Finite time stability and $L_2$-gain analysis for switched linear systems with state-dependent switching

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Abstract

This paper considers the finite time stability and finite time boundedness problems for switched linear systems subject to $L_2$ disturbances. Differently from the existing average dwell-time technique, state-dependent switching control strategy is used to design the switching rule, which does not require the switching instants to be known in advance. Sufficient conditions for the switched systems to be finite time stable and finite time bounded are derived; the occurrence of sliding motion will not destroy the stability with the proposed conditions. Moreover, $L_2$-gain analysis problem is also considered. The proposed conditions are given in terms of linear matrix inequalities. Several examples are given to illustrate the effectiveness of the proposed methods.

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1. Introduction

Switched systems consist of a family of continuous or discrete subsystems and a switching signal that orchestrates the switching among these subsystems; switched systems belong to a special class of hybrid systems. In practical applications, such as the automobile transmission systems, steeper motor drives, computer disc drives, network control and certain robot control systems, etc. [1], the switched systems are needed in order to get better control performance. Recently, the analysis and design of switched systems have attracted many authors’ attention [2–7], and many different issues have been considered such as model reduction [8,9], optimal switched impulsive control problems [10], sliding mode control [11], output feedback stabilization [12], and so on.

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Most of the literatures regarding the stability of switched systems focus on the Lyapunov asymptotic stability analysis, which is defined over an infinite time interval. However, in some practical applications, we are more interested in the system stability within a fixed short time interval, such as the problem of assuring that a space vehicle will remain in a specified orbit for a given fixed time in order to complete a set of experiments or in a chemical process, of keeping the temperature, pressure or some other parameters within specified bounds in a prescribed time interval [13]. Hence, from practical point of view, the finite time stability which was firstly introduced in [14] can be used. Specifically, a system is finite time stable if the system states remain within prescribed bounds during fixed time interval under bounded initial conditions; hence, it is different from the Lyapunov asymptotic stability. A system may be finite time stable but not Lyapunov asymptotic stable and vice versa [15].

Finite time stability (FTS) is used to study the transient performances of a system within a short time interval when there exist no exogenous disturbances, and when considering the exogenous disturbances, it becomes finite time boundedness (FTB) problem. Since the concept of short time stability was first introduced in [14], there have been many results with respect to finite time stability, especially in recent years. FTS and FTB problems of linear systems subject to parametric uncertainties and disturbances are studied in [15], a procedure is proposed to design dynamic output feedback controller [16]. In [17], sufficient conditions are given to guarantee the FTS of impulsive dynamical systems, and [18] considers the discontinuous linear time-varying systems. Furthermore, [19,20] propose necessary and sufficient conditions for time-varying switched systems respectively. Finite time stabilization problem for nonlinear dynamical systems is considered in [21]. Robust finite time $H_{\infty}$ control problems are considered for jump systems in [22,23], state feedback and dynamic observer-based state feedback are used for stochastic finite time stabilization and finite time $H_{\infty}$ control in [24,25] respectively. Recently, the concept of FTS is extended to input–output FTS [26,27], stochastic delayed reaction–diffusion genetic regulatory networks [28] and Markovian jumping neural networks [29].

However, there are few results about the FTS of switched systems due to the difficulty of dealing with the switching. [13] considers the FTS and controller design problem for switched linear systems, and in [30], the FTB and $L_2$-gain analysis problem are considered for switched delay systems. In [31], the authors consider the $H_{\infty}$ finite time stability for switched nonlinear discrete time systems with disturbances, and the finite time stability analysis is considered in [32]. A design approach is proposed for robust finite time $H_{\infty}$ control of a class of stochastic switching systems [33]. Note that in the existing literatures for the FTS and FTB of switched linear systems, the switching rule is time-dependent. Sometimes, the switching sequence cannot be prescribed in advance, and a switching rule based on current system states should be provided; hence, in this paper, we consider the state-dependent switching, based on which, sufficient conditions are proposed to guarantee the FTS and FTB of the considered switched systems.

For the sake of completeness, we should note that there is a different concept of FTS for nonlinear systems [34–38], based on the Lyapunov analysis, this type of FTS means that the system trajectories converge to the equilibrium point in finite time, and settling-time is used to describe the time interval. We can see that these two definitions of FTS are different, and both of them are important for different applications. In this paper, we focus on the former definition of FTS. To the best of the authors’ knowledge, there are no results available for finite time stability of switched linear systems with state-dependent switching rule. And the main contribution of this paper is that state-dependent switching rule is used for finite time
stability analysis of switched systems, sufficient conditions for the FTS and FTB of the considered switched linear systems are proposed, with these conditions the occurrence of sliding motion will not destroy the stability. Moreover, the $L_2$-gain analysis is also considered. In the end, examples are given to show that the proposed methods are effective.

The rest of the paper is organized as follows. In Section 2, some preliminaries are given and the problems are formulated. In Section 3, based on the state-dependent switching, sufficient conditions for the FTS and FTB of the considered systems are given, and sufficient conditions which guarantee that the system has finite time $L_2$-gain are presented. In Section 4, examples are given to illustrate the effectiveness of the proposed methods. And Section 5 concludes this paper.

The notations used in this paper are standard. $R^n$ denotes the $n$ dimensional Euclidean space, $S = [1, \ldots, N]$ denotes the set that includes integers from 1 to $N$. $P > 0$ denotes a symmetric positive definite matrix. $\lambda_{\text{max}}(P)\lambda_{\text{min}}(P))$ denotes the maximum (minimum) eigenvalue of matrix $P$. The identity matrix is denoted as $I$.

2. Problem formulation

Consider the switched linear systems of the form as follows

$$\dot{x}(t) = A_ix(t) + E_iw(t)$$

$$z(t) = C_ix(t)$$

where $x(t) \in R^n$ is the state, $w(t) \in R^n$ is the time-varying exogenous disturbance, $z(t)$ is the control output, $A_i, E_i$, and $C_i$ are constant real matrices with appropriate dimensions. $i \in S$ is the index which means that the current activated systems is the $i$th subsystem, and it is determined by the switching signal $\sigma(x(t)) : R^n \rightarrow S = [1, \ldots, N]$, say, $i = \sigma(x(t))$, where $N$ is the number of subsystems and $\sigma(x(t))$ depends on the current state $x(t)$, which means that the switching rule is state-dependent. In the rest of the paper, with slight notation abused, $x$ and $x(t)$ are equivalent according to the context.

The switching strategy used in this paper is the same as used in [3], such switching strategy is defined according to an array of appropriately symmetric matrices $Q_i \in R^{n \times n}, \ i \in S$. More specifically, the switching signal $\sigma(x(t))$ is defined as follows, which is referred to as the largest region function

$$\sigma(x) = \arg \max_{i \in S} x^T Q_i x$$

Suppose that there are $N$ regions $\Omega_i$, each one corresponding to a subsystem, which means that the subsystem $i$ is activated when the state belongs to $\Omega_i$, and some regions $\Omega_{ij}$ corresponding to the jump sets from subsystem $i$ to subsystem $j$. The following two properties should be satisfied for the well-defined switched system [3]

Covering property : $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N = R^n$

Switching property : $\Omega_{ij} \subseteq \Omega_i \cap \Omega_j \ i, j \in S$

The first property means that there are no regions in the state space where none of the subsystems is activated. The second property says that the switching from subsystem $i$ to subsystem $j$ only occurs for states where the regions $\Omega_i$ and $\Omega_j$ are adjacent. Specifically in this paper, based on the previous mentioned matrices $Q_i$, we define the regions $\Omega_i$ and $\Omega_{ij}$
as follows
\[
\Omega_i = \{x \in \mathbb{R}^n | x^T Q_i x \geq 0\}, \quad i \in S
\]
\[
\Omega_{ij} = \{x \in \mathbb{R}^n | x^T Q_i x - x^T Q_j x = 0\}, \quad i, j \in S
\] (4)

Hence, according to the switching signal (2), the subsystem \(i\) will be activated if the quadratic function \(x^T Q_i x\) is greater or equal to any other \(x^T Q_j x, \quad j \neq i\) [6]. The following lemma about the covering property is proposed in [3].

**Lemma 1. (Covering property).** If for every \(x \in \mathbb{R}^n\),
\[
\theta_1 x^T Q_1 x + \theta_2 x^T Q_2 x + \cdots + \theta_N x^T Q_N x \geq 0
\] (5)
where \(\theta_i > 0, i \in S\), then \(\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N = \mathbb{R}^n\).

In this paper, we consider a class of disturbances with bounded energies in finite time interval.
\[
W = \{w(t) : R \rightarrow R^n, \quad \int_0^T w^T(t)w(t)dt \leq d\} \tag{6}
\]

The following definitions are important for our main results.

**Definition 1. (FTS [13])** Given three positive constants \(c_1, c_2, T\) with \(c_1 < c_2\), a positive definite matrix \(R\), switching signal \(\sigma(x)\) which is determined by Eq. (2). The switched linear system (1) with \(w(t) = 0\) is said to be finite time stable with respect to \((c_1, c_2, T, R, \sigma)\), if
\[
x^T(t_0)Rx(t_0) \leq c_1 \Rightarrow x^T(t)Rx(t) < c_2, \quad \forall t \in (0, T) \tag{7}
\]

**Definition 2. (FTB [30])** Given four positive constants \(c_1, c_2, T, d\) with \(c_1 < c_2, \quad d \geq 0\), a positive definite matrix \(R\), switching signal \(\sigma(x)\) which is determined by Eq. (2). The switched linear system (1) is said to be finite time bounded with respect to \((c_1, c_2, T, R, d, \sigma)\), if
\[
x^T(t_0)Rx(t_0) \leq c_1 \Rightarrow x^T(t)Rx(t) < c_2, \quad \forall t \in (0, T), \quad \forall w(t) \in W \tag{8}
\]

**Definition 3. ([30])** For given \(T > 0\) and \(\gamma > 0\), system (1) is said to have finite time \(L_2\)-gain \(\gamma\), if under zero initial condition, it holds that
\[
\int_0^T z^T(\tau)z(\tau)d\tau \leq \gamma^2 \int_0^T w^T(\tau)w(\tau)d\tau \tag{9}
\]

**Remark 1.** Note that the concepts of FTB and reachable set are different. Reachable sets are defined as the set of states that a dynamical system attains given some bounded inputs and starting from some given initial conditions [15]. However, according to **Definition 2**, FTB is that given a set of non-zero bounded initial conditions and energy bounded disturbances, the state remains within a prescribed bounded set. In addition, it is assumed that the system is asymptotic stable when considering the reachable set analysis problem, while the asymptotic stability is unnecessary for FTB.

In this paper, we will focus on the following problems

1) The FTS and FTB analysis for system (1) based on state-dependent switching rule.
2) \(L_2\)-gain analysis for the system (1), where the exogenous disturbance \(w(t) \in W\).
3. Main results

3.1. FTS and FTB analysis

In this section, firstly, sufficient conditions for the system (1) to be finite time stable are proposed, and the sliding motion is also considered. Then the finite time boundedness problem is considered for system (1); sufficient condition are presented in the form of bilinear matrix inequalities (BMI) that can be reduced to linear matrix inequalities (LMI) with several scalar parameters fixed [3].

**Theorem 1.** Consider system (1). Suppose that there exist positive definite matrices $P_i > 0$, $Q_i = Q_i^T$, $\theta_i > 0$, and $\eta_{ij} > 0$ for any $i, j \in S$, $i \neq j$, and $\alpha > 0$, such that

$$A_i^T P_i + P_i A_i < \alpha \min_{i \in S} (\lambda_{\text{min}}(P_i)) I$$

$$P_i = P_j + \eta_{ij} (Q_i - Q_j)$$

$$\sum_{i = 1}^N \theta_i Q_i \geq 0$$

and the following condition is also satisfied

$$\mu \leq e^{-\alpha T} \frac{c_2}{c_1}$$

where $\mu = \lambda_1 / \lambda_2$, $\lambda_1 = \max_{i \in S} (\lambda_{\text{max}}(P_i))$, $\lambda_2 = \min_{i \in S} (\lambda_{\text{min}}(P_i))$, and $\bar{P}_i = R^{-1/2} P_i R^{-1/2}$, then the switched system (1) with $w(t) = 0$ is finite time stable with respect to $(c_1, c_2, T, R, \sigma)$ under the switching rule $\sigma(x)$

$$\sigma(x) = \arg \max_{i \in S} x^T Q_i x$$

Proof: by Lemma 1, condition 3 in Eq. (10), and the largest region function switching rule (2), it is guaranteed that the considered switched system is well defined.

Choose the Lyapunov functional candidate as follows

$$V(x) = V_i(x) = x^T P_i x$$

According to the switching rule, we will use the Lyapunov functional $V(x)$ to measure the system’s energy in the region $\Omega_i$. When $w(t) = 0$

$$\dot{V}(x) = \dot{V}_i(x) = x^T P_i A_i x + x^T A_i^T P_i x$$

From Eq. (10), we have

$$\dot{V}(x) < \alpha \min_{i \in S} (\lambda_{\text{min}}(P_i)) I \|x\|^2 \leq \alpha V(x)$$

As discussed in [6], $V(x) = V_i(x)$ in the region $\Omega_i$. Generally speaking, such function is not differentiable along the boundaries when switching occurs, i.e. when the activated subsystem jumps from $i$ to $j$. When the switching occurs between subsystems $i$ and $j$, we have $V_j(x) = V_j(x)$ on the boundary due to the condition 2 of Eq. (10), hence the condition in Eq. (13) is satisfied for any state not on the boundaries, and if no sliding motion occurs on any of the boundaries. Let $t_k$ denote the switching instant before the current time...
\( t(t \in [t_k, t_{k+1}]) \), and assume that \( \sigma(x(t_k^-)) = i, \sigma(x(t_k)) = j \), if sliding motion does not occur, then, integrating both sides of Eq. (13) from \( t_k \) to \( t \), we have

\[
V(x(t)) < e^{\alpha(t-t_k)} V_{\sigma(x(t_k))}(x(t_k))
\]  

(14)

Note that \( x(t_k) = x(t_k^-) \), then \( V_{\sigma(x(t_k))}(x(t_k)) < V_{\sigma(x(t_k^-))}(x(t_k^-)) \) from the condition 2 of Eq. (10) and the switching rule \( \sigma(x) \). And it follows from Eq. (14) that

\[
V(x(t)) < e^{\alpha t} V_{\sigma(x(0))}(x(0))
\]  

(15)

With \( \overline{P}_i \) defined in the theorem, we have

\[
V(x(t)) = x^T(t)P_{\sigma(x(t))}x(t)
= x^T(t)R_1^{-1}\overline{P}_{\sigma(x(t))}R_2^1x(t)
\geq \lambda_{min}(\overline{P}_{\sigma(x(t))})x^T(t)Rx(t)
\geq \lambda_2x^T(t)Rx(t)
\]  

(16)

and

\[
V(x(0)) = x^T(0)P_{\sigma(x(0))}x(0)
\leq \max(\overline{P}_{\sigma(x(0))})x^T(0)Rx(0)
\leq \lambda_1x^T(0)Rx(0)
\]  

(17)

Then, from Eqs. (15)–(17), it follows that

\[
x^T(t)Rx(t) < \frac{1}{\lambda_2} e^{\alpha t} \lambda_1x^T(0)Rx(0)
\]

Hence we can obtain \( x^T(t)Rx(t) < c_2 \) from Eq. (11), and the switched system (1) is finite time stable.

Because the sliding motion may occur in the switched systems with state-dependent switching rule, we will prove that with the given conditions, the occurrence of sliding motion will not destroy the finite time stability. Taking the sliding motion that occurs between subsystem \( i \) and \( j \) as example, the proof is similar for the sliding motion occurs between other subsystems. According to the switching rule, the sliding motion will occur along the hyper-surface \( \Omega_{ij} \), says \( x^TQ_i x = x^TQ_j x \geq 0 \). The sliding motion may lead to either stable or unstable dynamics along switching surface, hence, we need to analyze if the given conditions in Theorem 1 can guarantee that the switched system is still finite time stable when the sliding motion does occurs.

According to Filippov’s convex combination [39], the system on the switching surface can be represented as follows

\[
\dot{x} = \rho A_i x + (1-\rho)A_j x, \quad 0 \leq \rho \leq 1
\]  

(18)

Then, according to the analysis of sliding motions in [4,6], when the sliding motion occurs along \( \Omega_{ij} \), it implies that

\[
\begin{align*}
\begin{cases}
x^T A_i^T(Q_i - Q_j)x + x^T(Q_i - Q_j)A_i x < 0 \\
x^T A_j^T(Q_i - Q_j)x + x^T(Q_i - Q_j)A_j x > 0
\end{cases}
\end{align*}
\]

(19)
Remark 3.

There are some results about finite time stability of switched linear systems with Eq. (10) positive definite, and the asymptotically stability of the switched system can be obtained with partitioned into subspaces, each one corresponding to a subsystem and a Lyapunov matrix $N$. The difficulties of this research are mainly in two aspects, first of all, the state space must be time-dependent switching, but with state-dependent switching, few results have been obtained.

Remark 2.

There are some disturbances, because in practical systems the disturbances always exist. Hence, we next consider

It follows from Eq. (10) that

\[
\begin{align*}
 x^T A_i^T (P_j - P_i) x + x^T (P_j - P_i) A_i x &= 0 \\
 x^T A_j^T (P_j - P_i) x + x^T (P_j - P_i) A_j x &= 0
\end{align*}
\]

which implies that

\[
\begin{align*}
 x^T \left[ \rho (A_i^T P_j + P_i A_i) + (1 - \rho) (A_j^T P_j + P_j A_j) \right] x &< \min_{i \in S} \lambda_{\min}(P_i) I \\
 x^T \left[ \rho (A_j^T P_j + P_j A_j) + (1 - \rho) (A_i^T P_i + P_i A_i) \right] x &< \min_{i \in S} \lambda_{\min}(P_i) I
\end{align*}
\]

Hence, we can derive that the following conditions are satisfied

\[
\begin{align*}
 x^T \left[ \rho A_i^T + (1 - \rho) A_j^T \right] P_i + P_i (\rho A_i + (1 - \rho) A_j) x &< \min_{i \in S} \lambda_{\min}(P_i) I \\
 x^T \left[ \rho A_j^T + (1 - \rho) A_i^T \right] P_j + P_j (\rho A_i + (1 - \rho) A_j) x &< \min_{i \in S} \lambda_{\min}(P_i) I
\end{align*}
\]

Then, we can see that the conditions in Eqs. (13) and (15) are both satisfied for system (18) with the Lyapunov functional $x^T P_i x$ and $x^T P_j x$, and the switched system (1) is still finite time stable with the given conditions even if sliding motion occurs. This completes the proof.

**Remark 2.** Differently from the Lyapunov asymptotical stability, it is not required that the $V(x)$ is negative definite or negative semidefinite. If we let $\tau \leq 0$ in Eq. (10), then $\dot{V}(x)$ will be negative definite, and the asymptotically stability of the switched system can be obtained with Eq. (10).

**Remark 3.** There are some results about finite time stability of switched linear systems with time-dependent switching, but with state-dependent switching, few results have been obtained. The difficulties of this research are mainly in two aspects, first of all, the state space must be partitioned into $N$ subspaces, each one corresponding to a subsystem and a Lyapunov matrix $P_i$, that means besides the Lyapunov matrices $P_i$, the matrices $Q_i$ are also need to be designed. Then, with state-dependent switching, sliding motion may occur, and instability may be induced by sliding motion; so, it is necessary to analyze that if the proposed condition can guarantee the stability even if the sliding motion occurs. With the proposed conditions in Theorem 1, the state space can be partitioned and the occurrence of sliding motion will not destroy the finite time stability.

It is required that the switching sequence $(i_k, t_k)$ to be known beforehand and fast switching is not allowed (the average dwell-time has a lower bound) with time-dependent switching, but the state-dependent switching is based on the current value of the system state, which is more practical and the switching sequence $(i_k, t_k)$ does not need to be known in advance. Moreover, the switched systems are still finite time stable even if sometimes the fast switching occurs, which will be shown by a comparison result in the simulation section.

Another problem for the switched system that needs to be considered is exogenous disturbance, because in practical systems the disturbances always exist. Hence, we next consider
the situation when \( w(t) \neq 0 \) and \( w(t) \in W \), the following theorem gives sufficient conditions for the switched system (1) to be finite time bounded.

**Theorem 2.** Suppose that there exist positive definite matrices \( P_i > 0 \), \( Q_i = Q_i^T \), \( \theta_i > 0 \), and \( \eta_{ij} > 0 \) for any \( i,j \in S,i \neq j \), and \( \alpha > 0 \), \( \beta > 0 \), positive definite matrix \( H \), such that

\[
\begin{bmatrix}
A_i^T P_i + P_i A_i - \alpha \min_{i \in S} (\lambda_{\max}(P_i)) I & P_i E_i \\
E_i^T P_i & -\beta H
\end{bmatrix} < 0
\]

\[P_i = P_j + \eta_{ij}(Q_j - Q_i)\]

\[\lambda_1 c_1 + \beta d \lambda_{\max}(H) \leq e^{-\alpha T} \lambda_2 c_2\]

\[
\sum_{i = 1}^{N} \theta_i Q_i \geq 0
\]

(19)

where \( \lambda_1 = \max_{i \in S} (\lambda_{\max}(\bar{P}_i)) \), \( \lambda_2 = \min_{i \in S} (\lambda_{\min}(\bar{P}_i)) \), and \( \bar{P}_i = R^{-\frac{1}{2}} P_i R^{-\frac{1}{2}} \), then the switched system (1) with \( w(t) \in W \) is finite time bounded with respect to \((c_1,c_2,T,d,R,\sigma)\) under the switching rule \( \sigma(x) \)

\[
\sigma(x) = \arg\max_{i \in S} x^T Q_i x
\]

(20)

Proof: the well posedness is the same as discussed in Theorem 1. Choose the Lyapunov function \( V(x) = V_f(x) = x^T P_i x \). Multiplying condition 1 in Eq. (19) by \( \begin{bmatrix} x^T(t) & w^T(t) \end{bmatrix} \) in the left and by its transpose in the right, we can obtain

\[\dot{V}(x) - \alpha V(x) < \beta w^T H w\]

(21)

Multiplying both sides of the Eq. (21) with \( e^{-\alpha t} \), which implies that

\[\frac{d(e^{-\alpha t} V(x))}{dt} < \beta e^{-\alpha t} w^T H w\]

(22)

Then, integrating both sides of the Eq. (22) from \( t_k \) to \( t \), we have

\[V(x(t)) < e^{\alpha (t-t_k)} V(x(t_k)) + \beta \int_{t_k}^{t} e^{2(t-s)} w^T(s) H w(s) ds\]

From condition 2 of Eq. (19) and the switching rule (20), \( V(x(t_k)) < V(x(t_{k-1})) \), then it is derived that

\[V(x(t)) < e^{\alpha t} V(x(0)) + \beta \int_{0}^{t} e^{2(t-s)} w^T(s) H w(s) ds\]

\[\leq e^{\alpha T} V(x(0)) + \beta e^{2T} \lambda_{\max}(H) \int_{0}^{T} w^T(s) w(s) ds\]

It follows from the similar procedures as in the proof of Theorem 1 and note that \( w(t) \in W \), then

\[x^T(t) Rx(t) < \frac{e^{2T}}{\lambda_{\min}(\bar{P}_{\sigma(x(t))})} (\lambda_{\max}(\bar{P}_{\sigma(x(0))})) x^T(0) Rx(0) + \beta \lambda_{\max}(H) d\]

From \( x^T(0) Rx(0) \leq c_1 \) and condition 3 of Eq. (19), we can see that \( x^T(t) Rx(t) < c_2, \forall t \in (0,T] \).
We next consider the situation when sliding motion occurs, with the similar procedure as in the proof of Theorem 1, we can obtain the following conditions

\[
\begin{aligned}
&x^T\left[(\rho A_i^T + (1-\rho)A_j^T)P_i + P_j(\rho A_i + (1-\rho)A_j)\right]x \\
&+w^T(\rho E_i^T + (1-\rho)E_j^T)P_i x + x^T P_j(\rho E_i + (1-\rho)E_j)w \\
&< \gamma x^T P_i x + \beta w^T H w \\
&x^T\left[(\rho A_i^T + (1-\rho)A_j^T)P_j + P_i(\rho A_i + (1-\rho)A_j)\right]x \\
&+w^T(\rho E_i^T + (1-\rho)E_j^T)P_j x + x^T P_i(\rho E_i + (1-\rho)E_j)w \\
&< \gamma x^T P_j x + \beta w^T H w
\end{aligned}
\]

It can be proved that the switched system (1) with \(w(t) \in W\) is finite time bounded which follows from the proof of Theorem 1 and Definition 2, whether the sliding motion occurs or not. This completes the proof.

**Remark 4.** With the proposed sufficient conditions, the switched system (1) with exogenous disturbance \(w(t) \in W\) is finite time bounded under the state-dependent switching rule, even if the sliding motion occurs.

### 3.2. Finite time \(L^2\)-gain analysis

The finite time \(L^2\)-gain is defined in Definition 3, in this section, we will give sufficient conditions under which the switched linear system (1) has \(L^2\)-gain less than or equal to \(\gamma\).

**Theorem 3.** Suppose that there exist positive definite matrices \(P_i > 0, Q_i = Q_i^T, \theta_i > 0\), and \(\eta_{ij} > 0\) for any \(i, j \in S, i \neq j\), and \(\alpha > 0, \beta > 0, \gamma > 0\), positive definite matrix \(H\), such that

\[
\begin{bmatrix}
A_i^T P_i + P_i A_i - \alpha \min_{i \in S} (\lambda_{\min}(P_i)) I_i & P_i E_i & C_i^T \\
E_i^T P_i & -\beta H & 0 \\
C_i & 0 & -\gamma^2 I
\end{bmatrix} < 0
\]

\[
P_i = P_j + \eta_{ij}(Q_j - Q_i) \\
\beta d \lambda_{\max}(H) \leq e^{-x^T \lambda_2 c_2} \\
\sum_{i=1}^{N} \theta_i Q_i \geq 0
\]

where \(\lambda_{\max}(\overline{P}_i)\), \(\overline{P}_i = R^{-1/2} P_i R^{-1/2}\), then the switched system (1) is finite time bounded with respect to \((0, c_2, T, d, R, \sigma)\) under the switching rule \(\sigma(x)\)

\[
\sigma(x) = \arg \max_{i \in S} x^T Q_i x
\]

and the \(L^2\)-gain of the switched system is \(\gamma = \gamma \sqrt{\beta \lambda_{\max}(H)}\).
Proof. The well posedness is the same as discussed in Theorem 1. Choose the Lyapunov function \( V(x) = V_i(x) = x^T P_i x \), the finite time boundedness can be easily proved following the proof procedure of Theorem 2, and we have

\[
\dot{V}(x) - z V(x) < -\frac{1}{\gamma} z^T(t) z(t) + \beta w^T(t) H w(t)
\]

Then it can be derived that

\[
V(x(t)) < e^{\gamma t} V(x(0)) + \int_0^t e^{\gamma (t-s)} \left[ -\frac{1}{\gamma} z^T(s) z(s) + \beta w^T(s) H w(s) \right] ds
\]

Note that \( V(x(t)) > 0 \) and \( V(x(0)) = 0 \) with the given conditions in Theorem 3, then we have

\[
\int_0^T \left[ -\frac{1}{\gamma} z^T(s) z(s) + \beta \lambda_{\max}(H) w^T(s) w(s) \right] ds > 0
\]

\[
\int_0^T z^T(s) z(s) ds < \frac{1}{\gamma^2} \beta \lambda_{\max}(H) \int_0^T w^T(s) w(s) ds
\]

Let \( \frac{1}{\gamma^2} = \frac{\pi^2}{\beta \lambda_{\max}(H)} \), then according to Definition 3, the \( L_2 \)-gain of the system \( (1) \) is \( \gamma = \frac{1}{\sqrt{\beta \lambda_{\max}(H)}} \). This completes the proof.

From the conditions (10)(19) and (23), we can see that they are not strict LMIs, hence it is not easy to design the switching rule directly with these conditions. However, as discussed in [3], if we fix some parameters beforehand, these conditions will be translated into LMIs, and then it can be solved by the LMI tool effectively.

Take the Theorem 2 as an example, given \( x, \beta, \eta_{ij}, \theta_i \), and we restrict the value ranges of matrices \( P_i, H \) to be lower and upper bounded, i.e., let \( \lambda_2 R < P_i < \lambda_1 R \), \( 0 < H < \lambda_H I \). Then if \( \lambda_1 c_1 + \beta d \lambda_H \leq e^{-\gamma T} \lambda_2 c_2 \) is satisfied, the condition 3 of Eq. (19) can be guaranteed. Hence, the conditions (19) are translated into LMIs, if Eq. (19) has feasible solution, then with the switching rule \( \sigma(x) = \arg \max_i x^T Q_i x \), the switched system \( (1) \) is finite time bounded. In the simulation section, a numerical example will be given to illustrate the method.

For the \( L_2 \)-gain analysis problem, the solving method is similar as discussed above. However, generally speaking, we want to compute a minimal \( L_2 \)-gain \( \gamma \); with the above discussion only feasible solutions can be obtained. Due to the product of scalar and matrix in Eq. (23), we can use the path-following algorithm [40] to adjust parameters \( H \) and \( \gamma \). Then, based on the feasible solution, more optimal \( L_2 \)-gain \( \gamma \) can be obtained.

4. Numerical examples

In this section, several examples are given to illustrate the effectiveness of the proposed methods. For simplicity, in this paper, we consider the switching between two subsystems, in the case of which we can set \( Q_1 = Q, Q_2 = -Q, \theta_1 = \theta_2 = 1, \) and \( \eta_{12} = \eta_{21} = \eta \), then the constraint \( \sum_{i=1}^N \theta_i x^T Q_i x \geq 0 \) is satisfied.

Example 1 (a). Finite-time stability.
Consider the switched linear system with \( w(t) = 0 \), the system matrices that are used in [13] are given as follows

\[
A_1 = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}
\]

Both \( A_1 \) and \( A_2 \) are unstable. The values \( c_1 = 1, \ c_2 = 20, \ T = 10, \ R = I \) are the same as in [13], as discussed in Section 3, let \( \lambda_2 = 0.3 \), and fix \( \alpha = 0.1, \ \eta = 0.01 \), then apply Theorem 1 we obtain the following feasible solutions

\[
P_2 = \begin{bmatrix} 0.5505 & 0.0115 \\ 0.0115 & 1.1130 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0.1130 & -0.0115 \\ -0.0115 & 0.5505 \end{bmatrix}
\]

\[
Q_1 = -Q_2 = \begin{bmatrix} -28.1292 & 1.1508 \\ 1.1508 & 28.1292 \end{bmatrix},
\]

The system responses are shown in Fig. 1. In Fig. 1(a), the gray area denotes that the subsystem 2 is activated and the asterisk ‘*’ stands for the start point \( [1 \ 0]^T \) of the

![Fig. 1. System response under state-dependent switching rule with \( w(t) = 0 \): (a) phase plot of \( x(t) \), (b) switching signal, and (c) \( x^T(t)Rx(t) \).](image-url)
system trajectory. We can see that when the states belong to the gray area, according to the state-dependent switching rule, subsystem 2 is activated, and it is independent of time. With time-dependent switching in [13], it is shown that the switching signal is satisfied with the dwell-time $\tau > \tau^* = 3.1539$, then the system is finite time stable. From Fig. 1(b) we can see that with the state-dependent switching rule in this paper, the dwell-time is about 1.33 s. Moreover, from Fig. 1(c), according to Definition 1, with the initial condition $x^T(0)Rx(0) \leq 1$, the system states satisfy $x^T(t)Rx(t) \leq 20$; that means the system is finite time stable.

**Example 1 (b). Finite-time boundedness.**

Consider the switched linear system with $w(t) \neq 0$, the system matrices are the same as in (a), and $E_1, E_2$ are given as follows

$$E_1 = \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 0.15 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.4 & -1 \\ 0.7 & 0.3 \end{bmatrix}$$

![Fig. 2. System response under state-dependent switching rule with $w(t) \neq 0$: (a) phase plot of $x(t)$, (b) disturbance $w(t)$, and (c) $x^T(t)Rx(t)$.](image-url)
Now, we let $c_1 = 0.5, c_2 = 50$ as in [30] did. Then parameters are the same as in (a). With Theorem 2, we obtain the following feasible solutions with respect to $w(t) = \begin{bmatrix} 0.3\sin(t) & -0.1\cos(4t - \pi/2) \end{bmatrix}$.

$$P_2 = \begin{bmatrix} 0.0103 & 0.0002 \\ 0.0002 & 0.0206 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0.0218 & -0.0006 \\ -0.0006 & 0.0114 \end{bmatrix}$$

$$Q_1 = -Q_2 = \begin{bmatrix} -0.5725 & 0.0436 \\ 0.0436 & 0.4622 \end{bmatrix}, \quad H = \begin{bmatrix} 8.9502 & -0.1978 \\ -0.1978 & 8.6295 \end{bmatrix}$$

System trajectory starts from $[0.5 \ 0.3]^T$ and the exogenous energy bounded disturbance $w(t)$ are shown in Fig. 2(a) and (b) respectively. From Fig. 2(c) and according to Definition 2, we can see that the switched system is finite time bounded.

**Example 1 (c). $L_2$-gain analysis.**

This is to show that the proposed $L_2$-gain analysis method is effective. Given the matrices $C_1, C_2$ as follows.

$$C_1 = \begin{bmatrix} -0.1 & 0.03 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.04 & 0.1 \end{bmatrix}$$

The other parameters are the same as given in (b). With Theorem 3, the initial feasible solutions are computed as

$$P_2 = \begin{bmatrix} 0.0103 & 0.0002 \\ 0.0002 & 0.0205 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0.0209 & -0.0007 \\ -0.0007 & 0.0110 \end{bmatrix}$$

$$Q_1 = -Q_2 = \begin{bmatrix} -0.5321 & 0.0438 \\ 0.0438 & 0.4734 \end{bmatrix}, \quad H = \begin{bmatrix} 9.4754 & -0.0448 \\ -0.0448 & 9.4061 \end{bmatrix}$$

$\gamma = 41.1383, \quad \lambda_{\max}(H) = 9.4973$, and the $L_2$-gain $\gamma$ is computed as $\gamma = 12.6779$. In order to obtain a more optimal $L_2$-gain, we use the path-following algorithm as discussed in Section

Fig. 3. System response under state-dependent switching rule: (a) phase plot of $x(t)$, (b) disturbance signal $w(t)$ and control output $z(t)$. 

Fig. 3. System response under state-dependent switching rule: (a) phase plot of $x(t)$, (b) disturbance signal $w(t)$ and control output $z(t)$. 

3 to adjust the parameters $H$ and $\gamma$, then we have

$$H = \begin{bmatrix} 9.9034 & 0.0012 \\ 0.0012 & 9.9078 \end{bmatrix}, \quad \bar{\gamma} = 9.8290, \quad \gamma = 3.0939$$

We can see that, after the adjustment with the path-following algorithm, more optimal $L_2$-gain $\gamma$ is obtained, which demonstrates that the proposed methods are effective. The system trajectory start from $[0 \ 0]^T$ is shown in Fig. 3(a), the disturbance signal and the output $z(t)$ are shown in Fig. 3(b).

Remark 5. Compared with the time-dependent switching, the effectiveness of state-dependent switching is presented in Example 1(a) with $w(t)=0$, and when the disturbance $w(t)\neq 0$, the effectiveness of state-dependent switching for finite time boundedness and $L_2$-gain analysis is demonstrated with Example 1(b),(c).

In order to further illustrate the effectiveness of the proposed methods, a simple practical application example is given as follows.

Example 2. We consider a simplified liquid level control system, namely, three-tank system, which is considered in [41] with discretization method and often encountered in chemical process. The switched system is shown in Fig. 4; the liquid outflows from Tank 1 and flows to Tank 2 or Tank 3 by choosing the appropriate switching. Here, we consider
state-dependent switching; the states are the liquid height for Tank 1, Tank 2 and Tank 3—$h_1$, $h_2$, and $h_3$ respectively. The inflow rate is $p_0$ and the outflow rates are $p_1$, $p_2$, $p_3$, correspondingly. The control objective is to keep the liquid level of the whole system in a range in finite-time.

The inflow rate is dependent on the system states $h_1$, $h_2$, and $h_3$, when the liquid flows from Tank 1 to Tank 2, $p_0 = F_1 h$, where $h = [h_1 \ h_2 \ h_3]^T$, and $p_0 = F_2 h$ when the liquid flows from Tank 1 to Tank 3. The system model is as follows with two subsystems

$$
\Sigma_1 : \begin{cases}
\frac{dh_1}{dt} = p_0 - p_1 h_1 \\
\frac{dh_2}{dt} = p_1 h_1 - p_2 h_2 \\
\frac{dh_3}{dt} = -p_3 h_3
\end{cases} \quad \Sigma_2 : \begin{cases}
\frac{dh_1}{dt} = p_0 - p_1 h_1 \\
\frac{dh_2}{dt} = -p_2 h_2 \\
\frac{dh_3}{dt} = p_1 h_1 - p_3 h_3
\end{cases}
$$

With given parameters $p_1$, $p_2$, $p_3$, $F_1$, and $F_2$, the switched system can be represented as

$$
\dot{h}_{\Sigma_1} = \begin{bmatrix}
-0.45 & 0.5 & 0.5 \\
0.7 & -0.9 & 0 \\
0 & 0 & -0.7
\end{bmatrix} h, \quad \dot{h}_{\Sigma_2} = \begin{bmatrix}
-1 & 0.4 & 0.3 \\
0 & -0.9 & 0 \\
0.7 & 0 & -0.7
\end{bmatrix} h
$$

The parameters $c_1$, $c_2$, $T$, and $R$ are given as $c_1 = 10$, $c_2 = 20$, $T = 5$, and $R = 1$, and as discussed in Section 3, let $\lambda_2 = 0.01$, and fix $x = 0.01$, $\eta = 0.01$, with the method proposed in

Fig. 5. State trajectory of the three-tank liquid level control system with state-dependent switching, the blue circle is the start point. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
this paper, we obtain

$$P_1 = \begin{bmatrix} 21.0914 & 2.8970 & 6.3089 \\ 2.8970 & 20.4095 & 1.2757 \\ 6.3089 & 1.2757 & 21.9953 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4.2193 & -6.3390 & 1.3571 \\ -6.3390 & 25.6966 & 3.5425 \\ 1.3571 & 3.5425 & 12.3830 \end{bmatrix}$$

$$Q_1 = -Q_2 = \begin{bmatrix} -843.6075 & -461.8010 & -247.5894 \\ -461.8010 & 264.3566 & 113.3388 \\ -247.5894 & 113.3388 & -480.6121 \end{bmatrix}$$

The state trajectory starts from $x_0 = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix}$ is shown in Fig. 5. We can see that with the state-dependent switching rule, the switched system is finite-time stable. This example illustrates that the proposed methods in this paper have the potential to be used in some practical applications.

5. Conclusion

The finite time stability, finite time boundedness and $L_2$-gain analysis problems are analyzed for switched linear systems based on state-dependent switching, sufficient conditions for the finite time stability and finite time boundedness of the considered systems are proposed, even if the sliding motion occurs, the finite time stability will not be destroyed with the proposed conditions. Then sufficient conditions that can guarantee the switched system to have $L_2$-gain $\gamma$ are also proposed based on the state-dependent switching. In the end, several examples are given to illustrate that the proposed methods are effective.

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