PID pitch attitude control for unstable flight vehicle in the presence of actuator delay: Tuning and analysis

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Abstract

In the realm of flight control, proportional–integral–derivative (PID) control is still widely used in practice due to its simple structure and efficiency. The robustness and dynamic performance of PID controller can be evaluated by stability margins. Based on the empirical knowledge about the unstable flight dynamics, the analytical tuning formulas of the PID pitch attitude control with actuator delay are derived with the help of several proper approximations. These tuning formulas can meet the increasing gain and phase margins (iGPM) requirement and avoid time-consuming trial-and-error tuning process. The feasible iGPM area is established in 2-D plane subject to several conditions, especially taking the decreasing gain margin into account, wherein the numerical polynomial solving approaches are employed. The relationship between an existing PD tuning scheme and the proposed PID tuning method is also revealed. The applicable area of the tuning rule is then investigated on the basis of a crucial assumption. Furthermore, the achievable decreasing gain and phase margins (dGPM) area is obtained when the decreasing gain margin is critical; and another tuning rule is derived according to the dGPM specifications. The effect of the actuator delay on the achievable GPM area is demonstrated in a straightforward manner such that the reasonable criteria can be specified. Finally two numerical paradigms are presented to validate the proposed method; and the robustness and dynamic performance of the PID control are also reexamined for unstable flight dynamics.

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1. Introduction

The control design for an unstable flight vehicle has attracted lots of attention since 1960s. The intrinsic characteristic of the unstable flight dynamics is that the center-of-gravity (COG) locates behind the neutral point [1]. Therefore, a tight autopilot is necessary to keep the angle of attack (AOA) from diverging from the reference, and the major difficulty lies in that the actuator restrains the realization of good closed-loop performance due to its phase delay [2] or rate limit [3–5]. Up to now, many researchers have concentrated on air-to-air or surface-to-air agile missiles to investigate this problem [6,7]. In this field, the three-loop acceleration control [8–10] is the primary technique to guarantee sufficient agility and maintain robust stability. Nevertheless in some specific scenarios, attitude control is indispensable. Although the angle of attack is generally employed as a regulated variable in many modern control methods based on the state space representations [11–14], there is a requirement for accurate pitch angle control under certain circumstances [15–17]. For example, many modern launch vehicles with heavy boosters exhibit static instability [18]; and gust disturbance is a severe factor that should be dealt with carefully in the low dynamic pressure phase. The direct AOA control is sensitive to the gust disturbance, and the pitch angle regulation is preferred. Another example is that the accurate pitch control is also necessary when designing an automatic landing system for an aircraft [17]. However, in the literature there were few reports on pitch attitude control with practical requirements. It is urgent to fill this gap, especially for the unstable flight vehicles.

In the realm of aerospace industry, the PID-type controllers have been widely used for several decades [19] since the PID controller structure is simple and its principle is easier to understand than most of other advanced controllers. As there are omnipresent uncertainties in the aerodynamic coefficients, the analysis of robustness is very crucial for a flight control system. Gain and phase margins (GPM) are served as important measures of robustness. Although several modern robust analysis methodologies, for instance the singular value analysis [20,21], have emerged recently to provide refined descriptions of robustness, these philosophies are not easily accepted by the engineers who are familiar with GPM. Moreover, the GPM is recognized as an industrial standard for robust evaluation, which is also another reason that the engineers prefer GPM to other robust analysis methodologies. But the GPM specifications based PID tuning is still an open topic [22–27]. The conventional loop-shaping method in the frequency domain is a trial-and-error approach which is time-consuming and its precision cannot be ensured. In [23 and 24], the approximation of tan-1 function was employed to obtain the analytical tuning rules, and it can be used to find the feasible GPM region. Among these investigations, there were tuning methods for unstable processes [24,26,27].

In [2], an analytical PD tuning rule and a feasible GPM region were presented for an unstable short-period longitudinal dynamics. However, a rather long settling time is inevitable when the PD pitch attitude control is implemented because of an apparent lag between the pitch and the elevation angle [28]. In reality, an integral action can be added to the PD control to produce a PID pitch attitude control which can improve the dynamic performance remarkably. However, to the best of our knowledge, there has been no report on PID tuning for unstable flight dynamics so far. In this paper, an analytical PID tuning rule is to be provided based on both the increasing and the decreasing gain margins by means of numerical polynomial solving approach. Its accuracy and relationship with the PD tuning rule of Sun et al. [2] need to be investigated; and it is necessary to obtain the area of effective GPM combination for the previous PD tuning rule within the PID control framework. It would be better to give a feasible GPM region in a graphical manner. As a useful and practical method, the controller tuning rule should be provided step by step.
The rest parts of the paper are organized as follows. Section 2 formulates the PID control problem for an unstable longitudinal dynamics. Section 3 obtains the PID tuning rule according to the iGPM specifications and carries out the corresponding investigations. Section 4 conducts the study further based on the dGPM specifications. In Section 5, two illustrative examples are presented to validate the tuning rules, plot the available GPM regions and make comparisons. The concluding remarks are summarized in the last section.

2. Formulation of PID pitch attitude control for unstable longitudinal dynamics

The longitudinal model for a generic flight vehicle is chosen as a baseline for this study. Most of the important dynamic attributes can be preserved by the linearized time-invariant equations of motion. According to the small perturbation theory, a linearized model can be established as follows:

\[
\begin{align*}
\dot{\theta} &= m_\alpha \dot{\alpha} + m_\delta \delta_e, \\
\dot{\gamma} &= c_\alpha \alpha + c_{\delta_e} \delta_e, \\
\theta &= \gamma + \alpha, \\
q &= \dot{\theta}.
\end{align*}
\]

when the angle of attack \( \alpha \) is small. Here \( \gamma \) is the flight path angle, \( \theta \) and \( q \) are the pitch angular position and rate respectively, \( \delta_e \) is the elevator deflection, \( m_i(i = \alpha, q, \delta_e) \) are the pitch moment coefficients due to \( i \), and \( c_i(i = \alpha, \delta_e) \) are the lift coefficients due to \( i \). \( \delta_e \) is driven by a control voltage \( \delta \), and represented as \( \delta_e = \delta e^{-sL} \), where \( L \) is the time-delay or the reciprocal of the bandwidth of the actuator. The aerodynamic derivatives, \( m_\alpha, m_q, m_\delta, c_\alpha \) and \( c_{\delta_e} \), depict the sole characteristics of a specific flight vehicle. For an unstable flight vehicle, \( m_\alpha > 0 \).

After some algebraic manipulations, the unstable flight dynamical transfer function can be modeled as follows:

\[
\frac{\theta(s)}{\delta(s)} = \frac{k_m(T_n s + 1)}{s(T_p^2 s^2 + 2\zeta_p T_p s + 1)} e^{-sL}. \tag{2}
\]

where

\[
\begin{align*}
k_m &= \frac{m_\alpha c_\alpha - m_q c_{\delta_e}}{m_q c_\alpha + m_\alpha}, \\
T_p &= \frac{1}{\sqrt{m_q c_\alpha + m_\alpha}}, \\
T_n &= \frac{m_\delta T_p^2}{k_m}, \\
\zeta_p &= \frac{T_p(c_\alpha + m_q)}{2}.
\end{align*}
\]

According to our empirical knowledge [2], the following two relations hold true for unstable flight dynamics: (a) \( T_n \gg T_p \) and (b) the damping factor \( \zeta_p \) is a tiny value, generally less than 0.1. Eq. (2) can be reformulated as follows:

\[
G_p(s) = \frac{\theta(s)}{\delta(s)} = \frac{k_m(T_n s + 1)}{s(\tau s - 1)(T_d s + 1)} e^{-sL}. \tag{4}
\]

where \( \tau \) is slightly larger than \( T_d \). It should be noted that \( k_m < 0 \) for flight vehicles. However to facilitate the description, we set \( k_m = |k_m| \) and the related signal also switches its sign to tally with the notations in common without further interpretations.
The traditional PID control for the pitch angle is shown in Fig. 1 and represented as follows:

$$\delta = k_p e + k_i \int e dt + k_d q$$  \hspace{1cm} (5)

where $e = \theta_r - \theta$, and $\theta_r$ is a pre-specified reference. Note that the classical derivative term on the pitch error or $e$ is seldom used in practice due to two factors: (1) parts of the reference $\theta_r$ are calculated by using onboard measurement signals such as the altitude, whose measurement noises can generate corrupted derivative components; and (2) the entirely smooth $\theta_r$ cannot be guaranteed such that large signal is also difficult to be avoided when implementing differentiation. Therefore, the pitch angular rate feedback is utilized instead of a pitch error rate in order to achieve satisfactory damping characteristic. From the GPM point of view, Eq. (5) is equivalent to the controller $G_c(s)$ shown in Fig. 2 as follows:

$$G_c(s) = k_p + \frac{k_i}{s} + k_d s$$  \hspace{1cm} (6)

In practice, $k_i$ and $k_d$ are selected far less than $k_p$ to ensure: (a) no integral windup; (b) tiny oscillation caused by the integral action; and (c) insensitivity to the measurement noises from the rate gyro which can sense $q$ directly. Therefore, Eq. (6) can be rewritten as follows:

$$G_c(s) = \frac{k_i(T_\alpha s + 1)(T_\beta s + 1)}{s}$$  \hspace{1cm} (7)

where both $T_\alpha$ and $T_\beta$ are positive. This is a pivotal step to continue the following investigations or to generalize the approaches used in [2] to PID control. Combining Eqs. (6) and (7) yields

$$\begin{cases} k_p = k_i(T_\alpha + T_\beta), \\ k_d = k_i T_\alpha T_\beta. \end{cases}$$  \hspace{1cm} (8)

According to Fig. 2, the open-loop transfer function is as follows:

$$G_c(s)G_p(s) = \frac{k_i k_m(T_n s + 1)(T_n s + 1)(T_b s + 1)}{s^2 (\tau s - 1)(T_d s + 1)} e^{-sL}.$$  \hspace{1cm} (9)

![Fig. 1. Block diagram of PID pitch attitude control.](image1)

![Fig. 2. Reformulation of Fig. 1 within the GPM framework.](image2)
In the sequel, $A_r$, $A_l$, $A_m$, and $\phi_m$ are defined as the increasing gain margin, the decreasing gain margin, the specified gain margin and the specified phase margin of Eq. (9), respectively. It is evident that $A_r > 1$ and $A_l < 1$. Considering the symmetry, the critical gain margin is $A_m$ or

$$A_m = \min(A_l^{-1}, A_r)$$

(10)

The problems to be solved are:

(a) when $A_m = A_r$, obtaining the PID tuning rule and the feasible iGPM region as well as analyzing the accuracy;
(b) when $A_m = A_l^{-1}$, obtaining the PID tuning rule and the feasible dGPM region.

3. PID tuning rules based on iGPM

3.1. PID tuning rules

Herein, the explicit analytical PID tuning rules will be derived based on the iGPM specifications.

According to the definitions of $A_r$ and $\phi_m$, the following equations are obtained as follows:

$$\begin{align*}
\arg [G_c(j\omega_p)G_p(j\omega_p)] &= -\pi, \\
A_r &= \frac{1}{(G_c(j\omega_p)G_p(j\omega_p))}, \\
|G_c(j\omega_g)G_p(j\omega_g)| &= 1, \\
\phi_m &= \arg [G_c(j\omega_g)G_p(j\omega_g)] + \pi.
\end{align*}$$

(11)

where $\omega_p$ and $\omega_g$ are the crossover frequencies corresponding to the iGM and PM, respectively. Substituting Eq. (9) into (11) gives

$$\begin{align*}
-A_r + \arctan \omega_p T_a + \arctan \omega_p T_b + \arctan \omega_p T_n + \arctan \omega_p \tau - \arctan \omega_p T_d - \omega_p L &= 0, \\
A_r k_l k_m &= \omega_p^2 \sqrt{\frac{(\omega_p^2 \tau^2 + 1)(\omega_p^2 T_a^2 + 1)}{(\omega_p^2 T_a^2 + 1)(\omega_p^2 T_b^2 + 1)(\omega_p^2 T_n^2 + 1)}}, \\
k_l k_m &= \omega_g^2 \sqrt{\frac{(\omega_g^2 \tau^2 + 1)(\omega_g^2 T_a^2 + 1)}{(\omega_g^2 T_a^2 + 1)(\omega_g^2 T_b^2 + 1)(\omega_g^2 T_n^2 + 1)}}, \\
\phi_m &= -\pi + \arctan \omega_g T_a + \arctan \omega_g T_b + \arctan \omega_g T_n + \arctan \omega_g \tau - \arctan \omega_g T_d - \omega_g L.
\end{align*}$$

(12)

The analytical tuning rules are not available in the presence of $\arctan$ function. To bypass this problem, the following approximation, which was proposed in [24], can be employed

$$\arctan(x) = \begin{cases} 
x & 0 \leq x \leq 1 \\
\frac{x}{x^2 + 1} - \frac{1}{x} & x > 1
\end{cases}$$

(13)

From the engineering experience, the numerical solutions of Eq. (12) for unstable flight dynamics show that all of $\omega_p T_a$, $\omega_p T_b$, $\omega_p T_n$, $\omega_p \tau$, $\omega_p T_d$, $\omega_g T_a$, $\omega_g T_b$, $\omega_g T_n$, $\omega_g \tau$, and $\omega_g T_d$ are
far more than 1, which can be used to approximate Eq. (12) as follows:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{\varphi}{2} - \frac{1}{\omega_0 T_a} - \frac{1}{\omega_0 T_b} - \frac{1}{\omega_0 T_d} - \frac{1}{\omega_0 \tau} + \frac{1}{\omega_0 L} - \omega_p L \approx 0, \\
A_r \kappa k_m \approx \omega_p \sqrt{\frac{T_a}{T_b T_a}}, \\
k_i k_m \approx \omega_g \sqrt{\frac{T_d}{T_a T_a}}, \\
\phi_m \approx \frac{\varphi}{2} - \frac{1}{\omega_0 T_a} - \frac{1}{\omega_0 T_b} - \frac{1}{\omega_0 \tau} + \frac{1}{\omega_0 L} - \omega_g L.
\end{array} \right.
\]

(14)

Solving Eq. (14) gives rise to

\[
\begin{aligned}
\omega_g &= \frac{\phi_m + (\pi/2)(A_r - 1)}{(A_r^2 - 1)L}, \\
\omega_p &= A_r \omega_g, \\
T_a T_b &= \frac{\phi_m + (\pi/2)(A_r - 1)}{(A_r^2 - 1)L} \sqrt{\frac{T_d}{k_m T_a}}, \\
T_a + T_b &= \frac{\phi_m + (\pi/2)(A_r - 1)}{(A_r^2 - 1)L} \sqrt{\frac{T_d}{k_m T_a}} \left( A_r \frac{(\pi/2)(A_r - 1) + \phi_m}{(A_r^2 - 1)^2 L} - \frac{1}{\tau'} \right).
\end{aligned}
\]

(15)

where

\[
\frac{1}{\tau'} = \frac{1}{\tau} + \frac{1}{T_a} - \frac{1}{T_d}.
\]

(16)

For an unstable flight vehicle, the time constants of the right hand side (RHS) of Eq. (16) make \(\tau' < 0\) in most cases. Combining Eqs. (8) and (15) obtains

\[
\begin{aligned}
k_p &= \frac{\phi_m + (\pi/2)(A_r - 1)}{(A_r^2 - 1)L} \sqrt{\frac{T_d}{k_m T_a}} \left( A_r \frac{(\pi/2)(A_r - 1) + \phi_m}{(A_r^2 - 1)^2 L} - \frac{1}{\tau'} \right), \\
k_d &= \frac{\phi_m + (\pi/2)(A_r - 1)}{(A_r^2 - 1)L} \sqrt{\frac{T_d}{k_m T_a}}.
\end{aligned}
\]

(17)

Eqs. (15) and (17) play the key role in the sequel: when \(T_a T_b, T_a + T_b, k_p, \) and \(k_d \) emerge, the substitutions are implemented to obtain the available \((A_r, \phi_m)\) region.

As can be seen from Eq. (17), a reasonable integral control has nothing to do with the iGPM specifications. Taking the approximation factor into account, the effect of the integral control on the iGPM is still neglectable, which will be verified by the numerical examples. It should be noted that the results of Eq. (17) are completely identical to the ones obtained with a PD control in [2], which is a crucial point and should be emphasized.

### 3.2. Feasible iGPM region

Although \(k_i\) has little effect on the iGPM specifications, its impact on the available iGPM range should be investigated.

There are comprehensive constraints imposed on the choice of iGPM pairs. First of all \(k_p\) should be positive and one can obtain

\[
A_r \left( \frac{\pi/2(A_r - 1) + \phi_m}{(A_r^2 - 1)^2 L} \right) > \frac{1}{\tau'}
\]

(18)

Based on Eq. (17), Eq. (18) can be rearranged into a polynomial form with regard to \(\phi_m\) as follows:

\[
A_r^2 \phi_m^2 + \frac{\pi A_r (A_r - 1)^2}{2} \phi_m + \frac{(A_r^2 - 1)^2 L}{\tau'} - \frac{\pi^2 A_r (A_r - 1)^2}{4} < 0.
\]

(19)
With the help of the software developed in [2], the feasible intervals of $\phi_m$ can be obtained for each specific $A_r$. For the sake of legibility, a symbol $\Phi_{19}(A_r)$ is used to denote the $\phi_m$ interval solution for $A_r$, where 19 represents the sequence number of the inequality in the paper. Note that the reasonable interval for $\phi_m$ should be within $(0, \pi/2)$.

In addition, another condition requires

$$ (T_a + T_b)^2 > 4T_a T_b $$

(20)

to guarantee the existence of pairs of $T_a$ and $T_b$, that is

$$ \frac{\phi_m + \pi/2(A_r - 1)}{(A_r^2 - 1)L} \cdot \frac{\pi T_d}{k_j k_m T_n} \left( A_r \left( \frac{\pi/2(A_r - 1) + \phi_m}{(A_r^2 - 1)^2 L} - \frac{1}{T} \right)^2 > 4 \right. $$

(21)

Based on Eq. (17). Similar to Eq. (19), the inequality (21) can also be reformulated as follows:

$$ \sum_{i=0}^{5} \mu_{1,i}^i \phi_m < 0 $$

(22)

where $\mu_{1,i}$ is a polynomial in terms of $A_r$.

So far, only $A_r$ has been considered. The unstable plants demonstrate two gain margins: $A_r$ and $A_l$. The former one corresponds to the divergent oscillation because of a tight control, while the latter one corresponds to the control capability deficiency. The definition of GM refers to the critical GM, therefore $A_l$ should also be analyzed. On the basis of the PID tuning rules obtained earlier, Padé approximation for the time-delay can be utilized to calculate $A_l$ because $A_l$ corresponds to a very low frequency band which falls within the effective bound of Páde approximation. The simplest Páde approximation

$$ e^{-Ls} \approx \frac{1}{Ls + 1} $$

(23)

is employed amid the low frequencies around $A_l$. Its accuracy will be tested by several illustrative examples later. Generally speaking, the time constant $T_d \gg L$ and the open-loop transfer function Eq. (9) can be approximated as follows:

$$ G_p(s)G_c(s) \approx \frac{k_j k_m (T_d + L)(T_d s + 1)}{s^2 (\tau s - 1)(T_d s + 1)} \frac{1}{Ls + 1} $$

$$ \approx \frac{k_j k_m (T_d s + 1)(T_d s + 1)(T_b s + 1)}{s^2 (\tau s - 1)((T_d + L)s + 1)} $$

(24)

Designating the multiplicative gain uncertainty of $G_p(s)$ as $A_s$ yields

$$ G_o = A_s G_p(s)G_c(s) = \frac{A_s k_j k_m (T_n s + 1)(T_o s + 1)(T_b s + 1)}{s^2 (\tau s - 1)((T_d + L)s + 1)} $$

(25)

and the corresponding closed-loop characteristic equation is as follows:

$$ p_c(s) = \tau(T_d + L)s^4 + ((\tau - T_d - L) + A_s k_j k_m T_n T_a T_b)s^3 $$

$$ + (A_s k_j k_m (T_n T_a + T_n T_b + T_a T_b) - 1)s^2 + A_s k_j k_m (T_n + T_a + T_b)s + A_s k_j k_m $$

(26)

When $A_r$ is critical, one has $A_l < A_r^{-1}$ and $A_m = A_r$. Hence, the closed-loop system should be stable for $\forall A_s \in (1/A_r, 1]$. According to the Routh stability criterion, the necessary and sufficient
condition is as follows:
\[
\begin{align*}
\begin{cases}
 k_kk_m(T_nT_a + T_nT_b + T_aT_b) > A_r \\
 (T_n + T_a + T_b)(A_xk_kk_m(T_nT_a + T_nT_b + T_aT_b) - 1)
\end{cases}
\end{align*}
\]
\[
-\tau(T_d + L)A_xk_kk_m(T_n + T_a + T_b)^2 > ((\tau - T_d - L) + A_xk_kk_mT_nT_aT_b)
\]
\]
\[
(27)
\]
It is clear that the first inequality of Eq. (27) is free of $A_x$ and the second inequality is a quadratic polynomial in terms of $A_x$ as follows:
\[
p_c(A_x) = aA_x^2 + bA_x + c > 0
\]
where
\[
a = k_k^2k_m^2T_nT_aT_b((T_n + T_a + T_b)(T_nT_a + T_nT_b + T_aT_b) - T_nT_aT_b) > 0,
\]
\[
b = k_kk_m((T_n + T_a + T_b)((\tau - T_d - L)(T_nT_a + T_nT_b + T_aT_b) - T_nT_aT_b)
\]
\[
-\tau(T_d + L)(T_n + T_a + T_b)^2 - 2T_nT_aT_b(\tau - T_d - L)
\]
\[
c = -(T_n + T_a + T_b)(\tau - T_d - L) - (\tau - T_d - L)^2.
\]
Defining
\[
\begin{align*}
T_aT_b &= \alpha_1\phi_m + \alpha_0 \\
T_a + T_b &= \beta_2\phi_m^3 + \beta_2\phi_m^2 + \beta_1\phi_m + \beta_0
\end{align*}
\]
where $\alpha_i(i = 0, 1)$ and $\beta_j(i = 0, \cdots, 3)$ are the rational functions with regard to $A_r$, respectively, the first inequality of Eq. (27) can be reformulated as follows:
\[
k_kk_mT_n\beta_2\phi_m^3 + k_kk_mT_n\beta_2\phi_m^2 + k_kk_m(T_n\beta_1 + \alpha_1)\phi_m + k_kk_m(T_n\beta_0 + \alpha_0) - A_r > 0.
\]
After thorough examination, it can be found that Eq. (33) is independent of $k_i$ because it appears as a factor of each denominator of $\alpha_i$ and $\beta_i$. The $\phi_m$ solution interval for Eq. (33) is $\Phi_{35}(A_r)$. For the second inequality of Eq. (27), two cases should be taken into account separately: $c \leq 0$ and $c > 0$. In both cases, the nominal stability when $A_x = 1$ can be guaranteed as follows:
\[
p_c(1) > 0 \iff a + b + c > 0
\]
whose $\phi_m$ solution is $\Phi_{34}(A_r)$. Letting
\[
c \leq 0
\]
gives the $\phi_m$ solution as $\Phi_{35}(A_r)$. It is obvious that $p_c(A_r) = 0$ has two different real roots: one is positive and the other is negative. The positive root, $(\sqrt{b^2 - 4ac} - b)/(2a)$, is associated with $A_l$ as follows:
\[
A_l = \frac{\sqrt{b^2 - 4ac} - b}{2a} < \frac{1}{A_r}
\]
which can be transformed into
\[
2a + A_rb > 0
\]
and
\[
A^2_c(b^2 - 4ac) < (2a + A_rb)^2
\]
For another case of \(c > 0\), \(p_c(A_r) = 0\) should have two real roots and both must locate on \((0, 1)\) to ensure the nominal stability condition of Eq. (34) as shown in Fig. 3. Therefore, two inequalities

\[
c > 0
\]  
(39)

and

\[
b^2 - 4ac > 0
\]  
(40)

are added for this case to serve as complementarities to Eq. (36). In conclusion, the feasible \(\phi_m\) solution for Eq. (27) is as follows:

\[
\Phi_{iGPM}(A_r) = \Phi_{19}(A_r) \cap \Phi_{22}(A_r) \cap \Phi_{27}(A_r).
\]  
(42)

When \(A_r\) varies on a given interval, the envelope of \(\Phi_{iGPM}(A_r)\) generates a feasible iGPM region.

### 3.3. Applicability analysis

In the aforementioned deduction, the assumptions are made as follows:

\[
k_p + \frac{k_i}{s} + k_ds = \frac{k_i(T_a s + 1)(T_b s + 1)}{s}
\]  
(43)

and both \(\omega g T_a\) and \(\omega g T_b\) are much greater than 1. Here, these conditions need to be quantified to obtain an applicable curve for the PID tuning rule or even the PD one proposed in [2].

The precision of the assumptions can be measured by a threshold \(V_T\) as follows:

\[
\min(T_a, T_b)\omega_g > V_T
\]  
(44)

and according to (43), that is

\[
\frac{2k_d}{k_p + \sqrt{k_p^2 - 4k_d k_i}} \omega_g > V_T.
\]  
(45)
It implies that
\[ k_i < \frac{k_p^2}{4k_d} \]  
which is a preliminary estimation for the range of \( k_i \). Rearranging Eq. (45) results in
\[ \sqrt{k_p^2 - 4k dk_i} < \frac{2k_d\omega_g}{V_T} - k_p \]  
which can be decomposed into
\[ \frac{2k_d\omega_g}{V_T} - k_p > 0 \]  
as well as
\[ k_i > \frac{k_p\omega_g}{V_T} - \frac{k_d\omega_g^2}{V_T^2}. \]  
From Eq. (48), an applicable curve \( \Phi_{48}(A_r) \) can be obtained. As already mentioned above, moderate \( k_i \) has little effect on GPM, therefore \( \Phi_{48}(A_r) \) is also the boundary of the effective area for the PD tuning rule proposed in [2]. The applicability area of the PD tuning rule was not solved in [2], but it is obtained within the PID control framework using the proposed method.

4. PID tuning rules based on dGPM

The critical gain margin can be achieved at either \( A_r \) or \( A_l \). In this section, we attempt to investigate the dGPM case on the basis of the iGPM specification formulae.

According to Eq. (36), \( A_l \) is critical provided that
\[ A_l = \frac{\sqrt{b^2 - 4ac} - b}{2a} > \frac{1}{A_r} \]  
which can be reformulated as follows:
\[ 2a + A_r b < 0 \cup \left\{ \begin{array}{l} 2a + A_r b > 0 \\ A_t^2 (b^2 - 4ac) > (2a + A_r b)^2. \end{array} \right. \]  
It is the essential point in the following study. The task is to find a feasible \( A_r \) range for a specific \( \phi_m \) to meet the requirement of \( A_r > A_t^{-1} \) and then to determine the value domain of
\[ g(A_r) = \frac{2a}{\sqrt{b^2 - 4ac} - b} \]  
for the parts below the line of \( g(A_r) = A_r \) within this \( A_r \) range. It should be noted that there is a fundamental difference between iGPM and dGPM investigations: \( A_r \) and \( \phi_m \) are fixed respectively to plot the feasible GPM regions.

First of all, the concerned intervals of \( A_r \) for \( A_t^2 (b^2 - 4ac) > (2a + A_r b)^2 \) are to be solved.

To facilitate the research, several terms are represented by the rational functions with regard to \( A_r \) in advance as follows:
\[ T_a T_b = \frac{\sum_{i=1}^{l} \xi_1 A_i}{A_t^2 - 1} \]
and
\[
T_a + T_b = \sum_{i=0}^{5} \frac{\xi_{2,i} A^i_j}{(A^i_j - 1)^3}
\]
(54)

where \(\xi_{n,i}(n = 1, 2)\) are functions of \(\phi_m\). Note that these coefficients can be numerically calculated for a specific \(\phi_m\) via recursion by using conv function in Matlab and it is unnecessary to present them with analytically complicated forms. From Eqs. (53) and (54), one can obtain

\[
\begin{align*}
    a &= \frac{\sum_{i=0}^{12} \xi_{5,i} A^i_j}{(A^i_j - 1)^7} \\
    b &= \frac{\sum_{i=0}^{12} \xi_{4,i} A^i_j}{(A^i_j - 1)^6} \\
    c &= \frac{\sum_{i=0}^{6} \xi_{5,i} A^i_j}{(A^i_j - 1)^3}
\end{align*}
\]
(55)

This is the foundation for the subsequent analysis. Now, we have

\[
(2a + bA_r)^2 < A^2_r(b^2 - 4ac)
\]
\[
\Leftrightarrow \left(2 \sum_{i=0}^{12} \frac{\xi_{5,i} A^i_j}{(A^i_j - 1)^7} + \sum_{i=0}^{12} \frac{\xi_{4,i} A^i_j}{(A^i_j - 1)^6} A_r \right)^2 < A^2_r \left( \sum_{i=0}^{12} \frac{\xi_{4,i} A^i_j}{(A^i_j - 1)^6} \right)^2
\]
\[
- 4 \sum_{i=0}^{12} \frac{\xi_{5,i} A^i_j}{(A^i_j - 1)^7} \sum_{i=0}^{6} \frac{\xi_{5,i} A^i_j}{(A^i_j - 1)^3}
\]
(56)

or

\[
(2 \sum_{i=0}^{12} \xi_{5,i} A^i_j + A_r(A^2_r - 1)(\sum_{i=0}^{12} \xi_{4,i} A^i_j))^2
\]
\[
< A^2_r(A^2_r - 1)^2 \left( \sum_{i=0}^{12} \xi_{4,i} A^i_j \right)^2 - 4(A^2_r - 1)^2 \left( \sum_{i=0}^{12} \xi_{5,i} A^i_j \right)^2 \left( \sum_{i=0}^{6} \xi_{5,i} A^i_j \right)
\]
(57)

Although Eq. (57) seems to be a 30th-order polynomial, it can be found that the two highest-order terms are produced by \(\sum_{i=0}^{12} \xi_{4,i} A^i_j\) on both sides and they can be canceled each other totally, and Eq. (57) is a 28th-order polynomial in nature. Here the symbol \(\Omega_57(\phi_m)\) is used to denote the \(A_r\) interval solution for a given \(\phi_m\), where Eq. (57) represents the sequence number of the inequality in the paper. Note that the reasonable interval for \(A_r\) should be within \((1, \infty)\).

On the sign of \(2a + A_r b\), we have two parts as follows:

\[
2a + A_r b < 0
\]
(58)

and

\[
2a + A_r b > 0
\]
(59)

respectively. Herein, a preliminary solution to Eq. (51) is given as follows:

\[
\Omega_{51,p}(\phi_m) = \Omega_{58}(\phi_m) \cup (\Omega_{59}(\phi_m) \cap \Omega_{57}(\phi_m)).
\]
(60)

Several other constraints should be supplemented as aforesaid, such as Eqs. (19), (21), and (27). The first inequality of Eq. (27) is reformulated as follows:

\[
\sum_{i=0}^{6} \xi_{6,i} A^i_j < 0
\]
(61)
Similar to Section 3.2, the exact solution to Eq. (27) is as follows:

$$\Omega_{27}(\phi_m) = \Omega_{61}(\phi_m) \cap (\Omega_{34}(\phi_m) \cap \Omega_{37}(\phi_m) \cap \Omega_{38}(\phi_m) \cap (\Omega_{35}(\phi_m) \cup (\Omega_{39}(\phi_m) \cap \Omega_{40}(\phi_m))))$$

In this way, the true solution to (51) can be pieced together as

$$\Omega_{51}(\phi_m) = \Omega_{51,0}(\phi_m) \cap \Omega_{19}(\phi_m) \cap \Omega_{21}(\phi_m) \cap \Omega_{27}(\phi_m).$$

Next, the value domain of $g(\Lambda_r)$ on $\Omega_{51}(\phi_m)$ will be yielded. Since $\Omega_{51}(\phi_m)$ consists of a certain intervals, the entire value domain is a union set. For each interval belonging to $\Omega_{51}(\phi_m)$, its value domain is bounded by its minimal and maximal values according to the continuity of $g(\Lambda_r)$. These extrema must be achieved at either the boundary points or the local optimal points. Because $g(\Lambda_r)$ is a rational function with respect to $\Lambda_r$, all local optimal points can be sought in a straightforward way. One has

$$g'(\Lambda_r) \sim a'\sqrt{b^2 - 4ac - b} - a\left(\frac{bb' - 2a'c - 2ac'}{\sqrt{b^2 - 4ac}} - b\right)$$

where $\sim$ means that both sides have the same roots, and $x' = \partial x/\partial \Lambda_r$. Letting $g'(\Lambda_r) = 0$ obtains

$$a'b^2 - 2a'ac - abb' + 2a^2c' = (a'b - ab')\sqrt{b^2 - 4ac}$$

whose $\Lambda_r$ roots are entirely contained in the $\Lambda_r$ root set of

$$(a'b^2 - 2a'ac - abb' + 2a^2c')^2 = (a'b - ab')^2(b^2 - 4ac)$$

where

$$\left\{\begin{array}{l}
a' = \sum_{i=0}^{\Omega_{51}} \frac{\delta_{i,0} A_i'}{(A_i' - 1)^{1/4}} \\
b' = \sum_{i=0}^{\Omega_{51}} \frac{\delta_{i,1} A_i'}{(A_i' - 1)^{1/2}} \\
c' = \sum_{i=0}^{\Omega_{51}} \frac{\delta_{i,2} A_i'}{(A_i' - 1)^{3/4}}
\end{array}\right.$$  \hspace{1cm} (67)

After algebraic manipulations, a polynomial in terms of $\Lambda_r$ with at most 98th-order can be derived. Its roots on $\Omega_{51}(\phi_m)$, denoted as $\Xi_{66}$, can be obtained. For each element $r_i$ in $\Xi_{66}$, $r_i$ is a root of Eq. (65) if

$$(a'b^2 - 2a'ac - abb' + 2a^2c')(a'b - ab')\bigg|_{\Lambda_r = r_i} \geq 0.$$  \hspace{1cm} (68)

Those $r_i$ satisfying Eq. (68) and the terminal points of all interval solutions of $\Omega_{51}(\phi_m)$ form a set $\Xi_{56}$. Consequently, for each interval solution $I_a \in \Omega_{51}(\phi_m)$, the maximal value $V_{\text{max}}^{I_a}$ and the minimal value $V_{\text{min}}^{I_a}$ are determined based on those elements of $\Xi_{56}$ belonging to $I_a$.

Eventually, the feasible $\Lambda_r$ range for a given $\phi_m$ is

$$\Omega_{dGPM}(\phi_m) = \cup [V_{\text{min}}^{I_a}, V_{\text{max}}^{I_a}].$$  \hspace{1cm} (69)

When $\phi_m$ varies on $(0, \pi/2)$, the envelope of $\Omega_{dGPM}(\phi_m)$ generates the feasible dGPM region.

In the process of establishing the feasible dGPM region, a tuning rule based on the dGPM specifications is immediately obtained. For each point $(\Lambda_m, \phi_m)$ within the feasible dGPM region,
according to the definition of $\Omega_{\text{dGPM}}(\phi_m)$, there must be a $A_r > A_m$ satisfying

$$\frac{2a}{\sqrt{b^2 - 4ac - b}} = A_m$$

(70)

and the PD control parameters for a given $k_i$ are given by Eq. (17) in a direct fashion.

5. Numerical examples

In this section, two illustrative examples are presented to demonstrate the effectiveness and accuracy of the proposed method. The characteristic parameters of two operating points of a flight vehicle during different phases are presented in Table 1.

5.1. First example

For the first example, the feasible iGPM regions are shown in Fig. 4 with different $k_i$ of 0, 0.2, and 0.5. It is apparent that the region expands with the decrement of $k_i$. The design point $(A_m, \phi_m) = (5, 50')$ is selected and the corresponding $k_p = 0.5443$ and $k_d = 0.2278$ according to (17), the exact GPMs for these $k_i$ are $(5.0119, 51.2')$, $(5.0119, 50.6')$ and $(-3.3497, 49.6')$, respectively. Herein, a negative GM implies that $A_i$ is critical, i.e. $(-3.3497, 49.6')$ means that $A_m = A_r^{-1} = 3.3497$. These results are compatible with Fig. 4. The effects of the bandwidth of the actuator on the achievable iGPM region are shown in Fig. 5 for $k_i = 0.2$. It is evident that the achievable iGPM region monotonically contracts with the increment of the time-delay or the reciprocal of the bandwidth of the actuator, $L$, matching our empirical knowledge. According to Fig. 5, the bandwidth of the actuator should not be lower than $1/0.07 \approx 14$ Hz to meet the basic practical GPM requirements, such as $P_m \geq 45'$. This matching relationship between the achievable performance and the bandwidth of the actuator provides an overall perspective for designers to make a reasonable decision on both the macro and the micro levels for system design.

Next, the effects of $k_i$ on the GPM and the dynamic performance are explicitly revealed. A common feasible point $(A_m, \phi_m) = (4, 45')$ for these three mentioned $k_i$ is chosen to obtain $k_p = 0.8644$ and $k_d = 0.2811$, and the Bode diagrams are illustrated in Fig. 6, wherein the upper GPM values correspond to $k_i = 0.5$. It is evident that their high frequency parts exceeding the lower crossover frequencies match each other perfectly, which demonstrates that $k_i$ has little effect on the iGPM specifications. The exact tuning results are shown in Table 2, and the discrepancies among them are quite tiny in spite of remarkably varying values of $k_i$. Although several approximations are used to derive the tuning rules, the accuracy is still highly impressive. The step responses for these $k_i$ are shown in Fig. 7, where an apparently reduced settling time can be realized with a moderate integral action while the iGPM values are almost unchanged. This fact, originated from the unstable dynamics, contradicts the traditional conception of the

<table>
<thead>
<tr>
<th>Case/parameters</th>
<th>$\tau$ (s)</th>
<th>$T_d$ (s)</th>
<th>$T_n$ (s)</th>
<th>$k_m$</th>
<th>$L(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.7490</td>
<td>0.7448</td>
<td>279.74</td>
<td>0.0520</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>0.5561</td>
<td>0.4894</td>
<td>6.9538</td>
<td>0.15281</td>
<td>0.04</td>
</tr>
</tbody>
</table>
deterioration of robustness using an integral feedback for an open-loop stable plant. It should be noted that $L$ has indistinguishable effect on the time-domain property, thus it is abandoned when plotting the step response for simplicity. The PD control is unable to eliminate the long settling time \[3T_n\] or 800 s for this example, an intolerable order of magnitude. Nonetheless, a precondition should be pointed out that $k_i$ cannot be very large due to the constraint of Eq. (46). On the other hand, the windup phenomenon may also arise with a large $k_i$. As can be seen in Fig. 7, the overshoot increases with the increment of $k_i$. This problem is not critical since the true pitch command is quite continuous with an appropriate variation rate to guarantee moderate angle of attack in practical guidance design such that the overshoot can be effectively smoothed. According to Eq. (46), the possible largest $k_i$ for this design point is 0.6620, but it is a conservative estimation. In fact, the iGPM specifications are violated when $k_i$ is
larger than 0.95 which also triggers an aggressive control effort leading to excessive overshoot and it is generally not acceptable to practitioners.

Then the accuracy of the boundary of the feasible iGPM region is to be tested. For $k_i = 0.2$, we consider the design points $(A_m, \phi_m) = (4, 55)$ and $(4, 59)$, and both of them are close to the boundary. Their exact GPMs under the tuning rule are $(3.9811, 46.2)$ and $(3.9811, 45.6)$, respectively. It illustrates that good accuracy of the boundary is achieved and the tolerable errors are caused by using several approximations.

Now, it is time to establish the applicable area of this tuning rule. For brevity, we only consider the case of $k_i = 0.0$ or a PD control. This is because that Eq. (48) is independent of $k_i$, and we have demonstrated that $k_i$ has almost no impact on the exact GPMs. In Eq. (48), $V_T$ is set to two values of 2.0 and 3.0 for two cases, and the corresponding applicable curves are shown in Fig. 8 wherein the areas above the curves are the effective ones. To understand the physical meaning of the applicable area, the anticipated $A_m$ is fixed at 4.0 and the different values of $\phi_m$ are chosen as 10, 20, 30, 40, and 50 respectively. The exact GPMs are listed in Table 3. It is.

Table 2
Tuning results for example 1 with iGPM specifications.

<table>
<thead>
<tr>
<th>$k_i$</th>
<th>Specified GPM</th>
<th>Exact GPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(4, 45°)</td>
<td>(3.9811, 46.2°)</td>
</tr>
<tr>
<td></td>
<td>(−4, 45°)</td>
<td>(−4.0738, 48.9°)</td>
</tr>
<tr>
<td>0.2</td>
<td>(4, 45°)</td>
<td>(3.9811, 46.2°)</td>
</tr>
<tr>
<td></td>
<td>(−4, 45°)</td>
<td>(−4.0272, 46.7°)</td>
</tr>
<tr>
<td>0.5</td>
<td>(4, 45°)</td>
<td>(3.9811, 45.6°)</td>
</tr>
<tr>
<td></td>
<td>(−4, 45°)</td>
<td>(−3.9355, 45.5°)</td>
</tr>
</tbody>
</table>

Fig. 6. Bode diagrams for example 1 with iGPM specifications (loop transfer functions: 
\[ \frac{1.33x^2+4.09x+0.0146}{0.0181x^2+0.001366x-0.3252}e^{-0.05x} (k_i = 0.0), \]
\[ \frac{4.08x^4+12.59x^2+2.95x+0.0014}{0.5579x^4+0.0042x^2-x^2}e^{-0.05x} (k_i = 0.2) \]
and 
\[ \frac{0.5579x^4+0.0042x^2-x^2}{0.5579x^4+0.0042x^2-x^2}e^{-0.05x} (k_i = 0.5)). \]
obvious that the PM precision is higher within the area generated by a larger $V_T$. From another point of view, the higher accuracy can be achieved with a larger $\phi_m$. In practice, the desirable $\phi_m$ is normally larger than 30°, which is above $V_T = 3$ curve almost completely. To further validate this boundary, consider the case of $(A_m, \phi_m) = (2, 20)$, which has a smaller $\phi_m$ than the usual practice cases. The tuning result is shown in Fig. 9, where the achieved PM precision is enough. Therefore, $V_T = 3$ is the proper alternative to produce an applicable curve. It is worth mentioning that the applicable area for the PD control is obtained by a thorough consideration of an augmented PID strategy.
At last, we proceed with the dGPM case to make fair comparisons. The feasible dGPM regions are shown in Figs. 10 and 11 for \( k_i = 0.2 \) with \( L = 0.05s \) and \( L = 0.07s \), respectively. The singular part is due to the approximation errors together with the numerical errors generated in the course of seeking the roots of high-order polynomials using Matlab. Comparing these two figures, it is clear that the achievable dGPM region monotonically contracts with the decrement of the bandwidth of the actuator. The feasible point \((A_m, \phi_m) = (-4, 45')\) is selected with \( k_i = 0.2 \) and \( L = 0.05s \), the corresponding \( A_r = 6.2749 \) and the control parameters are: \( k_p = 0.4714 \) and \( k_d = 0.1813 \). According to Fig. 4, it can be seen that \((A_r, \phi_m) = (6.2749, 45')\) is quite close to the boundary of the feasible iGPM region. Comparing with the design point \((4, 45')\), the dGPM specifications can bring about lower feedback gains than the iGPM ones. Therefore, the dGPM specifications are less sensitive to the measurement noises. The Bode diagram is shown in Fig. 12. The step responses of the PID control and the related PD control are shown in Fig. 13. Comparing with Fig. 7, the lower feedback gains generate larger overshoot. Unlike the iGPM case, \( k_i \) has a strong effect on the dGPM specifications. For example, with the identical PD control gains as used in Fig. 12, the Bode diagrams are shown in

<table>
<thead>
<tr>
<th>Specified PM</th>
<th>Exact GM</th>
<th>Exact PM</th>
<th>PM error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3.8905</td>
<td>26.2</td>
<td>16.2</td>
</tr>
<tr>
<td>20</td>
<td>3.9355</td>
<td>30.4</td>
<td>10.4</td>
</tr>
<tr>
<td>30</td>
<td>3.9355</td>
<td>35.7</td>
<td>5.7</td>
</tr>
<tr>
<td>40</td>
<td>3.9811</td>
<td>42.3</td>
<td>2.3</td>
</tr>
<tr>
<td>50</td>
<td>3.9811</td>
<td>50.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Fig. 9. Bode diagram for example 1 with iGPM specifications of \((G_m, \phi_m) = (2, 20')\) (loop transfer function: \(0.9583s^2 + 7.143s + 0.02552e^{-0.05s} / 0.03257s + 0.0005637s^2 + 0.1342s\))
Figs. 14 and 15 with $k_i = 0$ (PD control) and $k_i = 0.5$, respectively. It is apparent that the role of $k_i$ in the tuning rule must be taken into account. In a word, Eqn. (70) needs to be solved for a specific $k_i$ to obtain PD gains. For the different values of $k_i$ in Table 2, the results of the dGPM-specifications-based tuning rule are also summarized in Table 2. The precision is acceptable. Furthermore considering the points $(A_m, \phi_m) = (-5, 50)$ and $(-5, 51)$, both are close to the boundary respectively, the exact GPM for the former one is $(-5.0119, 50.6)$ and the latter one has no solution, which validates the accuracy of the dGPM boundary.

![Fig. 10. Feasible dGPM region for example 1 with $k_i = 0.2$ and $L = 0.05s$.](image1)

![Fig. 11. Feasible dGPM region for example 1 with $k_i = 0.2$ and $L = 0.07s$.](image2)
5.2. Second example

Herein, the second example is provided to further validate the proposed method. The feasible iGPM regions are shown in Fig. 16 with different $k_i$ of 0 and 1.0. A common feasible point $(A_m, \phi_m) = (4, 45)$ is selected to obtain $k_p = 9.2463$ and $k_d = 2.3468$. The exact GPMs are shown in Table 4, respectively. The step responses for these $k_i$ are shown in Fig. 17. This
example, together with the previous one, turns out that $k_i$ should be chosen as the same order of magnitude as that of $k_d$ in order to ameliorate the dynamic performance while maintaining the iGPM unaltered.

Then the accuracy of the boundary is also to be checked. Fix $k_i$ at 1.0. We begin with the left boundary. Considering the points $(A_m, \phi_m) = (2, 45^\circ)$ and $(2, 46^\circ)$ (both are close to the boundary), the tuning rule results in a positive and a negative $k_p$, respectively, which implies

$$\text{Fig. 14. Bode diagram for example 1 with PD control (loop transfer functions: } 1.014 s^2 + 2.641 s + 0.009428 e^{-0.05s}). \text{)}$$

$$\text{Fig. 15. Bode diagram for example 1 with PID control and } k_i = 0.5(\text{loop transfer functions: } 2.635 s^3 + 6.866 s^2 + 7.298 s + 0.026 e^{-0.05s}). \text{)}$$
a switch from closed-loop stability to instability. Afterwards, we continue the right boundary investigation. Consider the points \( (A_m, \phi_m) = (5, 50) \) and \((5, 51)\) which are also close to the boundary, their exact GPMs are \((5.0119, 50.9)\) and \((-4.8978, 51.8)\), respectively.

Subsequently, the applicable area of the tuning rule is determined. \(V_T\) is set with two values of 2.0 and 3.0 as shown in Fig. 18. The expected \(A_m\) is fixed at 4.0 with different values of \(\phi_m\) at 10, 20, 30, 40 and 50, respectively. The exact GPMs are presented in Table 5. It is indicated again that the PM precision is enhanced with higher \(V_T\). Consider the case of \((A_m, \phi_m) = (2, 18)\) which has a smaller \(\phi_m\). The tuning result is shown in Fig. 19.

As the last step, we continue the dGPM case investigation. The feasible dGPM region is shown in Fig. 20. For two \(k_i\) in Table 4, the results of the dGPM-specifications-based tuning rules are also listed in Table 4. Furthermore considering the points \((A_m, \phi_m) = (-5, 49)\) and \((-5, 50)\) both closing to the boundary respectively, the exact GPMs are \((-5.0699, 50.2)\) and \((5.0119, 50.9)\). The latter one violates the specifications.

![Fig. 16. Feasible iGPM regions for example 2.](image-url)
6. Conclusions

The simple proportional–integral–derivative (PID) controller is still widely used to control the attitude of a flight vehicle, which may have unstable dynamics. Several reasonable approximations were employed to develop an analytical PID tuning rule for the pitch attitude control with specific gain and phase margins (GPM) requirement in the presence of actuator
delay. The feasible GPM regions, either the increasing gain margin or the decreasing gain margin being critical, were explicitly plotted in 2-D plane by means of numerical polynomial solving approach. Moreover, the proposed tuning rule is suitable for all the PID or PD control cases. The effects of the integral action on both robustness and dynamic performance were thoroughly revealed, and it was found that the moderate integral gain is helpful to improve dynamic performance yet sacrificing robustness for an unstable flight vehicle which is different from the traditional understanding. Most importantly, the matching relationship between the bandwidth of the actuator and the feasible GPM region may avoid improper index specifications for the system design. From the macropoint of view, the overall achievable GPM index set is visible for a specific actuator; and from the micropoint of view, the minimal demand for the bandwidth of the actuator can be found to realize anticipated GPM specifications. This is significant to prevent incompatible requirements and reduce the risk of design blindness. The numerical examples showed the efficiency and effectiveness of the proposed methods. In conclusion, the proposed methodology provides a good guideline for practitioners.

Table 5
Tuning results for example 2 to test applicable area.

<table>
<thead>
<tr>
<th>Specified PM</th>
<th>Exact GM</th>
<th>Exact PM</th>
<th>PM error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3.8905</td>
<td>26.4°</td>
<td>16.4°</td>
</tr>
<tr>
<td>20</td>
<td>3.9355</td>
<td>30.6°</td>
<td>10.6°</td>
</tr>
<tr>
<td>30</td>
<td>3.9355</td>
<td>35.8°</td>
<td>5.8°</td>
</tr>
<tr>
<td>40</td>
<td>3.9811</td>
<td>42.4°</td>
<td>2.4°</td>
</tr>
<tr>
<td>50</td>
<td>3.9811</td>
<td>50.6°</td>
<td>0.6°</td>
</tr>
</tbody>
</table>

Fig. 19. Bode diagram for example 2 with iGPM specifications of \((G_m, \phi_m) = (2, 18^\circ)\) (loop transfer functions: \(0.4287s^2 + 4.337s + 0.6148, 0.02729s^3 + 0.006689s^2 - 0.1003se^{-0.04s}\)).
Acknowledgments

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References


Fig. 20. Feasible dGPM regions for example 2.


