Multi-criteria decision-making based on hesitant fuzzy linguistic term sets: An outranking approach

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ABSTRACT
Hesitant fuzzy linguistic term sets (HFLTSs) are introduced to express the hesitance existing in linguistic evaluation as clearly as possible. However, most existing methods using HFLTSs simply rely on the labels or intervals of linguistic terms, which may lead to information distortion and/or loss. To avoid this problem, linguistic scale functions are employed in this paper to conduct the transformation between qualitative information and quantitative data. Moreover, the directional Hausdorff distance, which uses HFLTSs, is also proposed and the dominance relations are subsequently defined using this distance. An outranking approach, similar to the ELECTRE method, is constructed for ranking alternatives in multi-criteria decision-making (MCDM) problems, and the approach is demonstrated using a numerical example related to supply chain management. Because of the inherent features of the directional Hausdorff distance and the defined dominance relations, this approach can effectively and efficiently overcome the hidden drawbacks that may hamper the use of HFLTSs. Finally, the accuracy and effectiveness of the proposed approach is further tested through sensitivity and comparative analyses.

1. Introduction

The modeling and solving of multi-criteria decision-making (MCDM) problems under uncertain conditions has been a challenging topic in recent decades. However, the introduction of fuzzy sets to this task [1] has been very constructive, and various extensions of fuzzy sets have emerged to express the fuzziness and vagueness of information as clearly as possible. These extensions include type-2 fuzzy sets [2,3], type-n fuzzy sets [3], fuzzy multisets [4], intuitionistic fuzzy sets (IFSS) [5], interval-valued fuzzy sets (IVFSs) [5], hesitant fuzzy sets (HFSs) [7], and neutrosophic sets (NSs) [8], which mainly differ from each other by their descriptions of the membership degree and/or the non-membership degree of an element, and can properly depict quantitative information characterized by uncertainty. Nevertheless, assessing the alternatives using the linguistic expression is relatively convenient and preferred in reality, and, therefore, the related linguistic approaches are also essential. The fuzzy linguistic approach [3,9,10] was proposed during the 1970s, and has since received popular recognition. As such, linguistic approaches have been widely applied in a number of fields such as new product development [11], supply chain management (SCM) [12,13], emergency management evaluation [14], service quality evaluation [15], and performance evaluation [16]. Furthermore, extensions and improvements have been introduced, including a linguistic model based on discrete fuzzy numbers [17], type-2 fuzzy sets [18–20], the 2-tuple linguistic representation model [21–24], and extended 2-tuple fuzzy linguistic models [25]. Nevertheless, these linguistic models have a limited capability for describing fuzzy and vague information.

Rodríguez et al. [26], from the basis of the fuzzy linguistic approach [3,9], proposed hesitant fuzzy linguistic term sets (HFLTSs), which were based upon HFSs. In practice, the assessment of decision makers usually fluctuates between several possible linguistic values, and, as a result, a definite answer is not always provided. The primary characteristic of HFLTSs is that the approach can cope with such hesitance where more than a single linguistic term may be required for assessing a linguistic variable.

Several MCDM methods using HFLTSs have been introduced, but the methods retain certain deficiencies in their fundamental operations, which may degrade their credibility. These are outlined below.

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(1) Envelope-based approach. Firstly, in the MCDM model proposed by Rodríguez et al. [26], the linguistic intervals were calculated by means of aggregation operators, and used to obtain the final results. Moreover, this model has been applied in group decision-making [27]. Secondly, the HFLTS envelopes are the basic linguistic intervals defined by Rodríguez et al. [26]. These have been applied in document classification [28] and also integrated into a technique for order of preference by similarity to ideal solution (TOPSIS) model [29]. Thirdly, Zhang et al. [30] defined the aggregation operators of HFLTs, and then compared the envelopes of the aggregated results to rank the alternatives. Fourthly, the 0-cut of the HFLTs, which represents a revision of the HFLTS envelopes, was used for the fuzzy decision-making method [31]. Fifthly, Wang et al. [32] defined the dominance relations of HFLTSs, and established an outranking method to accommodate hesitant fuzzy linguistic information. The dominance relations being developed partially depend on the HFLTS envelopes. In these methods/models, the linguistic terms were transformed into intervals, and the operations mainly relied upon the labels of the linguistic variables. However, such designs ignored the fact that the linguistic term set is not a simple array with equal distances between neighbors.

(2) Fuzzy envelope. The fuzzy envelope of an HFLTS [33] was constructed using a fuzzy membership function, which was aggregated using the given HFLTSs and the ordered weighted averaging (OWA) operator [34], and then combined with a fuzzy TOPSIS model [35–37] to solve MCDM problems. To a certain extent, the fuzzy envelope can retain the vagueness of comparative linguistic expressions. Nevertheless, determining the parameters of the fuzzy membership function and OWA weights is fairly complicated, and the requisite calculations are considerable for an MCDM problem in the context of HFLTs, which, for example, may contain four alternatives, four criteria, and seven linguistic terms that are used for assessment. Nevertheless, while the different levels of importance of the linguistic terms of HFLTs was recognized in the design of fuzzy envelopes, the approach continued to ignore the possibility of different distances between adjacent linguistic terms.

(3) Derivatives of HFLTs. Based on a convex combination of HFLTs and the possibility degree formulae, Wei et al. [38] proposed the corresponding aggregation operators and introduced some MCDM methods. The necessary calculations also frequently involved the labels of the linguistic variables. However, the results calculated using these operators may vary because of different orders of operations or operands, which will be illustrated in Section 2.

(4) Distance measures for HFLTs. Liao et al. [39] proposed distance and similarity measures of HFLTs, and also considered their extensions in a continuous case; moreover, they developed the TOPSIS method using a relative distance measure. Zhu and Xu [40] developed hesitant fuzzy linguistic preference relations (HFLPRs), which consisted of HFLTSs. The distance between two HFLTSs was defined in [40], which is equivalent to the Hamming distance given in [39]. However, similar to the rules on the extension of HFSs that were defined by Xu and Xia [41,42], the normalization of HFLTSs was necessary in [39,40] because two HFLTSs must be of the same length to guarantee a correct ranking. Such an addition mainly relies on the subjectivity of decision makers, but the determination of risk preference is an intractable task indeed.

(5) To extend HFLTSs, hesitant fuzzy linguistic sets (HFLSs) have been defined on the basis of a continuous linguistic term set [43]. Although inconsecutive values are allowed in a hesitant fuzzy linguistic element (HFLE) [43], such an extension has certain limitations and the proposed aggregation operators involve complicated calculations, which will be discussed in detail in Section 2.

Due to the described limitations of the existing methods using HFLTSs, the directional Hausdorff distance is proposed in this paper. This distance can effectively accommodate HFLTSs, and is therefore further integrated into the outranking approach. The calculation involving intervals and labels is not adopted by this distance. Although cloud models have been shown to be capable of describing the inherent relation between randomness and fuzziness, and, therefore, have been successfully applied to linguistic decision making to transform linguistic variables into quantitative data [44–46], the parameters being involved, i.e., expected value, entropy, and hyper entropy, still cannot reveal various semantics. To carefully and comprehensively process the mapping from the linguistic terms to numerical data, the linguistic scale functions [47] are employed in this paper to conduct the related transformation. In this way, the original essence of vague evaluations can be properly retained and the accuracy and reliability of final results can be further increased.

In addition, a relation model is chosen in this paper because function models (e.g., TOPSIS and VIKOR) mainly rely on various distance measures, and different results may arise based on different operators and methods. ELECTRE, originally proposed by Benayoun and Roy [48,49], is a popular relation model by means of which the relevant alternatives can generally be ranked based on the defined outranking relations. ELECTRE and its extensions have been widely studied [48–51] and applied in various MCDM problems [52–58]. In this paper, an outranking approach to solving MCDM problems is developed, which is based upon the features of the directional Hausdorff distance and the elicitation of the ELECTRE methods used by Devi and Yadav [59] and Figueira et al. [60]. This approach will be further tested and compared with some of the above described methods employing HFLTSs.

The remainder of this paper is organized as follows. Section 2 contains the definitions of HFLTSs and HFLSs, and related discussion, as well as the definition of linguistic scale functions. In Section 3, the directional Hausdorff distance and dominance relations are developed together with some properties and propositions. In Section 4, using the proven proposal, an outranking approach to solving MCDM problems in the context of HFLTSs is constructed. Section 5 includes an illustrative example relevant to SCM, and the proposed approach is also validated through sensitivity and comparative analyses. The final conclusions and proposals for future work are summarized in Section 6.

2. Preliminaries

In this section, the definitions and some basic operations of HFLTSs are briefly reviewed, and the effectiveness of other existing methods/models using HFLTSs is discussed with the use of relevant examples. The definitions of HFLSs and their related operations are also discussed. Moreover, the linguistic scale functions are reviewed.

2.1. Hesitant fuzzy linguistic term sets

Assuming $S = \{s_i \mid 0 = 1, 2, \ldots, 2g, g \in \mathbb{N}\}$ is a linguistic term set with odd cardinality, where $s_i$ denotes a possible value for a linguistic variable, the following must be satisfied [61,62]:
(1) the set is ordered: \( \alpha > \beta \iff s_\alpha > s_\beta \);  
(2) there is a negation operator: \( \text{neg}(s_\alpha) = s_{2g-\alpha} \).

**Definition 1** [26]. Let \( S = \{s_0, s_1, \ldots, s_{2g}\} \) and \( H \) be an HFLTS comprised of an ordered finite subset of the consecutive linguistic terms of \( S \).

The set of all HFLTSs on the linguistic term set \( S \), is denoted by \( \mathcal{H}(S) \). The empty and full HFLTSs for a linguistic variable \( \alpha \) are defined as follows:

(1) the empty HFLTS: \( H_\alpha(\emptyset(x)) = \emptyset \);  
(2) the full HFLTS: \( H_\alpha(S) = S \).

**Definition 2** [26]. Let \( H_1, H_2, \) and \( H_3 \) be three arbitrary HFLTSs on \( S \).

The following then can be defined:

(1) the upper bound \( h_S \) and the lower bound \( h_S \) of \( H_1 \): \( h_S = \max\{s|s_\alpha \in H_1\} \), where \( h_S < h_S \);  
(2) the intersection between \( H_1 \) and \( H_2 \): \( H_1 \cap H_2 = \{s|s_\alpha \in H_1 \text{ and } s_\alpha \in H_2\} \);  
(3) the union of \( H_1 \) and \( H_2 \): \( H_1 \cup H_2 = \{s|s_\alpha \in H_1 \text{ or } s_\alpha \in H_2\} \).

**Example 1.** Assuming \( S = \{s_0 = \text{Very Poor}, s_1 = \text{Poor}, s_2 = \text{Medium Poor}, s_3 = \text{Fair}, \ldots, s_{2n} = \text{Good}, s_{2n+1} = \text{Very Good}\} \), \( H_1 = \{s_1, s_2, s_3\} \), \( H_2 = \{s_1, s_3, s_4\} \), and \( H_3 = \{s_3, s_5\} \), the following can be calculated in terms of **Definition 2**:

(1) \( h_1 \leq s_0 \) and \( h_1 \leq s_5 \);  
(2) \( H_1 \cap H_2 = \{s_1, s_3\} \);  
(3) \( H_1 \cup H_2 = \{s_1, s_2, s_3, s_4, s_5\} \).

Originally, the envelopes of HFLTSs were employed to undertake a comparison among HFLTSs [26].

**Definition 3** [26]. The envelope of an HFLTS, denoted by \( \text{en}(H_\alpha) \), is the linguistic interval of \( H_\alpha \): \( \text{en}(H_\alpha) = [h_S, h_S] \). A comparison using the envelopes of two HFLTSs, \( H_1 \) and \( H_2 \), is defined as follows:

(1) \( H_1 \) is superior to \( H_2 \), denoted by \( H_1 > H_2 \), iff \( \text{en}(H_1) > \text{en}(H_2) \);  
(2) \( H_1 \) is inferior to \( H_2 \), denoted by \( H_1 < H_2 \), iff \( \text{en}(H_1) < \text{en}(H_2) \);  
(3) \( H_1 \) is indifferent to \( H_2 \), denoted by \( H_1 = H_2 \), iff \( \text{en}(H_1) = \text{en}(H_2) \).

Rodríguez et al. [26] revised the acceptability function introduced by Sengupta and Kumar Pal [63], and used the approach presented by Wang et al. [64] to compare two numerical intervals. However, if two HFLTSs have common elements but different envelopes, it is incorrect to state that one HFLTS is absolutely superior to another based on the comparison between two linguistic intervals.

**Example 2.** Using the data of **Example 1**, the following can be calculated based on **Definition 3**:

\[
\text{en}(H_1) = [s_0, s_2]; \quad \text{en}(H_2) = [s_2, s_4]; \quad \text{en}(H_3) = [s_4, s_5].
\]

By means of the preference relation between two numerical intervals that was proposed by Wang et al. [64], the preference degrees can be calculated as follows:

\[
\begin{align*}
P(H_2 > H_1) &= \frac{0.4 - 0.2}{1 - 0.4 - 0.2} = 1, \text{ and, thus, } H_2 > H_1; \\
P(H_1 > H_2) &= \frac{0.2 - 0.4}{1 - 0.2 - 0.4} = 1, \text{ and, thus, } H_1 > H_2.
\end{align*}
\]

\( H_2 > H_1 \) is clearly an acceptable result, but \( H_3 > H_2 \) and \( H_2 > H_3 \) must be discussed further because these two pairs each contain a common element. According to the definition of HFLTSs, each element in an HFLTS represents one possible choice when decision makers hesitate in expressing their opinions. Therefore, the common element, e.g., \( s_j \) existing in both \( H_2 \) and \( H_3 \), may lead to \( H_2 = H_3 \). In this situation, it is not appropriate to treat equally the relations between \( H_2 \) and \( H_3 \) and \( H_3 \) and \( H_2 \) and \( H_1 \) and \( H_2 \).

Due to the limitations of the comparison method discussed above, Wei et al. [38] proposed two aggregation operators (i.e., the hesitant fuzzy linguistic weighted average operator and the hesitant fuzzy linguistic ordered weighted average operator) based on a convex combination of HFLTSs.

**Definition 4** [65]. For two linguistic terms \( s_i \) and \( s_j \) in \( S \), the convex combination of \( s_i \) and \( s_j \) is defined as follows:

\begin{equation}
C^2(w_1, s_i, w_2, s_j) = w_1 \circ s_i \circ w_2 \circ s_j = s_k.
\end{equation}

Here, \( w_1 \geq 0, w_2 \geq 0, w_1 + w_2 = 1 \), and \( k = \text{round}((w_1)s_i + (w_2)s_j) \), where \( \text{round} \) is the standard round operation.

**Definition 5** [38]. Let \( H_1 \) and \( H_2 \) be two HFLTSs on \( S \). A convex combination of \( H_1 \) and \( H_2 \) is defined as follows:

\begin{equation}
C^2(w_1, H_1, w_2, H_2) = w_1 \circ H_1 \circ w_2 \circ H_2 = \{C^2(w_1, a_1, w_2, a_2) | a_1 \in H_1, a_2 \in H_2\}.
\end{equation}

Based on **Definition 5**, Wei et al. [38] defined the following hesitant fuzzy linguistic operator.

**Definition 6** [38]. Let \( H_1, H_2, \ldots, H_n \) be \( n \) HFLTSs on \( S \), and \( W = (w_1, w_2, \ldots, w_n) \) be a weighting vector of \( H_j = \{1, 2, \ldots, n\} \), where \( w_j \geq 0 \) and \( \sum_{j=1}^{n} w_j = 1 \). Then, the hesitant fuzzy linguistic weighted average (HLWA) operator is defined as follows:

\begin{equation}
\text{HLWA}(H_1, H_2, \ldots, H_n) = C^w(w_j, H_j, j = 1, 2, \ldots, n)
= w_1 \circ H_1 \circ (1 - w_1) \circ C^w \left( \sum_{k=2}^{n} \frac{w_k}{\sum_{i=1}^{n-1} w_i} H_i, i = 2, 3, \ldots, n \right)
\end{equation}

The following example demonstrates that the convex combination of two HFLTSs, given in **Definition 5**, cannot satisfy the associative law, and that the HLWA operator in **Definition 6** must fix the order of operands.

**Example 3.** Using the data of **Example 1** and assuming \( W = (0.3, 0.2, 0.5) \) is the associated weighting vector, the convex combinations of \( H_1, H_2, \) and \( H_3 \) given in **Definition 5** can be obtained in the following ways:

\[
\begin{align*}
&\text{en}(H_1) = [s_0, s_2]; \quad \text{en}(H_2) = [s_2, s_4]; \quad \text{en}(H_3) = [s_4, s_5]; \\
&= 0.5 \circ \frac{3}{5} \circ \{s_0, s_1, s_2\} \circ \frac{2}{5} \circ \{s_2, s_3, s_4\} \circ 0.5 \circ \{s_4, s_5\} \\
&= 0.5 \circ \{s_1, s_2, s_3\} \circ 0.5 \circ \{s_4, s_5\} = \{s_3, s_4\}
\end{align*}
\]
w₁ ⊕ H₁ ≺ (w₂ ⊕ H₂ ⊕ w₃ ⊕ H₃) 
= 0.3 ⊕ \{s₀, s₁, s₂\} ⊕ 0.7 ⊕ 2/7 \{s₂, s₃, s₄\} ⊕ 5/7 \{s₄, s₅\} 
= 0.3 ⊕ \{s₀, s₁, s₂\} ⊕ 0.7 ⊕ \{s₃, s₄, s₅\} = \{s₂, s₃, s₄\}.

Likewise, the calculation among H₁', H₂', and H₃' using the HLWA operator given in Definition 6 can be shown as follows.

\[
\begin{align*}
\text{HLWA}(H₁', H₂', H₃') &= Cᵢ(w₁, H₁', w₂, H₂', w₃, H₃') \\
&= 0.3 ⊕ H₁' ⊕ 0.7 ⊕ Cᵢ \left(\frac{2}{7} \cdot H₂', \frac{5}{7} \cdot H₃'\right) \\
&= 0.3 ⊕ \{s₀, s₁, s₂\} ⊕ 0.7 ⊕ \{s₃, s₄, s₅\} = \{s₂, s₃, s₄\}.
\end{align*}
\]

Example 3 shows that the operators in Definitions 5 and 6 may produce different results depending upon either the order in which the operations are performed, or the order or sequence of the operands. Therefore, the calculation or evaluation using such operators is inconsistent, and may lead to an inaccurate conclusion.

In addition, having considered the complexity of the multi-criteria linguistic decision-making model proposed by Rodríguez et al. [26], Lee and Chen [31] defined the 0-cut of HFLTSs, utilized the membership functions of the linguistic terms, and finally proposed the comparison relations of HFLTSs, which were founded on the method of likelihood-based comparison relations between intervals [66].

Definition 7 [31]. Let Hᵢ' (i = 1, 2) be two HFLTSs on S. The 0-cut of any HFLTS Hᵢ is Hᵢ(0) = [vᵢ, vᵢ₀], where vᵢ = \{sᵢ|n = \max(t(Hᵢ(0)), 1, 0)\} and vᵢ₀ = \{sᵢ|n = \min(t(Hᵢ(0)), 1, 2)\}. The likelihood-based comparison relation between H₁' and H₂', denoted by ψ(H₁' ≻ H₂'), is defined as follows:

\[
\psi(H₁' ≻ H₂') = \max \left(1 - \frac{t(\bar{v}_₂') - t(\bar{v}_₁')}{L(H₁')(0) - L(H₂')(0)} \cdot 0, 0\right),
\]

where L(Hᵢ')(0) = t(\bar{v}_ᵢ') - t(\bar{v}_ᵢ) and t(sᵢ) = i denotes the label of the linguistic term sᵢ.

Example 4. Let S and H₁' be that defined in Example 1, and H₂ = \{s₁, s₂\}. Using Formula (4), H₁(0) = \{s₀, s₁\}, H₂'(0) = \{s₀, s₁\}, L(H₁(0)) = L(H₂'(0)) = 3, and

\[
\psi(H₁ ≻ H₂) = \psi(H₁' ≻ H₂') = \max \left(1 - \frac{3}{3 + 3} \cdot 0, 0\right) = 0.5.
\]

The result of Example 4 indicates H₁ = H₁', which is clearly unfeasible. Given H₁' and H₂', the drawbacks of Definition 7 can be briefly summarized:

(1) H₁'' = H₁', H₂'' = S₀, and H₃'' = S₁ ⇒ ψ(H₃'' ≻ H₂'') = ψ(H₂'' ≻ H₁'') = 0.5 ⇒ H₁'' ≻ H₂'';
(2) H₁'' = H₁', H₂'' = S₂, and H₃'' = S₃ ⇒ ψ(H₃'' ≻ H₁'') = ψ(H₁'' ≻ H₃'') = 0.5 ⇒ H₁'' ≻ H₃''.

Thereby, the fuzzy decision-making method based on Definition 7 is also unfeasible.

2.2. Hesitant fuzzy linguistic sets

To preserve all the given information, the discrete term set S can be extended to a continuous linguistic term set \(S = \{sₖ | s₀ ≤ sₖ ≤ s₈, k ∈ [0, 2g]\}\) [67]. If sₖ ∈ S, then sₖ is denoted as the original linguistic term and is used for evaluating the alternatives; otherwise, sₖ is denoted as the virtual linguistic term and only appears in operations.

Similarly based on the concept of HFSs [7] in which several values are used to define the membership of an element, S was employed and linguistic terms were permitted to be inconsistent in an HFLE [43].

Definition 8 [43]. Let X be a reference set, \(S = \{sₖ | s₀ ≤ sₖ ≤ s₈, k ∈ [0, 2g]\}\), and H₁₃ be an HFLS on X defined in terms of a function h₁₃ that returns a subset of Sₚ.

For convenience, h₁₃(x) = h₁₃ is denoted as an HFLE [43], and h₁₃ is the set of all HFLEs. An HFLTS is clearly an HFLE, whereas it is not certain whether an HFLE is an HFLTS [43].

Definition 9 [43]. Let h₁₃, h₂₃, and h₃₃ be three HFLEs on S, and \(λ ∈ [0, 1]\). The following operations can then be defined:

(1) \(h₁₃^λ\) = \(∪_{h₁₃ ∈ h₁₃} \{sₖ\}\)
(2) \(∩ h₁₃ = \{sₖ | h₁₃(sₖ) = 1\}\)
(3) \(h₁₃ ⊕ h₂₃ = \{sₖ | h₁₃(sₖ) = h₂₃(sₖ)\}\)
(4) \(h₁₃ ⊗ h₂₃ = \{sₖ | h₁₃(sₖ) = h₂₃(sₖ)\}\)

The operations of HFLEs given above are mainly dependent on the labels of the linguistic terms, and the distances of adjacent linguistic terms are assumed to be identical, which would not be the case in practice. For example, the distance between ‘Very Poor’ and ‘Poor’ may be different from that between ‘Medium Poor’ and ‘Fair’.

Example 5. Suppose \(S = \{s₀, s₁, s₂\}\), \(h₁₃, h₂₃,\), and h₃₃ are three HFLEs on X. Employing the data of Example 1 with h₁₃ = H₁₃ = \{s₀, s₁, s₂\}, h₂₃ = H₂₃ = \{s₂, s₃, s₄\}, and h₃₃ = H₃₃ = \{s₄, s₅\}, the following can be calculated based on Definition 9:

\[
\begin{align*}
\text{h₁₃} ⊕ \text{h₂₃} &= \cup_{sₖ ∈ \text{h₁₃} ∩ \text{h₂₃}} \{sₖ\} = \{s₂, s₄, s₅, s₆\}; \\
\text{h₁₃} ⊗ \text{h₂₃} &= \cup_{sₖ ∈ \text{h₁₃} ∩ \text{h₂₃}} \{sₖ\} = \{s₆, s₁₀, s₁₂, s₁₃, s₁₆, s₂ₙ\}.
\end{align*}
\]

The results of Example 5 reveal the complexities of Definition 9: the number of operations greatly relies on the number of elements in each related HFLE, e.g., because three elements are in each unit, \(3 \times 3 = 9\) additive operations are conducted in \(h₁₃ ⊕ h₂₃\); the cardinalities of the results of operations are also dependent on the number of elements in each related HFLE, and are no more than their product, e.g., the result of \(h₁₃ ⊕ h₂₃\) contains \(3 \times 2 = 6\) elements; however, the number of operations and the cardinalities of results will exponentially increase if more HFLEs are involved in the operations. The deterioration in computational efficiency caused by these complexities may limit the application of the hesitant fuzzy linguistic aggregation operators.

The allowance of inconsistent linguistic terms in an HFLE may lead to unusual conditions. For the HFLE h₁₃ = \{s₀, sₖ\}, it is intractable to interpret the hesitation between s₀ and sₖ, which may be
the result of a poor elicitation of preferences or inconsisteny of the decision-makers. Moreover, the calculation results of Definition 9 may exceed the domain of \( \overline{S} \), e.g., any element \( s_i \) in the result of \( h_{12}^S \oplus h_{12}^G \) in Example 5 does not exist in \( \overline{S} \). Therefore, the extension of HFLTSs in Definition 8 is neither complete nor sufficiently accurate.

### 2.3. Linguistic scale functions

The mapping from the linguistic terms to numerical data must involve careful and comprehensive processing because it can greatly affect the accuracy and reliability of the final results. The operations directly based on the labels of linguistic terms are quite commonly employed, but are impractical because a simple transformation from linguistic terms to real numbers cannot properly retain the original essence of vague evaluations. To cope with qualitative data properly, the means of transforming them into quantitative data is crucial, for which linguistic scale functions have been developed.

Taking into consideration the differences between adjacent linguistic terms and their common processing, two instances of the linguistic scale function are reviewed and used in subsequent analysis.

**Definition 10** [47]. Given the linguistic term \( s_i \) in \( S \), the linguistic scale function \( f : s_i \rightarrow \theta_i \) conducting the mapping from \( s_i \) to \( \theta_i \) \((i = 0, 1, \ldots, 2g)\), where \( 0 < \theta_0 < \theta_1 < \cdots < \theta_{2g} \leq 1 \).

The function \( f \) in Definition 10 is a strictly monotonically increasing function, and some possible alternatives are listed below.

1. **On the basis of the label function of linguistic terms**, the simplest form of \( f \) [47] is
   \[
   f(s_i) = \frac{x}{2g} \quad (x = 0, 1, \ldots, 2g).
   \]

2. **A scale function that can achieve a bidirectional increase in the geometric progression of the scale value** [23, 46, 47] is defined as \( f : s_i \rightarrow \theta_i(y = 0, 1, \ldots, 2g) \), where \( \theta_i \) is given as follows.
   \[
   \theta_i = \begin{cases} \frac{a^y}{a^{2g} - 1} & (0 \leq y \leq g) \\ \frac{a^y - a^{2g} - 1}{a^{2g} - 1} & (g < y \leq 2g) \end{cases}
   \]

Considerable experimental research has determined that \( a \) usually lies in the interval [1.36, 1.4] [68]. Moreover, \( a \) can also be determined by the following subjective set method. Supposing the alternative \( A_1 \) is far more important than the alternative \( A_2 \), and the importance ratio is \( m \), then \( a^x = m \) or \( a^x = \sqrt[2g]{m} \), where \( k \) represents the scale level. At present, it is generally accepted that \( m = 9 \) is the upper limit of the importance ratio.

**Example 6.** Supposing that \( S \) is defined according to Example 1, then the mappings given in Definition 10 may be calculated as follows.

1. \( \theta_0 = 0, \theta_1 = \frac{1}{6} = 0.1667, \theta_2 = \frac{1}{3} = 0.3333, \theta_3 = 0.5, \theta_4 = 0.6667, \theta_5 = 0.8333, \theta_6 = 1 \) are the mapping results based on Formula (5). Clearly, this is an arithmetic progression.

2. **Given** \( g = 3 \) and \( a = \sqrt[2g]{m} = 1.3687 \) for \( k = 7, \theta_1 = \frac{a^y}{a^{2g} - 1} = 0.2208 \) and \( \theta_2 = \frac{a^y - a^{2g} - 1}{a^{2g} - 1} = 0.6179 \). Based on Formula (6), the mapping results are therefore \( \theta_0 = 0, \theta_1 = 0.2208, \theta_2 = 0.3821, \theta_3 = 0.5, \theta_4 = 0.6179, \theta_5 = 0.7792, \) and \( \theta_6 = 1 \). The following features of this function can be verified: \( |\theta_2 - \theta_3| = |\theta_3 - \theta_4|, |\theta_1 - \theta_3| = |\theta_4 - \theta_5|, |\theta_0 - \theta_5| = |\theta_3 - \theta_6|, \) and \( |\theta_0 - \theta_1| > |\theta_1 - \theta_2| > |\theta_2 - \theta_3| \).

### 3. Dominance relations between hesitant fuzzy linguistic term sets

HFLTSs can clearly express the hesitance arising in decision-making using linguistic expressions in a straightforward way. In this section, a directional Hausdorff distance for HFLTSs is introduced on the basis of some prerequisites, and will be further employed to construct the dominance relations of HFLTSs. Additionally, some useful properties and propositions are discussed and sufficiently proven, which will be valuable to subsequent analysis.

#### 3.1. The directional hausdorff distance of hesitant fuzzy linguistic term sets

It is necessary to construct an appropriate comparison method for HFLTSs, and the unacceptable results demonstrated in Example 4 can be effectively avoided when the following statements in Definition 11 are taken into account.

**Definition 11.** Let \( H^S_1 \) and \( H^S_2 \) be two arbitrary HFLTSs on \( S \), and \( s^m_i \) be the \( m \)-th value in \( H^S_1 \). Then, a comparison method for HFLTSs can be defined as follows:

1. **(1)** \( H^S_1 \geq H^S_2 \) if \( s^m_1 \geq s^m_2 \) and \( h^L_1 \leq h^L_2 \), where \( s^m_i \in H^S_1, s^m_2 \in H^S_2, m = 1, 2, \ldots, l, l = \min \{ |H^S_1|, |H^S_2| \} \), and \( |H^S_1| \) denotes the number of linguistic terms in \( H^S_1 \) \((i = 1, 2)\);

2. **(2)** \( H^S_1 \geq H^S_2 \) if \( H^S_1 \) and \( H^S_2 \) are of the same HFLTS.

A complete ordering of all HFLTSs cannot be inferred in terms of the comparison method defined in Definition 11. Nevertheless, this comparison method is adequate to demonstrate the properties of the proposed directional Hausdorff distance.

**Example 7.** Let \( H^S_1, H^S_2, \) and \( H^S_3 \) be those given in Example 1, and, based on the comparison method given in Definition 11, \( H^S_2 \geq H^S_1, H^S_3 \geq H^S_2, \) and \( H^S_3 \geq H^S_1 \).

The directional Hausdorff distance of HFLTSs can now be introduced. The HLWA operator suffers from the considerable order or sequence of the operands.

**Definition 12.** Let \( H^S_1 \) and \( H^S_2 \) be two arbitrary HFLTSs on \( S \). As such, a hesitant directional Hausdorff distance \( D_{nah} \) from \( H^S_1 \) to \( H^S_2 \) can be defined as follows.

\[
D_{nah}(H^S_1, H^S_2) = \begin{cases} \frac{1}{|H^S_1|} \sum_{s^m_i \in H^S_1} \min \{ \max(0, f(s_i) - f(s_j)) \} & \text{if } h^L_1 \neq h^L_2 \\ \frac{1}{|H^S_2|} \sum_{s^m_i \in H^S_2} \min \{ \max(0, f(s_i) - f(s_j)) \} & \text{otherwise} \end{cases}
\]

Here, \( |H^S_i| \) denotes the number of the linguistic terms in \( H^S_i \) \((i = 1, 2)\).

**Example 8.** Let \( f, H^S_1, H^S_2, \) and \( H^S_3 \) be those given in Example 1. As such, the distances obtained using Formulae (5) and (7) can be given as follows:
$D_{hau}(H_1^3, H_2^3) = 0, D_{hau}(H_1^2, H_2^2) = 0, D_{hau}(H_1^1, H_2^1) = 0.1667, D_{hau}(H_1^2, H_2^2) = 0$, $D_{hau}(H_1^3, H_2^3) = 0.4167$, and $D_{hau}(H_1^2, H_2^2) = 0.0833$.

The distances obtained using Formulae (6) and (7) can be given as follows:

$D_{hau}(H_1^3, H_2^3) = 0, D_{hau}(H_1^2, H_2^2) = 0, D_{hau}(H_1^1, H_2^1) = 0.1179, D_{hau}(H_1^2, H_2^2) = 0, D_{hau}(H_1^3, H_2^3) = 0.3165$, and $D_{hau}(H_1^2, H_2^2) = 0.0807$.

Here, $D_{hau}(H_1^3, H_2^3) > D_{hau}(H_1^2, H_2^2) > D_{hau}(H_1^1, H_2^1)$, and the corresponding data clearly identify the relations/differences between $H_1^3$ and $H_2^3$, $H_1^2$ and $H_2^2$, $H_1^1$ and $H_2^1$, and $H_1^3$ and $H_2^3$. Accordingly, $D_{hau}$ is capable of successfully solving the problems highlighted in Example 2.

**Property 1.** Let $H_1^3$, $H_2^3$ and $H_3^3$ be three arbitrary HFLTSs on $S$, and require that Formula (7) satisfy the following properties:

1. $D_{hau}(H_1^3, H_2^3) = 0$;
2. $0 < D_{hau}(H_1^3, H_2^3) < 1$;
3. If $H_1^3 > H_2^3 > H_3^3$, then $D_{hau}(H_1^3, H_2^3) > D_{hau}(H_1^3, H_3^3)$ and $D_{hau}(H_1^3, H_2^2) > D_{hau}(H_1^3, H_2^3)$.

$D_{hau}$ depends on the direction taken, and, therefore, $D_{hau}(H_1^3, H_2^3) 
eq D_{hau}(H_2^3, H_3^3)$, which has been illustrated in Example 8, and $D_{hau}(H_1^3, H_2^3) = D_{hau}(H_2^3, H_3^3)$ is excluded from Property 1.

**Proof.** It is apparent that (1) and (2) in Property 1 are true, and the proof is therefore omitted here.

1. If $H_1^3 > H_2^3 > H_3^3$, then $h_1^3 > h_2^3 > h_3^3$ and $h_1^3 > h_2^3 > h_3^3$, based on the comparison method of HFLTSs given in Definition 11.

**Case 1:** $h_1^3 > h_2^3 > h_3^3$, $D_{hau}(H_1^3, H_2^3) = \frac{1}{\|P\|} \sum_{s_i \in H_1^3} \min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\}$ and $D_{hau}(H_1^3, H_2^3) = \frac{1}{\|P\|} \sum_{s_i \in H_1^3} \min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\}$.

If $s_i \in H_1^3$ and $s_i < h_2^3 < h_3^3$, i.e., $f(s_i) < f(h_2^3) < f(h_3^3)$, then $\max(0, f(s_i) - f(h_2^3)) = \max(0, f(s_i) - f(h_3^3)) = 0$ and $\min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\} = 0$. Thus, $D_{hau}(H_1^3, H_2^3) = D_{hau}(H_1^3, H_2^3)$ certainly holds.

If $s_i \in H_1^3$ and $h_2^3 < s_i < h_3^3$, i.e., $f(h_3^3) < f(s_i) < f(h_2^3)$, then $s_i > s_k$ for any $s_k \in H_3^3$, and, thus, $\min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\} = \min_{h_j \in H_2^3} \{f(s_i) - f(h_2^3)\} > 0$. Moreover, $\max(0, f(s_i) - f(s_k)) = f(s_i) - f(h_2^3) = f(h_3^3) - f(s_i) > 0$. Because $f(s_i) - f(h_2^3) > 0$, $f(s_i) - f(h_3^3) > 0$, $D_{hau}(H_1^3, H_2^3) > D_{hau}(H_1^3, H_2^3)$ certainly holds.

Similarly, $D_{hau}(H_1^3, H_2^3) > D_{hau}(H_1^3, H_2^3)$ can also be proven.

**Case 2:** $h_1^3 = h_2^3 = h_3^3$, $D_{hau}(H_1^3, H_2^3) = \frac{1}{\|P\|} \sum_{s_i \in H_1^3} \min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\}$ and $D_{hau}(H_1^3, H_2^3) = \frac{1}{\|P\|} \sum_{s_i \in H_1^3} \min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\}$.

If $s_k \in H_3^3$ and $h_2^3 < h_1^3 < s_k$, i.e., $f(h_2^3) < f(h_1^3) < f(s_k)$, then $\max(0, f(h_2^3) - f(s_k)) = \max(0, f(h_2^3) - f(s_k)) = 0$ and $\min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\} = \min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\} = 0$. Thus, $D_{hau}(H_1^3, H_2^3) > D_{hau}(H_1^3, H_2^3)$ certainly holds.

If $s_k \in H_3^3$ and $h_2^3 < s_k < h_1^3$, i.e., $f(h_2^3) < f(s_k) < f(h_1^3)$, then $s_k > s_i$ for any $s_i \in H_1^3$, and, thus, $\min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\} = \min_{h_j \in H_2^3} \{0\} = 0$. Thus, $D_{hau}(H_1^3, H_2^3) > D_{hau}(H_1^3, H_2^3)$ certainly holds.

An interesting characteristic of $D_{hau}$ is observed in the proof of Property 1: given $H_1^3 > H_2^3$ and $h_i^3 \neq h_i^3$, the value of $D_{hau}(H_1^3, H_2^3)$ is independent of $h_i^3$, for example, $D_{hau}(\{s_1\}, \{s_2, s_3\}) = D_{hau}(\{s_2\}, \{s_1, s_3\})$. This characteristic is reasonable and acceptable because $D_{hau}$ is not an absolute distance measure in nature. The proposed distance aims to reveal the superiority of $H_1^3$ against $H_2^3$, and is suitably accurate and feasible to accommodate comparative linguistic expressions. Indeed, this is precisely why the proposed distance is later employed to define the outranking relations of HFLTSs.

**Definition 13.** Let $H_1^3$ and $H_2^3$ be two arbitrary HFLTSs on $S$. As such, a generalized hesitant directional Hausdorff distance $D_{phd}(H_1^3, H_2^3)$ can be defined as follows.

$$D_{phd}(H_1^3, H_2^3) = \begin{cases} \frac{1}{\|P\|} \sum_{s_i \in H_1^3} \min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\} & \text{if } h_1^3 \neq h_3^3 \\ \frac{1}{\|P\|} \sum_{s_i \in H_1^3} \min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\} & \text{otherwise} \end{cases}$$

Here, $\lambda > 0$.

In particular, if $\lambda = 1$, then $D_{phd}$ is reduced to $D_{hau}$; if $\lambda = 2$, then $D_{phd}$ is reduced to the hesitant directional Euclidean-Hausdorff distance $D_{dheu}$, as follows.

$$D_{phd}(H_1^3, H_2^3) = \begin{cases} \frac{1}{\|P\|} \sum_{s_i \in H_1^3} \min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\} & \text{if } h_1^3 \neq h_3^3 \\ \frac{1}{\|P\|} \sum_{s_i \in H_1^3} \min_{h_j \in H_2^3} \{\max(0, f(s_i) - f(s_j))\} & \text{otherwise} \end{cases}$$

Formulae (8) and (9) clearly satisfy Property 1, and the proof is omitted in this case. For computational convenience, $D_{phd}$ is firstly utilized in the subsequent analysis, and $D_{phd}$ in the sensitivity analysis.

### 3.2. Dominance relations between hesitant fuzzy linguistic term sets

In terms of the features of the proposed distance, $D_{hau}(H_1^3, H_2^3)$, the distance from $H_1^3$ to $H_2^3$ also represents the degree to which
outranks $H^2_2$. Furthermore, supposing the differences of the defined values of adjacent linguistic terms are respectively calculated, the minimum among them, i.e., $\min_{s_i \in \langle f(s_{i-1}) - f(s_i) \rangle}$, is employed as the threshold for judging the dominance relations in the outranking approach.

**Definition 14.** Let $H^1_1$ and $H^2_1$ be two arbitrary HFLTs on $S$. The binary relations between $H^1_1$ and $H^2_1$ can be defined by the following four categories.

1. Strong dominance ($H^1_1$ strongly dominates $H^2_1$ or $H^2_1$ is strongly dominated by $H^1_1$): $H^1_1 H^2_1$ (or $H^2_1 H^1_1$) $\iff D_{abh}(H^1_1, H^2_1) = \min_{s_i \in \langle f(s_{i-1}) - f(s_i) \rangle}$ and $D_{abh}(H^2_1, H^1_1) = 0$.

2. Weak dominance ($H^1_1$ weakly dominates $H^2_1$ or $H^2_1$ is weakly dominated by $H^1_1$): $H^1_1 H^2_1$ (or $H^2_1 H^1_1$) $\iff 0 < D_{abh}(H^1_1, H^2_1) < \min_{s_i \in \langle f(s_{i-1}) - f(s_i) \rangle}$ and $D_{abh}(H^2_1, H^1_1) = 0$.

3. Indifference ($H^1_1$ is indifference to $H^2_1$, i.e., $H^1_1 = H^2_1$): $H^1_1 H^2_1$ $\iff D_{abh}(H^1_1, H^2_1) = D_{abh}(H^2_1, H^1_1) = 0$.

4. Incomparable relation: if none of the relations defined above exists between $H^1_1$ and $H^2_1$, then $H^1_1$ and $H^2_1$ are incomparable, as denoted by $H^1_1 \perp H^2_1$.

For the purpose of establishing the threshold, Formula (5) yields $\min_{s_i \in \langle f(s_{i-1}) - f(s_i) \rangle} = \frac{1}{2^p}$, whereas, using Formula (6), $\min_{s_i \in \langle f(s_{i-1}) - f(s_i) \rangle} = \frac{1}{2^{2^p}}$.

**Example 9.** Let $H^1_1$, $H^2_1$, and $H^3_1$ be those provided by Example 1, and $H^4_1 = (s_4)$. Using Formula (5), the threshold for judging dominance is $\frac{1}{2^p} = 0.1667$. Then, the calculations based on Definition 14 are given as follows:

$D_{abh}(H^1_1, H^2_1) = 0$ and $D_{abh}(H^1_1, H^2_1) = 0.1667$

$\Rightarrow H^1_1 H^2_1$, i.e., $\{s_2, s_3, s_4\} >_{s} \{s_2, s_1, s_3\}$;

$D_{abh}(H^1_1, H^2_1) = 0$ and $D_{abh}(H^1_1, H^2_1) = 0.4167$

$\Rightarrow H^1_1 H^2_1$, i.e., $\{s_4, s_5\} >_{s} \{s_3, s_1, s_3\}$;

$D_{abh}(H^1_1, H^2_1) = 0$ and $D_{abh}(H^1_1, H^2_1) = 0.3334$

$\Rightarrow H^1_1 H^2_1$, i.e., $\{s_4, s_5\} >_{s} \{s_3, s_1, s_3\}$;

$D_{abh}(H^1_1, H^2_1) = 0$ and $D_{abh}(H^1_1, H^2_1) = 0.0833$

$\Rightarrow H^1_1 H^2_1$, i.e., $\{s_2, s_3, s_4\} >_{s} \{s_2, s_1, s_4\}$;

$D_{abh}(H^1_1, H^2_1) = 0$ and $D_{abh}(H^1_1, H^2_1) = 0.1667$

$\Rightarrow H^1_1 H^2_1$, i.e., $\{s_4, s_5\} >_{s} \{s_3, s_1, s_3\}$;

$D_{abh}(H^1_1, H^2_1) = 0.0833$ and $D_{abh}(H^1_1, H^2_1) = 0$

$\Rightarrow H^1_1 H^2_1$, i.e., $\{s_4, s_5\} >_{s} \{s_2, s_3, s_4\}$;

Several propositions can be derived from Definition 14, and may facilitate subsequent analysis.

**Proposition 1.** Let $H^1_1$ and $H^2_1$ be two arbitrary HFLTs on $S$. If $h^1_1 >h^2_1$, then $H^1_1 H^2_1$.

Proof. Supposing $h^1_1 > h^2_1$, then $\min \{s_i | s_i \in H^1_1 \} > \max \{s_j | s_j \in H^2_1 \}$, which indicates $f(s_i) - f(s_j) > 0$, where $s_i \in H^1_1$ and $s_j \in H^2_1$. The proposed distance between $H^1_1$ and $H^2_1$ is given as follows.

$$D_{abh}(H^1_1, H^2_1) = \frac{1}{H^1_1} \sum_{s_j \in H^2_1} \min \{f(s_j) - f(s_i) \}$$

$$= \frac{1}{H^1_1} \sum_{s_j \in H^2_1} \{f(s_j) - f(h^2_1) \}$$

$$= \frac{1}{H^1_1} \sum_{s_j \in H^2_1} \{f(s_i) - f(h^2_1) \}$$

Given Formula (5), $\max_{s_j \in H^2_1} \{f(s_j) - f(s_i) \} = f(h^1_1) - f(h^2_1)$, and

$$\Rightarrow \frac{1}{H^1_1} \sum_{s_j \in H^2_1} \{f(s_i) - f(h^2_1) \} = f(h^1_1) - f(h^2_1)$$

$$\Rightarrow \min \{f(s_i) - f(s_j) \}$$

Therefore, $H^1_1 H^2_1$ can be obtained, based on the statement of the strong dominance relation in Definition 14.

**Proposition 2.** Let $H^1_1$ and $H^2_1$ be two arbitrary HFLTs on $S$. If $h^1_1 > h^2_1 \geq h^1_2$, then $H^1_1 H^2_1$ or $H^1_2 H^2_1$.

Proof. Suppose $s_i \leq h^2_1$, where $s_i \in H^1_1$, $\max \{0, f(s_i) - f(h^2_1) \} = 0$, and $\min_{s_j \in H^2_1} \{f(s_i) - f(s_j) \} = 0$. Also, suppose $s_i > h^1_2$, where $s_i \in H^1_1$ and $s_i > s_j$ for any $s_j \in H^2_1$. Then, $\min_{s_j \in H^2_1} \{f(s_i) - f(s_j) \} = f(s_i) - f(h^1_2)$, and the proposed distance between $H^1_1$ and $H^2_1$ is given as follows.

$$D_{abh}(H^1_1, H^2_1) = \frac{1}{H^1_1} \sum_{s_j \in H^2_1} \min \{0, f(s_i) - f(s_j) \}$$

$$= \frac{1}{H^1_1} \sum_{s_j \in H^2_1} \{f(s_i) - f(h^1_2) \} > 0$$

For any $s_j \in H^2_1$, if $s_i \leq h^2_1$, then $f(s_i) - f(h^1_2) < 0$ and $\max \{0, f(s_i) - f(h^1_2) \} = 0$. Thus, $\min_{s_j \in H^2_1} \{f(s_i) - f(s_j) \} = 0$, and the proposed distance between $H^1_1$ and $H^2_1$ is given as follows.

$$D_{abh}(H^1_1, H^2_1) = \frac{1}{H^1_1} \sum_{s_j \in H^2_1} \min \{0, f(s_i) - f(s_j) \}$$

$$= \frac{1}{H^1_1} \sum_{s_j \in H^2_1} \min \{0 \} = 0$$

Therefore, $D_{abh}(H^1_1, H^2_1) > 0$ and $D_{abh}(H^1_1, H^2_1) = 0 \Rightarrow H^1_1 H^2_1$ or $H^1_2 H^2_1$, in terms of the statements related to the strong and weak dominance relations given in Definition 14.
Proposition 3. Let $H^1_2$ and $H^2_2$ be two arbitrary HFLTs on $S$. If $h^1_2 > h^2_2$, then $H^1_2 > H^2_2$ (Proposition 1); if $h^1_2 > h^2_2$, then $H^1_2 > H^2_2$ (Proposition 2); if $h^1_2 = h^2_2$, then $H^1_2 = H^2_2$ (Proposition 3); if $h^1_2 < h^2_2$, then $H^1_2 < H^2_2$ (Proposition 4).

The proof of Proposition 3 is similar to that of Proposition 2, and is therefore omitted here.

The summary above covers all possible relations between two HFLTs, which indicates that incomparable pairs of HFLTs do not actually exist. Therefore, the proposed dominance relations in Definition 14 are comprehensive, which results in the correct order of all HFLTs in $\mathcal{R}(S)$.

The transitivity of the strong dominance relation is now discussed.

Proposition 4. Let $H^1_2$, $H^2_2$, and $H^3_2$ be three arbitrary HFLTs on $S$. If $H^1_2 > H^2_2$ and $H^2_2 > H^3_2$, then $H^1_2 > H^3_2$.

Proof. If $H^1_2 > H^2_2$ and $H^2_2 > H^3_2$, then $D_{\text{th}}(H^1_2, H^2_2) \geq \min_{s \in S} (f(s_{11}) - f(s_{21}))$ and $D_{\text{th}}(H^2_2, H^3_2) \geq \min_{s \in S} (f(s_{11}) - f(s_{21}))$ based on Definition 14.

Case 1: $H^1_2 \cap H^3_2 = \emptyset$.

Using Propositions 1–3, $H^1_2 > H^2_2$ and $H^2_2 > H^3_2$ can be deduced, and, therefore, $H^1_2 > H^3_2$ in terms of $h^1_2 > h^3_2$.

Case 2: $H^1_2 \cap H^3_2 \neq \emptyset$ and $h^1_2 \neq h^3_2$.

$h^1_2 > h^2_2$ and $h^2_2 > h^3_2$ can be deduced using the given conditions and Propositions 1 and 2. These expressions can be simplified as $f(h^1_2) > f(h^2_2) > f(h^3_2) > f(h^1_2)$, and $f(h^1_2) > f(h^3_2)$.

According to the proof of Proposition 2, $D_{\text{th}}(H^1_2, H^2_2) = \frac{1}{|H^1_2| \sum_{h^1 \in H^1_2} (f(s_{11}) - f(h^1_2))}$, which can also be given as $D_{\text{th}}(H^1_2, H^2_2) = \frac{1}{|H^1_2| \sum_{h^1 \notin H^1_2} (f(s_{11}) - f(h^1_2))}$ for simplicity. The proposed distance between $H^1_2$ and $H^2_2$ is given as follows.

$$D_{\text{th}}(H^1_2, H^2_2) = \frac{1}{|H^1_2|} \sum_{h^1 \in H^1_2} \max_{h^2 \in H^2_2} (f(s_{11}) - f(h^1_2))$$

$$= \frac{1}{|H^1_2|} \sum_{h^1 \in H^1_2} \left( f(h^1_2) - f(h^2_2) \right)$$

$$= \frac{1}{|H^1_2|} \sum_{h^1 \in H^1_2} (f(s_{11}) - f(h^1_2)) = D_{\text{th}}(H^1_2, H^2_2)$$

Thus, $H^1_2 > H^2_2$.

If $h^1_2 = h^2_2$ in Case 2, the proof is similar to the process provided above, and has therefore been omitted here.

Case 3: $H^1_2 \cap H^3_2 \neq \emptyset$ and $h^1_2 = h^3_2$. $H^1_2 \cap H^3_2$, and $h^1_2 = h^3_2$ can be deduced using the given conditions and Propositions 1–3. These expressions are simplified as $f(h^1_2) > f(h^2_2) \geq f(h^1_2)$ and $H^1_2 \cap H^3_2$; $H^1_2 \cap H^3_2$, and $h^1_2 = h^3_2$ can be deduced with the given conditions and Propositions 1–3. These expressions are simplified as $f(h^1_2) > f(h^2_2) \geq f(h^1_2)$.

$H^1_2 > H^2_2$ and $H^1_2 > H^3_2$ can be proven as well. Thus, $H^1_2 > H^3_2$.

Property 2. Let $H^1_2$, $H^2_2$, and $H^3_2$ be three arbitrary HFLTs on $S$.

1. The strong dominance relation has
   (a) irreflexivity: $\emptyset \neq H^1_2 \cap H^2_2$;
   (b) asymmetry: $H^1_2 \cap H^2_2 \neq H^1_2 \cap H^2_2$;
   (c) transitivity: $H^1_2 \cap H^2_2 \cap H^3_2 \neq H^1_2 \cap H^2_2$.

2. The weak dominance relation has
   (a) irreflexivity: $\emptyset \neq H^1_2 \cap H^2_2$;
   (b) asymmetry: $H^1_2 \cap H^2_2 \neq H^1_2 \cap H^2_2$;
   (c) non-transitivity: $H^1_2 \cap H^2_2 \cap H^3_2 \neq H^1_2 \cap H^2_2$.

3. The indifference relation has
   (a) reflexivity: $\emptyset \neq H^1_2 \cap H^2_2$;
   (b) symmetry: $H^1_2 \cap H^2_2 = H^1_2 \cap H^2_2$;
   (c) transitivity: $H^1_2 \cap H^2_2 \cap H^3_2 = H^1_2 \cap H^2_2$.

4. The incomparable relation has
   (a) irreflexivity: $\emptyset \neq H^1_2 \cap H^2_2$;
   (b) symmetry: $H^1_2 \cap H^2_2 = H^1_2 \cap H^2_2$;
   (c) non-transitivity: $H^1_2 \cap H^2_2 \cap H^3_2 = H^1_2 \cap H^2_2$.

Property 2 can be partially illustrated by the following example.

Example 10. Let $H^1_2$, $H^2_2$, and $H^3_2$ be those given in Example 9, and $H^2_2 = \{s_3, s_4\}$. Using Formula 5, the threshold for judging the dominance relations is $\frac{1}{|H^1_2|} = 0.1667$. According to Definition 14, the following can be obtained.

$$D_{\text{th}}(H^1_2, H^2_2) = 0.5834$$

$$D_{\text{th}}(H^2_2, H^3_2) = 0.2500$$

Then, the following can be demonstrated based on Property 2.

1. Irreflexivity: $H^1_2 \neq H^1_2$;
2. Irreflexivity: $H^1_2 \neq H^3_2$;
3. Irreflexivity: $H^1_2 \neq H^1_2$;
4. Irreflexivity: $H^1_2 \neq H^3_2$;
5. Irreflexivity: $H^1_2 \neq H^3_2$;
6. Irreflexivity: $H^1_2 \neq H^1_2$;
7. Irreflexivity: $H^1_2 \neq H^3_2$;
8. Irreflexivity: $H^1_2 \neq H^3_2$;
9. Irreflexivity: $H^1_2 \neq H^3_2$;
10. Irreflexivity: $H^1_2 \neq H^3_2$;
11. Irreflexivity: $H^1_2 \neq H^3_2$;
12. Irreflexivity: $H^1_2 \neq H^3_2$;
13. Irreflexivity: $H^1_2 \neq H^3_2$;


4. An outranking approach for solving MCDM problems based on the directional hausdorff distance

In this section, the defined dominance relations are unified into the ELECTRE method to provide an outranking approach in the context of HFLTSs.

A multi-criteria linguistic ranking or selection problem usually refers to assessing $n$ alternatives, denoted by $P = \{p_1, p_2, \ldots, p_n\}$, in terms of $m$ criteria, denoted by $R = \{r_1, r_2, \ldots, r_m\}$. The evaluation result of $p_i$ under $r_j$ is given using linguistic expressions, and denoted by $p_{ij}$, which can be transformed into an HFLTS using the linguistic term set $S = \{s_j|j = 0, 1, 2, \ldots, 2g, g \in N\}$ and the corresponding grammar and function given by Rodríguez et al. [26]. The weight of $r_j$ is $w_j$, satisfying $w_j \geq 0$ ($j = 1, 2, \ldots, m$) and $\sum_{j=1}^{m} w_j = 1$.

**Definition 15.** The set of labels for all criteria is $J = \{j | j = 1, 2, \ldots, m\}$. To determine the relationship between the alternatives $p_i$ and $p_k$ ($i, k = 1, 2, \ldots, n$), the following sets of criteria can be defined:

1. $C_{\text{co}}(p_i, p_j) = \{j | 1 \leq j \leq m, p_{ij} \succ p_{jk}\}$;
2. $C_{\text{wo}}(p_i, p_j) = \{j | 1 \leq j \leq m, p_{ij} \succ_w p_{jk}\}$;
3. $ID(p_i, p_j) = ID(p_j, p_i) = \{j | 1 \leq j \leq m, p_{ij} \sim u p_{ji}\}$;
4. $D_{\text{wo}}(p_i, p_j) = \{j | 1 \leq j \leq m, p_{ij} \sim p_{jk}\}$;
5. $D_{\text{wo}}(p_i, p_k) = \{j | 1 \leq j \leq m, p_{ijn} \sim w p_{kn}\}$.

The integration of the ELECTRE method and HFLTSs is utilized in this paper, and corresponding definitions are therefore required here.

The concordance index is modeled on the basis of the comprehensive concordance index [60], and has been revised to serve the directional Hausdorff distance defined in Section 3. The concordance index $c_{ik}$ between $p_i$ and $p_k$ is thus defined as follows:

$$c_{ik} = \sum_{j \in C_{\text{co}}(p_i, p_k)} w_j + \sum_{j \in D_{\text{wo}}(p_i, p_k)} w_j D_{\text{hdh}}(p_{ij}; p_{kj}),$$  

where $Q(p_i, p_k) = C_{\text{co}}(p_i, p_k) \cup C_{\text{wo}}(p_i, p_k) \cup ID(p_i, p_k)$, and the concordance matrix $C$ is:

$$C = \begin{pmatrix} c_{12} & c_{13} & \cdots & c_{1(n-1)} & c_{1n} \\ c_{21} & - & c_{23} & \cdots & c_{2(n-1)} & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{(n-1)1} & c_{(n-1)2} & \cdots & - & c_{(n-1)n} \\ c_{nn} & c_{n2} & c_{n3} & \cdots & c_{n(n-1)} & - \end{pmatrix}.$$ 

In $C$, $c_{ij}(i \neq k)$ denote the degree to which the evaluations of $p_k$ are at least as good as those of the competitor $p_i$, and the degree to which $p_k$ is inferior to $p_i$ decreases with increasing $c_{ij}$.

The discordance index is modeled on the basis of the discordance index proposed by Devi and Yadav [59], and has been revisited to serve the directional Hausdorff distance defined in Section 3. The discordance index $d_{ik}$ between $p_i$ and $p_k$ is thus defined as follows:

$$d_{ik} = \frac{\max_{j \in J: D_{\text{wo}}(p_i, p_j) \cup D_{\text{hdh}}(p_{ij}; p_{kj})} \sum_{j \in D_{\text{wo}}(p_i, p_j)} w_j D_{\text{hdh}}(p_{ij}; p_{kj})}{\max_{j \in J: D_{\text{wo}}(p_i, p_j) \cup D_{\text{hdh}}(p_{ij}; p_{kj})} \sum_{j \in D_{\text{wo}}(p_i, p_j)} w_j D_{\text{hdh}}(p_{ij}; p_{kj})},$$

where the discordance matrix $D$ is:

$$D = \begin{pmatrix} d_{12} & d_{13} & \cdots & d_{1(n-1)} & d_{1n} \\ d_{21} & - & d_{23} & \cdots & d_{2(n-1)} & d_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{(n-1)1} & d_{(n-1)2} & \cdots & - & d_{(n-1)n} \\ d_{nn} & d_{n2} & d_{n3} & \cdots & d_{n(n-1)} & - \end{pmatrix}.$$ 

In $D$, $d_{ij}(i \neq k)$ denotes the degree to which the evaluations of $p_k$ are worse than those of the competitor $p_i$, and the degree to which $p_i$ is superior to $p_k$ decreases with increasing $d_{ij}$.

The analysis given above indicates that both $c_{ik}$ and $d_{ij}$ include the weights of the criteria and the outranking relations among the alternatives. However, they measure different aspects of the relations, and the concordance indices and discordance indices are therefore not complementary.

To rank all alternatives, the net dominance index of $p_k$ is

$$c_k = \sum_{i=1}^{n} c_{ik} - \sum_{i=1}^{n} c_{ki},$$  

and the net disadvantage index of $p_k$ is

$$d_k = \sum_{i=1}^{n} d_{ik} - \sum_{i=1}^{n} d_{ki}.$$  

Here, $c_k$ is the sum of the concordance indices between $p_k$ and $p_l(l \neq k)$ minus the sum of the discordance indices between $p_l(l \neq k)$ and $p_k$, and reflects the dominance degree of the alternative $p_k$ among the relevant alternatives. Meanwhile, $d_k$ reflects the disadvantage degree of the alternative $p_k$ among the relevant alternatives. Therefore, $p_k$ obtains a greater dominance over the other alternatives that are being compared as $c_k$ increases and $d_k$ decreases.

**Definition 16.** The ranking rules of two alternatives are

1. if $c_i < c_k$ and $d_i > d_k$, then $p_i$ is superior to $p_k$, as denoted by $p_i > p_k$;
2. if $c_i = c_k$ and $d_i = d_k$, then $p_i$ is indifferent to $p_k$, as denoted by $p_i \sim p_k$;
3. if the relation between $p_i$ and $p_k$ does not belong to (1) or (2), then $p_i$ and $p_k$ are incomparable, as denoted by $p_i \perp p_k$.

A ranking of alternatives obtained by the rules defined above may be only a partial ranking, and greater detail is discussed by Wu and Chen [69].

The proposed approach for solving MCDM problems is described below.

**Step 1:** Construct and normalize the HFLTS decision matrix $P = (p_{ij})_{n \times n}$. For the benefit criterion $r_j$, no operations should be conducted on $p_{ij}$; for the cost criterion $r_j$, all elements of $p_{ij}$ should be normalized using the negation operator.

**Step 2:** Calculate the directional Hausdorff distance between $p_{ij}$ and $p_{ik}$. Determine the dominance relations between $p_i$ and $p_k$ for $r_j$ in terms of Definition 14, and with the help of Propositions 1–4 if possible. Calculate the sets of criteria labels in terms of Definition 15.

**Step 3:** Calculate the concordance index $c_{ij}(i \neq j$ and $j = 1, 2, \ldots, n$) based on Formula (10), and then construct the concordance matrix $C = (c_{ij})_{n \times n}$.

**Step 4:** Calculate the discordance index $d_{ij}(i \neq j$ and $j = 1, 2, \ldots, n$) based on Formula (11), and then construct the discordance matrix $D = (d_{ij})_{n \times n}$.

**Step 5:** Calculate the net dominance index of each alternative $c_i(i = 1, 2, \ldots, m)$ based on Formula (12), and the net disadvantage index of each alternative $d_i(i = 1, 2, \ldots, m)$ based on Formula (13).

**Step 6:** Formulate the ranking of all alternatives in light of the rules given by Definition 16.
5. An illustrative example

In this section, an illustrative example is provided to clearly highlight the strengths of the proposed approach. Moreover, its validity is tested by means of a sensitivity analysis, and its advantages are demonstrated through a comparative analysis with other existing methods that use HFLTSs.

5.1. Illustration of the proposed approach

In this subsection, the example used by Rodriguez [26] is employed to demonstrate the practicability of our proposal. However, to present a comprehensive analysis and application, this example is further adapted, wherein a criterion, i.e., the fourth criterion, is added, and the weight vector of the four criteria is assumed to be known. Therefore, in this illustration, the indifference relation will appear and the distinction of the defined concordance index will be explicitly shown. Except for the evaluation data, the background of this example is improved a little.

Example 11. With the rapid development of the economy and the increasing consumption of resources in China, it is vital to encourage domestic organizations to adopt green supply chain management (GSCM) and advocate green practices for the purpose of sustainable development for the entire country. In this context, suppose $P = \{p_1, p_2, p_3\}$ is a set of alternatives in a multi-criteria linguistic ranking or selection problem, $R = \{r_1, r_2, r_3, r_4\}$ is a set of criteria regarding greenness, and $S = \{s_0 = \text{VeryPoor}(V), s_1 = \text{Poor}(P), s_2 = \text{MediumPoor}(MP), s_3 = \text{Fair}(F), s_4 = \text{MediumGood}(MG), s_5 = \text{Good}(G), s_6 = \text{VeryGood}(VG)\}$ is a linguistic term set. The evaluation results $p_{ij}$ are given by a single expert or decision-maker who has uncertainty in their evaluation, and, therefore, most $p_{ij}$ are not single linguistic terms but several consecutive terms in $S$. Moreover, assuming that the weight vector of the four criteria is $W = (0.3, 0.4, 0.1, 0.2)$, the resulting $p_{ij}$ are listed in Table 1.

This hesitance in original evaluation information can be expressed using HFLTSs, and will be further processed to obtain a comprehensive ranking of the given alternatives. The proposed MCDM approach, as described in Section 4, is now utilized and the procedure is described as follows.

Step 1: The transformation from the linguistic expressions into HFLTSs can be completed based on the functions introduced in [26]. Moreover, all criteria are of the benefit type, and, thus, no normalization operations are required. The HFLTS decision matrix $P = (p_{ij})_{3 \times 4}$ is given below.

$$P = \left[ \begin{array}{cccc}
{s_1, s_2, s_3} & {s_4, s_5} & {s_4} & {s_4, s_5} \\
{s_2, s_3} & {s_3} & {s_0, s_1, s_2} & {s_4, s_5} \\
{s_4, s_5, s_6} & {s_1, s_2} & {s_4, s_5, s_6} & {s_5, s_4} \\
\end{array} \right]$$

Step 2: Calculate $D_{dbh}$ between $p_i$ and $p_{ij}$ to determine their dominance relations for $r_j$, and further identify the sets of criterion labels in terms of Definition 15. Firstly, the distance between each pair is obtained using Formulae (5) and (7). Secondly, the dominance relations between $p_i$ and $p_j$ are determined respectively for a threshold $\frac{1667}{1667} = 0.1667$. The dominance relations are determined as follows:

$$D_{dbh}(p_{11}, p_{21}) = 0.0555 \Rightarrow p_{21} > p_{11}$$

$$D_{dbh}(p_{12}, p_{22}) = 0.2500 \Rightarrow p_{12} > p_{22}$$

$$D_{dbh}(p_{13}, p_{23}) = 0$$

$$D_{dbh}(p_{14}, p_{24}) = 0.0833 \Rightarrow p_{14} > p_{24}$$

Similarly, $p_{11} > p_{13} \geq p_{12}, p_{13} > p_{14}, p_{13} > p_{12}, p_{13} > p_{12}, p_{13} > p_{12}, p_{13} > p_{12}$. The sets of criteria labels are given below:

$$C_{0d}(p_{11}, p_{21}) = \{2, 3\}, C_{0d}(p_{12}, p_{22}) = \{4\}, ID(p_{11}, p_{21}) = \emptyset$$

$$C_{0d}(p_{12}, p_{23}) = \emptyset, C_{0d}(p_{12}, p_{24}) = \{1\}, ID(p_{12}, p_{23}) = \emptyset$$

$$C_{0d}(p_{13}, p_{23}) = \{2\}, C_{0d}(p_{13}, p_{24}) = \emptyset$$

$$ID(p_{13}, p_{23}) = \emptyset, C_{0d}(p_{13}, p_{24}) = \{1, 3\}, ID(p_{13}, p_{23}) = \emptyset$$

Step 3: Based on Formula (10), the concordance matrix $C$ can be obtained as below:

$$C = \left[ \begin{array}{ccc}
- & 0.7167 & 0.6000 \\
& 0.3167 & - & 0.6000 \\
& 0.4167 & 0.6000 & - \\
\end{array} \right].$$

Step 4: Based on Formula (11), the discordance matrix $D$ can be obtained as below:

$$D = \left[ \begin{array}{ccc}
- & 0.1665 & 0.5999 \\
0.1665 & - & 1 \\
0.6670 & 1 & - \\
\end{array} \right].$$

Step 5: Based on Formulae (12) and (13), the net dominance index of each alternative $c_i$ ($i = 1, 2, 3$) and the net disadvantage index of each alternative $d_i$ ($i = 1, 2, 3$) can be obtained as shown below:

$$c_1 = 0.5833, c_2 = -0.4000, \text{ and } c_3 = -0.1833 \Rightarrow c_3 < c_2 < c_1$$

$$d_1 = -1.2334, d_2 = 1.1666, \text{ and } d_3 = 0.0668 \Rightarrow d_2 > d_3 > d_1$$

Step 6: According to the rules of Definition 16, the final ranking is $p_3 \succ p_2 \succ p_1$, and the best alternative is $p_1$.

If Formula (6) is utilized rather than Formula (5) in Step 2, the dominance relation of each pair is then given as follows:

$$D_{dbh}(p_{11}, p_{21}) = 0.0558 \Rightarrow p_{21} > p_{11}$$

$$D_{dbh}(p_{12}, p_{22}) = 0.1986 \Rightarrow p_{12} > p_{22}.$$
and the linguistic terms are not equal, the proposed approach is still conventional verification measure, and requires no further explanation. Moreover, both formulae in Definition 10 are also required here. The choice of different linguistic scale functions aims to confirm that, even if the differences between adjacent linguistic terms are not equal, the proposed approach is still competent. The sensitivity analysis is applied to Example 11. The results using Formulae (5) and (8) are summarized in Table 2, and the results using Formulae (6) and (8) are summarized in Table 3. The evaluation results are identical in Tables 2 and 3. This indicates that the final rankings of the alternatives remain unchanged regardless of the values of the parameter λ and the linguistic scale functions, which further verifies the robustness of the proposed approach.

5.3 Comparative analysis and discussion

The MCDM methods using HFLTSs discussed in Examples 2–4 are now used again for comparative purposes.

(1) The decision-making model including three phases (transformation, aggregation and exploitation) proposed by Rodríguez [26] is employed using the data of Example 11. In the aggregation phase, the linguistic intervals of three alternatives are built using two aggregation operators (min_upper and max_lower): ζ(p1) = [s1, s4], ζ(p2) = [s2, s3], and ζ(p3) = [s2, s4].

In the exploitation phase, similar to the calculation process of Example 2, the preference degrees can be calculated as follows:

\[ P(ζ(p_1) > ζ(p_2)) = 1, P(ζ(p_1) > ζ(p_3)) = 0.6667, \] and \[ P(ζ(p_2) > ζ(p_3)) = 0.6667. \]

The preference relations obtained here are equivalent to those obtained by Rodríguez [26], although the fourth criterion and the corresponding data have been added in Example 11. Hence, due to the nondominance choice degree of each alternative, the final ranking is \( p_1 > p_3 > p_2 \) and the best alternative is \( p_1 \).

The transformation from linguistic terms to intervals is by nature improper. The hesitance among the linguistic values directly produces a range that is applicable for both discrete and continuous values, and the fuzziness of the information provided is thus distorted. Furthermore, the comparison methods of HFLTSs [26], similar to those of intervals, are mainly dependent on the labels of the linguistic terms and ignore the different semantics that, in reality, exist. Therefore, these limitations of the envelopes of HFLTSs may hinder the development of the relevant methods.

(2) The MCDM method using the HLWA operator [38] is employed using the data of Example 11. This method is usable if the weighting vector of criteria is known. Firstly, the overall aggregation values of three alternatives are \( δ(p_1) = HLWA(p_{11}, p_{12}, p_{13}, p_{14}) = \{s_1, s_4\} \), \( δ(p_2) = HLWA(p_{21}, p_{22}, p_{23}, p_{24}) = \{s_1\} \), and \( δ(p_3) = HLWA(p_{31}, p_{32}, p_{33}, p_{34}) = \{s_1, s_4\} \). Secondly, \( δ(p_2) = \{s_1, s_4\} \) and \( δ(p_3) = \{s_1, s_4\} \) are constructed, and the possibility degree of \( p_1 \) being not less than \( p_2 \) is calculated as \( η(p_1 ≥ p_2) = 0.75 \). Thirdly, the aggregation results are ranked in terms of the possibility degree method, resulting in \( p_1 ≥ p_3 > 0.75p_2 \). It can be seen that both \( p_1 \) and \( p_3 \) are the best alternatives.

### Table 2

<table>
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<tr>
<th>( λ )</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( c_3 )</th>
<th>( d_1 )</th>
<th>( d_2 )</th>
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<td>-0.1833</td>
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<td>1.1666</td>
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### Table 3

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(3) The fuzzy decision-making method proposed by Lee and Chen [31], which uses the likelihood-based comparison relations of HFLTSs, is employed using the data of Example 11.

Firstly, in terms of Definition 7, the 0-cut of the linguistic term set \( S \) is \( S(0) = \{ \phi_1^0, \phi_2^0 \} \). Then, the likelihood-based comparison relation \( P' = (p_{ij}^{g}) \) is constructed using Formula (4), resulting in

\[
p_{ij}^{g} = \psi(p_{ij}) = \max \left( 1 - \max \left( \frac{t(\phi_i^g) - t(\phi_j^g)}{L(p_{ij}(0)) + L(S(0))}, 0 \right), 0 \right)
\]

\[
= \max \left( 1 - \max \left( \frac{2g - t(\phi_i^g)}{L(p_{ij}(0)) + 2g}, 0 \right), 0 \right).
\]

where \( p_{ij}(0) = [\phi_1^0, \phi_2^0] \). Thus,

\[
P' = \left[ \begin{array}{cccc}
0.4000 & 0.6667 & 0.6250 & 0.6667 \\
0.4444 & 0.5000 & 0.3333 & 0.5556 \\
0.6667 & 0.3333 & 0.6667 & 0.5556 \\
\end{array} \right].
\]

Secondly, according to \( R'(p_i) = \sum_{j=1}^{m} w_j p_{ij}^{g} \), the scores \( R'(p_1) = 0.5825, R'(p_2) = 0.4778, \) and \( R'(p_3) = 0.5111 \) are obtained. Finally, \( R'(p_1) > R'(p_2) > R'(p_3) \Rightarrow p_1 > p_2 > p_3 \), and \( p_1 \) is again the best alternative.

According to the results obtained using the aforementioned methods, the following advantages of the proposed approach can be accessed.

(1) When being compared to the methods above, the proposed approach yields equivalent or more accurate ranking results, and can clearly identify the differences among alternatives using the net dominance indices and the net disadvantage indices. Moreover, even if the original evaluation information remains unchanged, the evaluation results of the proposed approach vary with changing semantics, which cannot be achieved using the existing methods of HFLTSs.

(2) The formulae appearing in the proposed approach are sufficiently adequate to exclude the incorrect or inaccurate rankings among HFLTSs, which may hamper existing methods, as has been illustrated in Examples 2–4. This implies that the basis of the proposed approach is more robust. Thereby, the final ranking produced by the proposed approach is more conclusive than those produced by the other methods considered, and its accuracy and reliability is therefore evident according to the comparative analysis.

(3) In the proposed approach, the transformation from qualitative information to quantitative data is conducted by virtue of the linguistic scale functions and not simply by the labels of the linguistic terms. In this way, the fuzziness of the original information can be maintained and fully utilized for the ranking of all alternatives. Therefore, the proposed approach is more competent in linguistic decision-making than the other methods considered, wherein the linguistic term set is, by default, an array with identical distances between the adjacent elements.

(4) The transformation from HFLTSs to intervals may lead to information distortion and/or loss, and the final ranking obtained by the other methods considered is thereby not conclusive. By contrast, the directional Hausdorff distance regards the linguistic terms as discrete data, and also takes into account the differences between adjacent linguistic terms.

(5) The calculations required for the proposed approach are relatively straightforward and the burden of calculation can be greatly decreased with the help of the proven propositions and properties.

(6) HFLTSs can permit the value of a linguistic evaluation for an alternative under a given criterion to be several consecutive linguistic terms rather than a single term, which is particularly helpful for conditions where hesitance in evaluation occurs due to uncertain information and/or incomplete knowledge. In view of this, the possible applications of the proposed approach include new product development, SCM, emergency management evaluation, service quality evaluation, performance evaluation, and other similar linguistic decision-making problems that require the hesitance in the original evaluation information to be retained using HFLTSs. The strengths of the proposed approach over existing MCDM methods with HFLTSs have been analyzed, and the results reflect the enhanced applicability of our proposed method.

6. Conclusions and further study

In this paper, an outranking approach for solving MCDM problems with HFLTSs has been proposed on the basis of linguistic scale functions, the directional Hausdorff distance of HFLTSs, and the ELECTRE method. Firstly, the directional Hausdorff distance of HFLTSs was proposed, which fixes the order or sequence of the operands, and employs linguistic scale functions to retain the merits of linguistic evaluation. The illustration given in Section 3 demonstrated that the proposed distance can clearly identify the differences among HFLTSs, and avoid the hidden problems that hamper existing methods in the context of HFLTSs. Secondly, the dominance relations were defined based on the proposed distance, and have been proven to be comprehensive and feasible. The calculations involved with identifying the dominance relations of alternatives are relatively manageable with the help of valid propositions and properties. Thirdly, the integration of the directional Hausdorff distance and the ELECTRE method was tested using a numerical example, and was further validated through sensitivity and comparative analyses. The accuracy and reliability of the proposed approach was demonstrated by comparisons with other recently proposed methods/models employing HFLTSs.

In future research, the proposed approach will be applied to more practical cases to demonstrate its efficiency and effectiveness. Moreover, the extension of HFLTSs based on multi-hesitant fuzzy sets will be explored and the related models based on aggregation operators, distance measures, or dominance relations will also be developed. In particular, the application of the directional Hausdorff distance to the extension of HFLTSs will be studied.

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