\[ = 5.2. \]

Similarly, we can obtain the preference degrees for all the identified groups of decision alternatives regarding each criterion as follow (we will not mention the groups of decision alternatives if its preference degrees is 0 for the reason that all of those groups are not the focal elements with respect to each criterion):

- for the price (p) criterion:
  \( \rho_p([a_1, a_2]) = 2.34, \rho_p([a_1]) = 6, \rho_p([a_3]) = 2, \rho_p([a_5]) = 4.5; \)

- for the quality (q) criterion:
  \( \rho_q([a_1]) = 6, \rho_q([a_2]) = 8, \rho_q([a_3]) = 4.54, \rho_q([a_5]) = 4.5; \)

- for the service level (sl) criterion:
  \( \rho_{sl}([a_1]) = 4, \rho_{sl}([a_2]) = 2.5, \rho_{sl}([a_3]) = 6, \rho_{sl}([a_5]) = 2.34; \)

- for the external environment (ee) criterion:
  \( \rho_{ee}([a_1]) = 4.79, \rho_{ee}([a_2]) = 4.5, \rho_{ee}([a_3]) = 4.5, \rho_{ee}([a_4]) = 8, \rho_{ee}([a_5]) = 4; \)

- for the investment profit (ip) criterion:
  \( \rho_{ip}([a_1]) = 5.2, \rho_{ip}([a_2]) = 5.34, \rho_{ip}([a_3]) = 4.34, \rho_{ip}([a_4]) = 6.51, \rho_{ip}([a_5]) = 5.29; \)

- for the style (s) criterion:
  \( \rho_s([a_1]) = 6.5, \rho_s([a_2]) = 6, \rho_s([a_3]) = 4.5, \rho_s([a_4]) = 8. \)
Now without loss generality, it is assumed that the criteria weights based on the decision maker’s judgment on importance of each criterion as follows:

\[
\omega(Price) = 0.2, \omega(\text{Quality}) = 0.3, \omega(\text{Service Level}) = 0.1,
\]

\[
\omega(\text{External Environment}) = 0.1,
\]

\[
\omega(\text{Investment Profit}) = 0.2, \omega(Style) = 0.1.
\]

Then the mass function for the focal elements of each criterion can be obtained by formulas (8) and (9). For the price \( (p) \) criterion, the decision maker can obtain

\[
m_p([a_1, a_2]) = \frac{\rho_p([a_1, a_2])\omega_p}{(\rho_p([a_1, a_2]) + \rho_p([a_3]) + \rho_p([a_4]) + \rho_p([a_5]))\omega_p + \sqrt{d}}
\]

\[
= \frac{2.34 \times 0.2}{(2.34 + 6 + 2 + 4.5) \times 0.2 + \sqrt{4}}
\]

\[
= 0.094.
\]

\[
m_p([a_3]) = \frac{\rho_p([a_3])\omega_p}{(\rho_p([a_1, a_2]) + \rho_p([a_3]) + \rho_p([a_4]) + \rho_p([a_5]))\omega_p + \sqrt{d}}
\]

\[
= \frac{6 \times 0.2}{(2.34 + 6 + 2 + 4.5) \times 0.2 + \sqrt{4}}
\]

\[
= 0.242,
\]

\[
m_p([a_4]) = \frac{\rho_p([a_4])\omega_p}{(\rho_p([a_1, a_2]) + \rho_p([a_3]) + \rho_p([a_4]) + \rho_p([a_5]))\omega_p + \sqrt{d}}
\]

\[
= \frac{2 \times 0.2}{(2.34 + 6 + 2 + 4.5) \times 0.2 + \sqrt{4}}
\]

\[
= 0.081,
\]

\[
m_p([a_5]) = \frac{\rho_p([a_5])\omega_p}{(\rho_p([a_1, a_2]) + \rho_p([a_3]) + \rho_p([a_4]) + \rho_p([a_5]))\omega_p + \sqrt{d}}
\]

\[
= \frac{4.5 \times 0.2}{(2.34 + 6 + 2 + 4.5) \times 0.2 + \sqrt{4}}
\]

\[
= 0.181,
\]

\[
m_p(\Theta) = \frac{\sqrt{d}}{(\rho_p([a_1, a_2]) + \rho_p([a_3]) + \rho_p([a_4]) + \rho_p([a_5]))\omega_p + \sqrt{d}}
\]

\[
= \frac{\sqrt{4}}{(2.34 + 6 + 2 + 4.5) \times 0.2 + \sqrt{4}}
\]

\[
= 0.402.
\]
Similarly, by formulas (8) and (9), the decision maker can obtain that mass function for the focal elements of other criteria:

- for the quality (q) criterion:
  \[ m_q(\{a_1\}) = 0.202, m_q(\{a_2\}) = 0.269, m_q(\{a_3\}) = 0.153, m_q(\{a_4\}) = 0.151, m_q(\Theta) = 0.225; \]
- for the service level (sl) criterion:
  \[ m_{sl}(\{a_1\}) = 0.115, m_{sl}(\{a_2\}) = 0.072, m_{sl}(\{a_3\}) = 0.172, m_{sl}(\{a_4\}) = 0.067, m_{sl}(\Theta) = 0.574; \]
- for the external environment (ee) criterion:
  \[ m_{ee}(\{a_1\}) = 0.099, m_{ee}(\{a_2\}) = 0.094, m_{ee}(\{a_3\}) = 0.094, m_{ee}(\{a_4\}) = 0.166, m_{ee}(\{a_5\}) = 0.083, m_{ee}(\Theta) = 0.464; \]
- for the investment profit (ip) criterion:
  \[ m_{ip}(\{a_1\}) = 0.137, m_{ip}(\{a_2\}) = 0.141, m_{ip}(\{a_3\}) = 0.115, m_{ip}(\{a_4\}) = 0.172, m_{ip}(\{a_5\}) = 0.14, m_{ip}(\Theta) = 0.295; \]
- for the style (s) criterion:
  \[ m_s(\{a_1\}) = 0.144, m_s(\{a_2\}) = 0.133, m_s(\{a_3\}) = 0.1, m_s(\{a_4\}) = 0.178, m_s(\Theta) = 0.445. \]

Then, the overall mass function can be obtained by using Dempster’s combination rule (i.e., formulas (3) and (4)). As the rule satisfies commutative and associative, in the real estate investment problems, the mass function corresponding to different criteria can be combined in any order. Without loss of generality, the decision maker combines the mass functions of the price criterion and the quality criterion first as follows:

\[
k = 1 - \sum_{x_i \cap y_j = \emptyset} m_p(x)m_q(y)
\]

\[
= 1 - m_p(\{a_1, a_2\})m_q(\{a_4\}) - m_p(\{a_1, a_2\})m_q(\{a_5\}) - m_p(\{a_3\})m_q(\{a_1\}) - \ldots - m_p(\{a_5\})m_q(\{a_1\}) - m_p(\{a_5\})m_q(\{a_2\}) - m_p(\{a_5\})m_q(\{a_4\})
\]

\[
= 1 - 0.094 \times 0.153 - 0.094 \times 0.151 - 0.242 \times 0.202 - \ldots - 0.181 \times 0.202 - 0.181 \times 0.269 - 0.181 \times 0.153
\]

\[
= 0.6205,
\]

\[
m_{p,q}(\{a_1, a_2\}) = \frac{\sum x_i \cap y_j = \{a_1, a_2\} m_1(x_i)m_2(y_j)}{k}
\]

\[
= \frac{m_p(\{a_1, a_2\})m_q(\Theta)}{0.6204}
\]

\[
= \frac{0.094 \times 0.225}{0.6204}
\]

\[
= 0.034.
\]
Similarly, we can obtain

\[ m_{p,q}(\{a_1\}) = 0.162, m_{p,q}(\{a_2\}) = 0.215, m_{p,q}(\{a_3\}) = 0.088, m_{p,q}(\{a_4\}) = 0.148, \]
\[ m_{p,q}(\{a_5\}) = 0.207, m_{p,q}(\{a_1, a_2\}) = 0.034, m_{p,q}(\Theta) = 0.146. \]

Repeat this process until all the mass functions of the given criteria are combined and then an overall mass function regarding all criteria is obtained as follows:

\[ m_{\text{house}}(\{a_1\}) = 0.214, m_{\text{house}}(\{a_2\}) = 0.198, \]
\[ m_{\text{house}}(\{a_3\}) = 0.107, m_{\text{house}}(\{a_4\}) = 0.272, \]
\[ m_{\text{house}}(\{a_5\}) = 0.161, m_{\text{house}}(\{a_1, a_2\}) = 0.009, m_{\text{house}}(\Theta) = 0.039. \]

Furthermore, from the overall mass function, the measures of belief (Bel) function can be obtained by formula (1) as follows:

\[ Bel_{\text{house}}(\{a_1\}) = \sum_{X \subseteq \{a_1\}} m_{\text{house}}(X) = m_{\text{house}}(\{a_1\}) = 0.214, \]

Similarly, by formula (1), we can obtain

\[ Bel_{\text{house}}(\{a_2\}) = 0.198, Bel_{\text{house}}(\{a_3\}) = 0.107, \]
\[ Bel_{\text{house}}(\{a_4\}) = 0.272, Bel_{\text{house}}(\{a_5\}) = 0.161. \]

Moreover, the plausibility (Pl) function can be obtain by formula (2) as follows:

\[
Pl_{\text{house}}(\{a_1\}) = \sum_{X \cap \{a_1\} \neq \emptyset} m_{\text{house}}(X) \\
= m_{\text{house}}(\{a_1\}) + m_{\text{house}}(\{a_2\}) + m_{\text{house}}(\{a_1, a_2\}) \\
= 0.262,
\]

Similarly, by formula (2), we can obtain

\[ Pl_{\text{house}}(\{a_2\}) = 0.246, Pl_{\text{house}}(\{a_3\}) = 0.146, \]
\[ Pl_{\text{house}}(\{a_4\}) = 0.311, Pl_{\text{house}}(\{a_5\}) = 0.2. \]

Hence, by Definitions 9 and 10, we can obtain

\[
\delta_U(a_1) = \delta_U(a_2) = \frac{0.009 \times \log_2 2 + 0.039 \log_2 5}{\log_2 5} = 0.043,
\]
and similarly, by Definitions 9 and 10, we can obtain
\[ \delta_U(a_3) = \delta_U(a_4) = \delta_U(a_5) = 0.039. \]
Thus, by Theorem 5 and Definition 11, we can obtain the overall preference ordering as
\[ a_4 \succ a_2 \succ a_1 \succ a_5 \succ a_3. \]
So, \( a_4 \) is the optimal one.

7. RELATED WORK

Since 1970s, MCDM has been an active area of research, which aims to help the decision maker to solve the decision and planning problems regarding multiple criteria.\(^{26,27}\) MCDM problems can be classified into two categories: discrete and continuous.\(^{27}\)

For discrete MCDM problems, the focus is to elicit some alternatives that satisfy the decision maker’s goal from a finite number of decision alternatives. The examples of this kind include the selection of the best location for a dam,\(^{28}\) the health diagnosis of hydraulic metal structure healthy based on their multicriteria safety evaluation,\(^{29}\) the flood-defense resource allocation,\(^{30}\) the level classification of environmental pollution for all provinces in China,\(^{31}\) and automated multi-issue negotiation for an accommodation renting problem.\(^{32}\) Moreover, different schools of thought have developed for solving discrete MCDM problems:

(i) **The American school.** This school includes two well-known methods: multiattribute utility theory (MAUT)\(^{33}\) and AHP.\(^{7}\) MAUT is an extension of utility theory and developed to solve MCDM problems by two steps: first normalize evaluation matrix as individual utility regarding each criterion based on the decision maker’s preference and risk attitude; and then combine these individual evaluations by the weighted sum method to obtain an overall evaluation. Different from MAUT, using AHP the decision maker first needs to model the problem as a hierarchy, which consists of an overall goal, a group of alternatives for reaching the goal, and a group of criteria that relate the alternatives to the goal. Then through a series of pairwise comparisons of the criteria and the alternative against each criterion, the decision maker can obtain the criteria’s weights and the evaluation of each decision alternative against each criterion. Finally, by using the weighted sum method the decision maker can obtain an overall evaluation of each decision alternative considering all the criteria and thus has the proper preference ordering over them. Recently, Saaty proposed a more general form of AHP—the analytic network process.\(^{34}\) In addition, Yager proposed a model of ordered weighted averaging aggregate operator, which can cover a family of well-known aggregate operators for solving MCDM problems.\(^{35,36}\)

(ii) **The French school.** Based on either a partial or complete ordering of alternatives, the family of the outranking methods has been proposed by this school. For example, ELECTRE (ELimination and Choice Expressing REALity) is to construct one or several outranking relation(s) to pairwise compare actions comprehensively, and then elaborate an exploitation procedure based on the assessments obtained in the first phase; and the PROMETHEE (Preference Ranking Organization Method for Enrichment of Evaluations) procedure is also based on pairwise comparisons, but the decision maker needs to
obtain the aggregated preference indices and the outranking flows to set the partial or complete ranking of the decision alternatives.\textsuperscript{37}

Now we turn to continuous MCDM problems. Feasible alternative sets in such problems usually consist of a very large countable number or infinite and not countable number of decision alternatives. For such an MCDM problem, the focus of a decision maker is not on estimating each alternatives’ expected utility against each criterion, but on finding the optimal (most preferred) or good enough solution by an interactive decision procedure. In this process, the decision maker elicits and uses implicit evaluation about his preference to converge to a final solution in a reasonable number of iterations. The examples of these problems include the problems of portfolio selection,\textsuperscript{38} sustainable processes planning,\textsuperscript{39} least cost alternative providing for pollution control,\textsuperscript{40} and automated negotiation.\textsuperscript{41} The following categories contain the main approaches for solving continuous MCDM problems: (i) the goal programming that is the generalization of linear programming for multiple objective measures;\textsuperscript{38} (ii) the vector maximization method that is originally developed for multiple objective linear programming problems to approximate the nondominated set\textsuperscript{42}; (iii) the interactive programming that presents a man–machine interactive mathematical programming method;\textsuperscript{43} and (iv) the evolutionary multiobjective optimization that can capture a number of Pareto-optimal solutions concurrently in a single run by the inherent parallelism of evolutionary algorithms.\textsuperscript{44}

Most of these MCDM models are based on the accurate evaluations of each decision alternative against each criterion. However, in many real-life decision situations, as a result of the internal uncertain (caused by the hesitation of a decision maker on judgments) and the external uncertainties (caused by the imperfect evaluation), it is difficult to elicit accurate evaluations and sometimes it is even impossible.\textsuperscript{45} In such situations, it becomes necessary to deal with these uncertainties and thus some important methods for handling such uncertainties have been developed. For example, fuzzy sets approach\textsuperscript{46} is used to model and solve fuzzy MCDM problems;\textsuperscript{4, 47–54} rough set approaches are used to deal with the preferential evaluation causing inconsistencies in MCDM;\textsuperscript{55–57} and an MCDM method for eliciting the optimal alternatives in uncertain ranges has been proposed.\textsuperscript{58, 59} Nevertheless, these MCDM approaches are essentially based on traditional evaluation methods, which cannot well handle uncertainty of ambiguity, such as missing, imprecise, and uncertain evaluations of multiple criteria.

To solve such ambiguity problems in MCDM, some researchers incorporated the D-S evidence theory into the MCDM methods. Recently, this kind of D-S theory based methods became an active field of research in MCDM. For example,

(1) \textbf{The evidential reasoning (ER) approach}\textsuperscript{3, 60–64}. It is a generic evidence-based approach that can solve the MCDM problems involving various types of uncertainties such as incomplete evaluation, complete ignorance, and fuzziness. This method uses an ER analytical algorithm for aggregating multiple belief structures and constructs a belief decision matrix to represent the evaluation with uncertainty. Hence, some kinds of evidential reasoning algorithms such as the ER nonlinear optimization models will be given to aggregate criteria for generating distributed assessments (the expected utility). Our method also applies D-S theory to solve discrete MCDM problems that have some advantages over the above-mentioned
existing methods. (i) These existing approaches can just assign the degree of belief for the decision alternatives regarding the given criterion on a single element grade or the whole set of grades or the interval grades, whereas our method can assign the degree of belief for that on any subset of the whole set of grades. For example, in our scenario of real estate investment, for the decision alternative $a_2$ (see Table II), we assign the precise grade regarding *price* criterion, the whole set of grades regarding a *service-level* criterion, the interval grades regarding an *external environment* criterion, and also the subset grades regarding an *investment profit* criterion. (ii) For some of those existing method, for example, the decision maker cannot obtain the complete preference ordering over all decision alternatives and elicit the optimal decision alternatives, but our method can. And (iii) for some other existing methods (e.g., Ref. 63) that employ the minimax regret approach to rank decision alternative, in Ref. 2 we find that it violates the transitive axiom of the preference ordering and the Houthakker axiom.

(2) The DS-AHP method. It can solve the MADM problems with incomplete evaluation. Using this model, the decision maker first needs to identify all possible focal elements from the incomplete decision matrix and then employ the AHP method to find the criteria weights and calculate the mass function of each focal element by the method of weighted mean. After that, the mass function for all the criteria and the belief intervals can be obtained by D-S theory. Then, the preference relations among all decision alternatives are defined by comparing their belief intervals in the method that are employed by some ER methods as follows:

(i) $a_i \succ a_j$ iff $p(a_i > a_j) > 0.5$;
(ii) $a_i \sim a_j$ iff $p(a_i > a_j) = 0.5$; and
(iii) $a_i \prec a_j$ iff $p(a_i > a_j) < 0.5$.

where

$$p(a_i > a_k) = \frac{\max[0, Pl([a_i]) - Bel([a_k])] - \max[0, Bel([a_i]) - Pl([a_k])]}{[Pl([a_i]) - Bel([a_i])] + [Pl([a_k]) - Bel([a_k])]}.$$  

Compared with our method about the ranking order by the belief interval in Definitions 10 and 11, we find that the ranking method as mentioned above, which is adopted by the ER and DS-AHP, has some problems: (i) Their method cannot set the preference relations between two point valued probabilities because $[Pl([a_i]) - Bel([a_i])] + [Pl([a_k]) - Bel([a_k])] = 0$. As the probability function is actually a special case of mass function, a well-formed method should be able to cover this special case. (ii) Their method produces some results that violate our intuitions. For example, the Pl and Bel functions of decision alternative $a$ are both 0.5, the Pl and Bel functions of decision alternative $b$ are 0.99 and 0.01, respectively. And suppose that the plausibility function of decision alternative $b$ is obtained by two subsets of the discernment frame, which contain a large number of decision alternatives, that is, $m(\{b\}) = 0.01$, $m(\{\Theta/a\}) = 0.98$. Intuitively, as the subset $\Theta/a$ contains a large number of decision alternatives, although the Pl function of decision alternative $b$ is 0.99, we might still think that decision alternative $a$ is better because $Bel([a]) = 0.5 \succ Bel([b]) = 0.01$. However, by using the belief interval
method of DS-AHP, we obtain that

\[
p(a > b) = \frac{\max[0, Pl(a) - Bel(b)] - \max[0, Bel(a) - Pl(b)]}{[Pl(a) - Bel(a)] + [Pl(b) - Bel(b)]}
\]

\[
= \frac{\max[0, 0.5 - 0.01] - \max[0, 0.5 - 0.99]}{[0.5 - 0.5] + [0.99 - 0.01]}
\]

\[
= \frac{0.49 - 0}{0 + 0.98}
\]

\[
= 0.5.
\]

It means that by DS-AHP, decision alternatives \(a\) and \(b\) are indifferent. Moreover, the intuition that we have mentioned is similar to the result obtained by the Ellsberg paradox.\(^{23}\) And (iii) their method cannot consider the affection of the belief interval’s ambiguity in setting the preference ordering, while our method can.

Moreover, the DS-AHP method has also some other problems: (i) It claims that it can handle missing evaluation problem, which is its advantage over the DS/AHP method,\(^8,10\) but we find that the DS/AHP method actually already did that. In fact, as the missing evaluation for a group of decision alternatives is not the focal element against the given criterion in DS/AHP, by formulas (8) and (9), the mass function of this group of decision alternatives is zero. It means that the decision maker has completely no idea about the preference relation of the given decision alternative with missing evaluation. Therefore, the DS/AHP method can also solve the missing evaluation. (ii) As missing evaluations are just the first type of ambiguous evaluation, the DS-AHP method cannot handle the second and third types of ambiguous evaluation, but we can. And (iii) DS-AHP does not justify why the decision maker should employ the method of weighted mean rather than the AHP method to obtain the mass function of each focal element.

(3) Deng’s method.\(^{66}\) This method was proposed in 2011 and used fuzzy set theory and D-S theory to deal with a supplier selection problem. In Deng’s model, it first uses the distance of the ideal solution (IS) and the negative ideal solution (NS) to obtain the mass function. After that, this model uses the weight of a criterion to discount the mass function with respect to the criterion and employs the Dempster combination rule (i.e., formulas (3) and (4)) to aggregate the mass functions of all the criteria to get the overall evaluation of a decision alternative. If there is more than one decision maker, the model uses the weights to discount the mass functions of the overall criteria for each decision maker and also combine it by the Dempster combination rule. Finally, Deng’s model will employ the pignistic probability transformation to obtain the final decision results.

This method has also some problems: (i) It does not propose an approach to deal with the ambiguous evaluations that our model can do; and (ii) according to the Deng’s method, for the benefit attributes (the more amounts the better), we have

\[
m_1(IS) = \frac{d_1(NS)}{d_1(NS) + d_1(IS) + d_1(IS, NS)},
\]
where

\[ d_1(IS) = |a_1 - \max\{a_i\}|, \]
\[ d_1(NS) = |a_1 - \min\{a_i\}|, \]
\[ d_1(IS, NS) = |a_1 - \frac{\max\{a_i\} + \min\{a_i\}}{2}|. \]

From the above formulas, we can see that if \( a_1 = \max\{a_i\} = \min\{a_i\} \), \( d_1(IS) = d_1(NS) = d_1(IS, NS) = 0 \) and so no value for \( m_1(IS) \) cannot be determined. It means that Deng’s model cannot handle the situation that all the decision alternatives have the same point value of the evaluation against a given criterion.

(4) The DS/AHP method.\(^8,10\) It incorporates the D-S theory with the AHP process. Generally speaking, the various MCDM models based on D-S theory can be regarded as the solutions to MCDM problems, in which the point-valued probability of each single outcome is unknown or not available. However, all of the existing methods of this kind have the following problems: (i) They address little about the significant influence of the cognitive factor—ambiguity aversion—in MCDM problems when the available evaluations are ambiguous as many psychological investigations\(^{67,68}\) and experiments in behavioral economics\(^{23}\) have shown. (ii) All of these methods except the ER method cannot handle all the three types of ambiguous evaluations of multiple criteria, which our model can. Moreover, our model keeps the advantage of AHP, which is the most powerful model to solve a variety of complex decision-making problems, especially those with high stakes, involving human subjective judgments and understanding.\(^{69}\) And (iii) our model proposes a belief interval method to rank decision alternatives more reasonable than other MCDM method based on D-S theory.

8. SUMMARY

In this paper, we first reviewed D-S theory, the DS/AHP method, and our ambiguous aversion principle of minimax regret.\(^2\) Then we incorporated the expected utility interval and the ambiguous aversion principle of minimax regret into the DS/AHP method so that ambiguous evaluations of multiple criteria can be handled well in AHP. More specifically, first we give a formal definition of MCDM problems with missing, interval-valued, and ambiguous lottery evaluations. Then we point out that the two among three types of ambiguous evaluations cannot be handled by the DS/AHP method, but our model can do. After that we apply the mass function on the discernment frame that is a set of numeral assessment grades, to express these three types of ambiguous evaluations. On the basis of the expected evaluation interval and the ambiguity degree of groups of decision alternatives against a given criterion, we introduce the concept of preference degree of groups of decision alternatives against each criterion, which aggregates the effects of expected evaluation and ambiguity degree. Furthermore, after applying the DS/AHP method and Dempster’s combine rule to obtain the overall mass function of groups of decision alternatives against
all the criteria, we give a ranking method based on the Bel and Pl functions, which is induced by this overall mass function. Finally, we illustrate the applicability of our model in the domains of business by an example of the real estate investment problem.

In the future, it is interesting to conduct a number of psychological experiments to refine and validate our model, integrate more factors of emotions into our model further, and apply our model into a wider range of domains, especially intelligent agent systems such as multi-attribute negotiating agents.\textsuperscript{32,41,70} It is also interesting to apply our idea of this paper into other MCDM techniques such as fuzzy constraints\textsuperscript{49} and fuzzy aggregation operators.\textsuperscript{4,71,72} Moreover, to obtain the criterion weights, this paper still adopts the pairwise comparison matrix that is constructed as AHP does. This might cause a problem: Why do not we adopt the same method as the decision alternative comparison that identify a special group of focal elements (a set of single element) as favorable from the frame of discernment (of all criteria)? This is also the problem for the DS/AHP method. Therefore, it is required to develop a model that employs the mass function to model not only the preference ordering of the decision alternative but also the importance ordering of the criteria. In addition, in the real-life decision making, ambiguous evaluations might also affect the decision maker’s subjective judgment for the criteria weights. So, it is worth discussing how to deal with the ambiguous weights of criteria with ambiguous evaluation.

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