The Optimal Insurance Policy for the General Fixed Cost of Handling an Indemnity under Rank-Dependent Expected Utility

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Abstract
In this article, based on Bernard et al’s research, we focus on the Pareto optimal insurance design with the insured’s Rank-Dependent Expected Utility (RDEU). Compared with their previous work, our novelties are the more general fixed cost function of the insurer and the discussion of adverse selection and moral hazard. In particular, Bernard et al only consider the case in which the fixed cost function of handling an indemnity is the linear function. However, the fixed cost function is not just linear functions in real insurance market. So, we explore more general fixed cost function including both the linear and nonlinear functions. On the other hand, we consider adverse selection and moral hazard which are involved by Bernard et al. Leading adverse selection and moral hazard into our research makes our results more practical and meaningful. Moreover, we provide an insight into the sensitivity of an optimal solution for the insured’s initial wealth and the parameters related to the fixed cost function of handling an indemnity. We further compare the two different utility functions of the insured in terms of influence of optimal policy analysis.

Key words and phrases: Insurance policy, Praeto optimal solution, Prospect theory, Rand-Dependant Expected Utility.

1 Introduction
In an economic system, both companies and individuals have to face uncertainty in the future. In such a situation, the redistribution of risk among the firms and individuals is facilitated. Usually, persons use insurance policies for reallocating risk. In the late 1960s, J. Mossin, V. Smith and J. Gould initiated

*The article is supported by National Natural Science Foundation of China (71201051) the State Scholarship Fund (Grant No. 2014BQ11) and Young Talents Training Plan of Hunan Normal University(2014YX04), Philosophical and Social Science Fund of Hunan(No.14YBA264) and Social Science Found of China (12&ZD050,10BJL024)
the insurance decision analysis [21, 25, 16], where they focused on relationally purchasing insurance from the viewpoint of an individual who has to face a specific risk, given his preference structure and wealth level. Their appealing research builds on the assumption that the insurance policy is exogenously specified. However, K. Borch [6] and K. Arrow [2, 3] object to Mossin’s opinion and firmly believe that the insurance contract is not exogenous. Sappington [26] provides a complete justification of principal-agent modeling for the insurance problem. He sets up the basic framework and the key model of game relationship between the principal and the agent. Furthermore, he builds a monitoring mechanism based on the prisoner’s dilemma and builds an incentive mechanism by competition.

In contrast with Sappington’s principal-agent model, the insured corresponds to the agent and the insurer corresponds to the principal. Actually, our problem is different from his. In his model, the agent first decides whether to accept or reject the contract. After the agent signs the contract, he can change his own expected utility by choosing the efficient level of effort. In our model, the agent first determines whether to accept or reject the contract too. But, once the agent agrees to and signs the insurance contract, he can not change his own value function which is only determined by the random loss \( x \).

Although there are obvious differences between our models, our research is consistent with his basic framework. We can make a close link with his research. 1) As an agent, the insured should pay the upfront premium \( Q \) for sharing the loss with the insurer. This is as if the insured has to pay the “franchise fee” \( k \) for the right to work for the principal; 2) In his research, the agent will accept the contract offered by the principal if and only if the subsequent self-interested behavior under the terms of the contract provides the agent with a level of expected utility that exceeds his reservation level, \( \bar{U} \). The insured’s criterion of buying or not buying the insurance is

\[
U^R(w_0 - x - Q + I(x)) \geq U(w_0 - x),
\]

where \( U(w_0 - x) \) is equal to \( \bar{U} \) in his paper; 3) In his article, this contract promises payments, \( P \), to the agent that are precisely the principal’s valuation of the agent’s performance less some fixed constant \( k \). Formally, \( P(X) = V(X) - k \). This is consistent with the insurer’s safety loading. In particular, the insurer will price the indemnity in such a way that

\[
Q \geq E[I(x) + C(I(x))].
\]

In a competitive insurance market, we can understand \( E[I(x) + C(I(x))] \) as the minimum price of the indemnity \( I(X) \) for a risk-neutral insurer to participate in the business.

Beyond insurance policy analysis, expected utility theory (EUT) has an underlying assumption that the decision-maker is rational and uniformly risk
averse, only considering the objective probability rather than the subjective probability [22]. In reality, however, various decision makers’ behaviors deviate from the implications of expected utility. Substantial experimental and empirical evidence identifies that expected utility theory is incompatible with human observed behavior. The abundant paradoxes lead to the development of a more realistic theory. It is dominant in prominent paradigms that Kahneman and Tversky propose prospect theory (PT) [27]. Later, they develop their prospect theory to cumulated prospect theory (CPT) since CPT is consistent with the first-order stochastic dominance [28]. In the context of CPT, they incorporate human emotion into their investigation.

The incentive for the optimal decision, especially to the optimal insurance policy, is extensively accepted. Quiggin [23] revolutionized the classical expected utility theory (EUT) by rank dependent utility (RDU). His framework provides the theoretical background for the essentiality of design of insurance contracts. In 2000, Chateauneuf etc. [8] presented the Choquet expected utility framework and gave some results in the RDU framework as a special case. The Pareto efficient insurance contracts under RDU is illustrated in Dana and Carlier’s work [9, 10]. Dana and Scarsini [14] mention optimal risk sharing with background risk and briefly identify the case of RDU. Zhou etc. [17, 18] emphasize the optimal insurance contract when the distortion is convex. Subsequently, Carlier and Dana [11] investigate two-persons efficient risk-sharing problems about concave law-invariant utilities and give a characterized result which is valid for any RDU. Carlier and Dana [12] derive the optimal contingent claim for two significant decision frameworks, the RDU and the CP. However, these papers don’t obtain the explicit solution while Bernard et al. [19] do so, when the utility function of the insured is concave.

Bernard et al’s work inspires us to explore the optimal insurance design under Rank-Dependent Expected Utility. Compared with their research, our novelties are the generalization of the cost function and the discussion of adverse selection and moral hazard.

In detail, the key contribution of Bernard et al is to get the explicit solution of the optimal insurance contract. But this result implicitly relies on the concavity of the extreme point function $H_\lambda(z)$ (the equation (8) on page 15). However, $H_\lambda(z)$ is concave, only when the fixed cost of handling the indemnity $C(I(x))$ is a linear function of $I$. So they only discuss the case in which the fixed cost function is linear, i.e. $C(I(x)) = \rho I(x)$. However, in a real insurance market, the different insurers have various cost functions including linear functions and other nonlinear functions. So, we pay attention to the more general cost functions to make the results more practical. Generalizing the fixed cost of handling the indemnity brings us a divers obstacle from Bernard and Zhou’s work: we are not sure about the convexity or concavity of the extreme point function $J(z)$ (the equation (3.2) in section 3). In other words, this general cost
function results in uncertain monotonicity of the extreme point function \( J(z) \) which is different from the proposition of the extreme point function \( H_\lambda(z) \) in Bernard and Zhou’s article. Further, this uncertainty of monotonicity makes us have to discuss the different monotonic intervals of \( J(z) \) and five different relationships between \( J(z), F^{-1}_x(z) \) and \( k(z) \) (see Figure 2) while Bernard and Zhou only need to consider one monotonic interval of \( J(z) \) and one relationship between \( J(z), F^{-1}_x(z) \) and \( k(z) \) (see Figure 2). Although the various relationships lead to complicated discussion, these relationships make our novel results more general. In fact, Case 1 in Figure 2 coincides with Bernard and Zhou’s Figure 2. Through discussing five different cases and the different monotonic intervals, we attain the explicit solution which is the general result applying to both the linear cost functions and nonlinear functions.

Another novel contribution is the discussion about two critical issues which are adverse selection and moral hazard while Bernard and Zhou do not involve them. In particular, we use the following bonus-malus system

\[
Q = \begin{cases} 
\delta Q_0 & \text{if no accident occurred in the previous period} \\
\gamma Q_0 & \text{if an accident occurred in the previous period}
\end{cases}
\]

to determine the premium \( Q \), where \( Q_0 \) is the premium in the previous period. We can estimate \( \delta \) and \( \gamma \) by empirical data to decide the premium \( Q \). If \( Q \) is the premium in the first period and we have no empirical data and \( Q_0 \), we have to decide \( Q \) relying on the indexes associated with \( Q \), such as, age, gender, occupation, and so on. Based on this fixed \( Q \), we research the optimal problem for indemnity \( I(x) \) under Rank-Dependent Expected Utility. Although we only offer a brief thought of how to decide \( Q \) and not carefully research it, this significant thought not only makes close link between these critical issues and our research but also offers the basic framework for further research.

Recently, Dhiab investigates the demand for insurance under the non-expected utility theory [15]. He applies Rank Dependent Expected Utility (RDEU) to the insurance contract. In his insurance context, agents behave not only according to their probability distribution but also according to their attitude towards risk.

Although Ben Dhiab’s research is similar to mine, there are obvious differences between our research. The important difference is that we research the insurance problem from different angles. In particular, he researches the optimal insurance contract from a insured’s (or agent’s) point of view, so he only needs to maximize the RDEU of the insured without considering the utility function of a insurer and relative restrictive conditions. However, we study the optimal insurance policy from a insurer’s point of view. Thus, we set up the optimal insurance policy subjecting to the restrictive condition associated with the utility function of the insurer. Meanwhile, the insurer’s utility functions and the optimal solutions change due to different insurers’ cost functions.
Another difference is that he doesn’t present the quantified relationship between the indemnity function $I(x)$ and the random loss $x$ when the optimal insurance policies are partial insurance and over-insurance. However, I reveal the accurately quantified relationship between the indemnity function $I(x)$ and the random loss $x$ in Proposition 4.3.

Furthermore, there are two main differences on the technological detail. 1) He supposes the probability weighting function (probability distortions) is always concave or convex. But, the S-shape probability weighting function is more reasonable than the concave or convex probability weighting function, because Kahneman and Tversky used sufficient experiments and evidences to extensively demonstrate that not only people often overweight low-probability and certain outcomes but also the individual’s attitude to the risk always changes. So, we employ the S-shape probability weighting function; 2) Dhiab only discusses two states of the nature which are the loss $x$ with probability $p$ and no-loss with probability $1 - p$. We explore the more general and complicated case in which the loss $x$ is a random variable on $[0, w_0]$. The general $x$ means that we can not attain the optimal solution by directly calculating and simply discussing as Dhiab do. So we have to use quantile function for solving the optimal problem.

This paper is organized as follows: In section 2, we set up critical models. Section 3 explores the optimal solutions. In section 4, numerical analysis is performed. Section 5 summarizes the conclusions. In section 6, we introduce the further research. The paper ends with an Appendix containing the proofs.

2 The model

In this section, we focus on the Pareto optimal insurance contract where the insured has Rank-Dependent Expected Utility [23] preference.

2.1 The basic setting

2.1.1 The original insurance problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. An economic agent, called a policyholder or an insured, is endowed with the initial wealth $w_0$ and has to face the a non-negative random loss $x$ with support in $[0, w_0]$. The initial wealth of the insurer is $W_0$. The loss $x$ is a random variable with the probability density function $p(x)$. The insured should pay the upfront premium $Q$ for sharing the loss with the insurer. If the insured stands to the loss $x$, the insurer will pay out $I(x)$. We treat $I(x)$ as the indemnity function of the loss or coverage function. The indemnity principle compensates for the insured’s loss when an accident
happens. According to this rule, the policyholder cannot collect more money than his actual loss. Hence, we assume $0 \leq I(x) \leq x$. The constraint condition implies that if there is no loss there will be no reimbursement. As for the cost of the insurer, we state that it includes two part, the administrative expenses or other expenses and deadweight loss related to the insured and the insurer. So, we suppose the cost of the insurer consists of fixed and variable components, which depends on the size of the insurance payment. We denote the cost by $C(I(x))$. And assume that $C(0) = c \geq 0, C'(I) \geq 0$ and $C''(I) = a \geq 0$.

We further suppose that the utility function of the insurer is $v(\cdot)$. It is easy to see that the insurer’s final wealth is $W_0 + Q - I(x) - C(I(x))$. Here, we suppose $v'(\cdot) > 0$ and $v''(\cdot) = 0$.

Consider there exist finite states of the world: the wealth level $w_1, w_2, ... , w_n$. And further assume that you can assign probabilities to each of these outcomes. You are fairly optimistic about your future, so you assign a probability $p_1$ to the wealth level $w_1$, $p_2$ to the wealth level $w_2$, ..., $p_n$ to the wealth level $w_n$. \[ \sum_{i=1}^{n} p_i = 1. \] We call the the series of wealth outcomes a prospect $P_n$ and represent this situation using the following convenient format:

$$ P_n(p_1, w_1, p_2, w_2, ... , p_{n-1}, w_{n-1}, w_n). $$

The value function $V$ represents the preferences in the EU model. For a prospect $P_n$, preferences can be represented by a functional such that:

$$ V(P_n) = \sum_{i=1}^{n} p_i v(w_i), $$

where $v : \mathbb{R}^+ \to \mathbb{R}$ is an utility function, strictly increasing and unique up to an affine positive transformation function. The value function $V$ is linear in probabilities.

It is necessary to mention that when $v''(\cdot) = 0$, namely, the insurer is risk-neutral, the utility function $v(\cdot)$ has an important proposition that the expected utility function equals the utility of the expected value, i.e. $Ev(P) = v(E(P))$, where $P$ is a prospect (see [1]). We suppose, for simplicity, that there are only two states of the world: low wealth $w_l$ and high wealth $w_h$. And further assume that you can assign probabilities to each of these outcomes. You are fairly optimistic about your future, so you assign a probability $p_1$ to low wealth $w_l$ and a probability $1 - p_1$ to high wealth $w_h$. This situation can be represented by a prospect $P$ as the following convenient format:

$$ P(p_1, w_l, w_h). $$

Then, it is easy to write

$$ Ev(P) = p_1 v(w_l) + (1 - p_1) v(w_h) $$
Utility Function for a Risk-neutral insurer

![Utility Function Diagram]

and

\[ v(E(P)) = v(p_1w_1 + (1-p_1)w_h). \]

Due to a risk-neutral utility function, from above figure, we can attain \( Ev(P) = v(E(P)) \).

Generally, for a prospect \( P_n(p_1, w_1, p_2, w_2, \ldots, p_{n-1}, w_{n-1}, w_n) \),
a similar result is attained

\[ V(P_n) = Ev(P_n) = \sum_{i=1}^{n} p_i v(w_i) = v\left(\sum_{i=1}^{n} p_i w_i\right) = v(E(P_n)). \]

In the present article, because the random loss \( x \) has finite number of values,

\[ V(W_0 + Q - I(x) - C(I(x))) = E\left[v(W_0 + Q - I(x) - C(I(x)))\right] = v(E[W_0 + Q - I(x) - C(I(x))]) \quad (2.1) \]

holds.

Meanwhile, our research relies on another essential assumption that the utility function of the insured is \( u(\cdot) \) satisfying \( u'(\cdot) \geq 0 \) and \( u''(\cdot) \leq 0. \)

2.1.2 Rank-Dependent Expected Utility

Because the insured prefers Rank-Dependent Expected Utility, we use the decision weights instead of using simple probabilities as expected utility (see [1]).
Definition 2.1. Let $F_w(\cdot)$ be the cumulative distribution function (CDF) of a random variable $w$. The probability distortions are denoted by $G$. We define the probability weight function (distortions) $G : [0,1] \rightarrow [0,1]$:

$$G(F_w(y)) = \frac{F_w^\gamma(y)}{(F_w^\gamma(y) + (1 - F_w^\gamma(y)))^{1/\gamma}}, \quad \text{with} \quad 0.28 < \gamma < 1$$

Definition 2.2. Define the RDEU of the insured as (see [7] and [8]):

$$U^R(w) = \int_0^{+\infty} u(y)d(G(1 - F_w(y))),$$

where $w$ is the final wealth of the insured and $U^R(w)$ is the Choquet integral of $u(\cdot)$.

Since the insured’s final wealth is $w = w_0 - x - Q + I(x)$, the RDEU of the insured is as follows:

$$U^R(w_0 - x - Q + I(x)) = \int_0^{+\infty} u(y)d(G(1 - F_{w_0-x-Q+I(x)}(y))),$$

where $F_{w_0-x-Q+I(x)}$ is the cumulative distribution function (CDF) of $w$ which is the function of the random variable $x$. Thus, at this time, $F_w$ is replaced by $F_{w_0-x-Q+I(x)}$.

2.1.3 The optimal insurance problem

In order to attract insureds in the business competition, the insurer has to make the insureds gain as much profit as possible. To explicitly explain the influence of the competition on the optimal problem, we take an example. There two insurers, $A$ and $B$. Because of the different insure designs of $A$ and $B$, the same insured has different RDEUs for $A$ and $B$. RDEUs are respectively written as $U^R_A$ and $U^R_B$. If $U^R_A > U^R_B$, the insured prefers the insurer $A$ rather than $B$. Otherwise, the insured is willing to choose the insurer $B$ but not $A$. Since there exit many insurers in a limited insurance market, each insurer tries his best to maximize the RDEU of the insured by designing the optimal insurance contract for attracting more clients.

That is, the insurer will find the optimal solution of $U^R$

$$\max_{Q,I(x)} U^R(w_0 - x - Q + I(x)) = \int_0^{+\infty} u(y)d(G(1 - F_{w_0-x-Q+I(x)}(y))).$$

On the other hand, as shown in Raviv [24], a necessary condition for the insurer to offer such a policy is

$$E[v(W_0 + Q - I(x) - C(I(x)))] \geq v(W_0).$$
Considering the impact of adverse selection and moral hazard on the insurance contract \([4, 13, 20]\), we can use the following bonus-malus system

\[
\begin{align*}
Q &= \delta Q_0 \text{ if no accident occurred in the previous period} \\
&= \gamma Q_0 \text{ if an accident occurred in the previous period}
\end{align*}
\]

to determine the premium \(Q\), where \(Q_0\) is the premium in the previous period. We can estimate \(\delta\) and \(\gamma\) by empirical data to decide the premium \(Q\). If \(Q\) is the premium in the first period and we have no empirical data and \(Q_0\), we have to decide \(Q\) relying on the indexes associated with \(Q\), such as, age, gender, occupation and so on. Based on this fixed \(Q\), we research the optimal problem of indemnity \(I(x)\).

We denote by \(I\) the set of all indemnity functions, i.e, \(I := \{I(\cdot)|0 \leq I(x) \leq x, \forall x \in [0, w_0]\}\). The optimal problem is as follows:

**Model 2.1.**

\[
\begin{align*}
\text{Max}_{I(\cdot) \in I} & \left(U^R(w_0 - x - Q + I(x)) = \int_0^{+\infty} u(y)d(-G(1 - F_{w_0 - x - Q + I(x)}(y))) \right) \\
\text{Subject to} & \\
Q & \geq E[I(x) + C(I(x))].
\end{align*}
\]

As Raviv shown, if the insurer is risky-neutral, the insurer will price the indemnity in such a way that

\[
Q \geq E[I(x) + C(I(x))]. \tag{2.2}
\]

In a competitive insurance market, we can understand \(E[I(x) + C(I(x))]\) as the minimum price of the indemnity \(I(X)\) for a risk-neutral insurer to participate in the business (typically referred as the insurer’s safety loading).

Noticing the independent variable \(I(x) \in I\) of the function \(C(\cdot)\) is bounded and the function \(C(\cdot)\) is continuous, we can say \(C(\cdot)\) is bounded. So, there exits finite \(Q\) satisfying the restrict condition (2.2).

So, the Model 2.1 can be rewritten as

**Model 2.2.**

\[
\begin{align*}
\text{Max}_{I(\cdot) \in I} & \left(U^R(w_0 - x - Q + I(x)) = \int_0^{+\infty} u(y)d(-G(1 - F_{w_0 - x - Q + I(x)}(y))) \right) \\
\text{Subject to} & \\
Q & \geq E[I(x) + C(I(x))].
\end{align*}
\]
Let $R(x) = x - I(x)$. $R(x)$ is the part of loss shared by the insured and is the so-called retention function. Let
\[ R = \{ R(\cdot) \mid 0 \leq R(x) \leq x, \forall x \in [0, w_0] \}. \]

Then, the above model becomes

**Model 2.3.**
\[
\text{Max}_{R(x) \in R} U^R(w_0 - Q - R(x)) = \int_{0}^{+\infty} u(y)d(-G(1 - F_{w_0-Q-R(x)}(y))).
\]

Subject to
\[
Q \geq E[x - R(x) + C(x - R(x))].
\]

### 2.2 The objective model

Before we analyze this model, we introduce some indispensable assumption and lemmas.

**Assumption 2.4.** (see [5]) The loss $x$ has no atom, i.e. the cumulative distribution function $F(x)$ of $x$ is the continuous function. Accordingly, its quantile function $F^{-1}_x : (0, 1) \rightarrow \mathbb{R}_+$ is continuous.

**Assumption 2.5.** (see [1]) The probability weighting function $G(\cdot) : [0, 1] \rightarrow [0, 1]$ and satisfies $G(0) = 0, G(1) = 1, G'(\cdot) \geq 0$ and
\[
\begin{aligned}
G''(z) & \leq 0 \quad \text{if } z \in [0, z_0] \\
G''(z) & \geq 0 \quad \text{if } z \in (z_0, 1].
\end{aligned}
\]

**Lemma 2.6.** Suppose $A$ is a constant. $g(x)$ is a random variable with the probability density function $f(\cdot)$ and the cumulative distribution function $F_{g(x)}(\cdot)$. The according quantile function $F^{-1}_{A-g(x)}$ satisfies that
\[
F^{-1}_{A-g(x)}(z) = A - F^{-1}_{g(x)}(1 - z).
\]

The proof is seen in Appendix 7.1.

**Lemma 2.7.** (see [5]) With Assumption 2.4, if feasible solution $R(\cdot)$ to Model 2.3, then $\bar{R}(x) = F^{-1}_{R(x)}(F_x(x))$ is also feasible with respect to Model 2.3 and $\bar{R}(x)$ has the same the law as $R(x)$.

From Lemma 2.6 and Lemma 2.7, we can write the model as follows:
Model 2.8.

\[ \text{Max}_{k(z)} \int_0^1 u(w_0 - Q - k(z))G'(z)dz. \]

Subject to

\[ 0 \leq k(z) \leq F^{-1}_X(z), 0 < z < 1, \]
\[ k \in \mathcal{K} \]

and

\[ Q \geq E[x - k(z) + C(x - k(z))]. \]

Here, \( k(z) \) is a quantile function. \( \mathcal{K} \) represents the set of all quantile functions. That is, \( \mathcal{K} = \{ k : (0, 1) \rightarrow \mathbb{R} | k(\cdot) \text{ is non-decreasing and left-continuous} \} \).

Detailedly, form Lemma 2.6, we can transform RDEU of the insured into

\[
U^R(w_0 - Q - R(x)) = \int_0^{\infty} u(x)d(-G(1 - F_{w_0-Q-R(x)}(x))
\]
\[
= \int_0^1 u(F^{-1}_{w_0-Q-R(x)}(z))G'(1 - z)dz
\]
\[
= \int_0^1 u(w_0 - Q - F^{-1}_{R(x)}(1 - z))G'(1 - z)dz
\]
\[
= -\int_1^0 u(w_0 - Q - F^{-1}_{R(x)}(z))G'(z)dz
\]
\[
= \int_0^1 u(w_0 - Q - F^{-1}_{R(x)}(z))G'(z)dz.
\]

Let

\[ k(z) = F^{-1}_{R(x)}(z) = F^{-1}_{R(x)}(F_x(x)) = \tilde{R}(x). \]

Recalling Lemma 2.7, we can demonstrate that \( k(z) \) satisfies the constraints in Model 2.8.

With Lagrange dual method, we can obtain the auxiliary problem:

Model 2.9.

\[ \text{Max}_{k(z)} U_\lambda(k(z), z) = \int_0^1 u(w_0 - Q - k(z))G'(z) + \lambda k(z) - \lambda C(F^{-1}_X(z) - k(z))dz + \lambda(Q - Ex) \]

Subject to

\[ 0 \leq k(z) \leq F^{-1}_X(z), 0 < z < 1, \]
\[ k \in \mathcal{K}. \]
3 The results

3.1 The piecewise optimal solution

Now, we only consider the maximal value of
\[ u(w_0 - Q - k(z))G'(z) + \lambda k(z) - \lambda C(F_x^{-1}(z) - k(z)). \]

Set
\[
\frac{\partial}{\partial k(z)} [u(w_0 - Q - k(z))G'(z) + \lambda k(z) - \lambda C(F_x^{-1}(z) - k(z))]
\[
= -u'(w_0 - Q - k(z))G'(z) + \lambda (1 + C'(F_x^{-1}(z) - k(z)))
\[
= 0.
\] (3.1)

observing Taylor expansions of \( u'(\cdot) \) and \( C'(\cdot) \), we have
\[
u'(w_0 - Q - k(z)) \approx u'(w_0 - Q) - u''(w_0 - Q)k(z)
\]
and
\[
C'(F_x^{-1}(z) - k(z)) \approx C'(F_x^{-1}(z)) - C''(F_x^{-1}(z))k(z).
\]

The above results are applied to (3.1), it is showed that
\[-(u'(w_0 - Q) - u''(w_0 - Q)k(z))G'(z) + \lambda (1 + C'(F_x^{-1}(z)) - C''(F_x^{-1}(z))k(z)) \approx 0.\]

Then, not considering the other constraints in the Model 2.9, we can get the approximate optimal solution
\[
J(z) = \frac{\lambda (1 + C'(F_x^{-1}(z))) - u'(w_0 - Q)G'(z)}{\lambda C''(F_x^{-1}(z)) - G''(z)w''(w_0 - Q)}.
\] (3.2)

In order to satisfy the first constraint in the Model 2.9, we transform \( k(z) \) into
\[ K(z) = \max\{0, \min\{J(z), F_x^{-1}(z)\}\}, 0 \leq z \leq 1. \]

Noting \( u'(\cdot) \geq 0, u''(\cdot) \leq 0 \) and the Assumption 2.5, we can find that \( J(z) \) is non-decreasing on \([0, z_0]\) which satisfies the proposition of the quantile function. Hence, \( K(z) \) satisfies the second constraint in the Model 2.9 on \([0, z_0]\). However, it is regretful that \( J(z) \) is not non-decreasing on \((z_0, 1)\) and we can’t make sure that \( K(z) \) non-decreasing on \((z_0, 1)\). So, we hope to achieve a new solution suitable for all constraints in Model 2.9 from \( K(z) \). We only pay attention to the case of \( J(z) \) in \((z_0, 1)\). Since in \((z_0, 1]\), \( J(z) \) is not always decreasing
Figure 1
Figure 2: Different cases
or increasing, we divide \((z_0, 1]\) by monotonicity. If \((z_0, 1]\) has \(N\) an increasing ranges and \(M\) decreasing ranges, we denote increasing range by \((d_i, e_i] (i = 0, 1, \ldots , N - 1)\) and a decreasing range by \((e_j, d_{j+1}] (j = 0, \ldots , M - 1)\) (see Figure 1).

In the first case in Figure 1, we can safely claim that \(J(z)\) is non-decreasing on \([0, z_0]\) and \([z_0, e_0]\) \(([z_0, e_0]\) is equivalent to \([d_0, e_0]\)). So, we can look \([0, z_0]\) and \([z_0, e_0]\) as one whole, namely, we can deal with \(J(z)\) on \([0, e_0]\) similar to on \([0, z_0]\). At this time, the rest \([e_0, 1]\), \(J(z)\) first decreases on \([e_0, d_1]\) and then increases on \([d_1, d_2]\), which is the same as the second case in Figure 1. \(J(z)\) first decreases and then increases in \((e_0, 1]\) and is similar to the second case in Figure 1.

We denote the intersection point of \(F^{-1}_x(z)\) and \(J(z)\) in \((e_0, d_1]\) by \(s_1\). And write the intersection point of \(k(z)\) and \(J(z)\) on \([s_1, d_1]\) as \(s_2\). Now, we discuss each possible cases (see Figure 2):

**Case 1:** There exist \(s_1\) and \(s_2\) in \((e_0, d_1]\). At this time, we denote the intersection point of \(k(z)\) and \(F^{-1}_x(z)\) by \(m_0\). Denote the intersection point of the horizontal line through \(m_0\) and \(J(z)\) on \([e_0, d_1]\) by \(l_0\) and on \([d_1, e_1]\) by \(n_0\);

**Case 2:** There exists \(s_1\) and doesn't exist \(s_2\) in \((e_0, d_1]\). In this case, we denote the intersection point of \(k(z)\) (here \(k(z)\) is equivalent to \(F^{-1}_x(z)\)) and the horizontal line through \(d_1\) by \(m_0\). And, set \(l_0 = n_0 = d_1\);

**Case 3:** \(F^{-1}_x(z) \geq J(z)\) and there does not exist a intersection of \(J(z)\) and \(k(z)\) on \([z_0, d_1]\). In other words, there does not exist \(s_1\). But we can look \(e_0\) as \(s_1\), then this case is similar to the first case. We denote the intersection of \(J(z)\) and \(k(z)\) on \([z_0, d_1]\) by \(s_2\). And, we denote the intersection point of \(k(z)\) and the horizontal line through \(s_2\) by \(m_0\). It is valuable noticed that \(k(z)\) is \(J(z)\) at this time. In fact, \(m_0\) is the intersection point of \(J(z)\) and the horizontal line through \(s_2\). Besides, denote the intersection point of the horizontal line through \(s_2\) and \(J(z)\) on \([d_1, e_1]\) by \(n_0\) and \(l_0 = s_2\);

**Case 4:** \(F^{-1}_x(z_0) \geq J(z_0)\) and there doesn’t exist a intersection of \(J(z)\) and \(k(z)\) on \([z_0, d_1]\). At this time, we denote the intersection point of \(k(z)\) which is \(J(z)\) in this case and the horizontal line through \(d_1\) by \(m_0\). And, let \(l_0 = n_0 = d_1\);

**Case 5:** \(F^{-1}_x(d_1) < J(d_1)\). We denote the intersection of \(F^{-1}_x(z)\) and the vertical line through \(d_1\) by \(m_0\). And, let \(n_0 = l_0 = m_0\).

Let

\[K_{m_0}(z) = K(z)I_{0 < z < m_0} + K(m_0)I_{n_0 > z > m_0}, 0 < z < n_0.\]

Then, we can achieve an important lemma as follows.

**Lemma 3.1.** For and feasible solution \(k(z)\) of Model 2.9, \(K_{m_0}(z)\) satisfies

i) \(U_\lambda(k(z), z) \leq U_\lambda(K_{m_0}(z), z), 0 < z < n_0;\)

ii) The equality holds if and only if \(k(z) = K_{m_0}(z), 0 < z < n_0.\)

The proof is seen in Appendix 7.2.
3.2 The global optimal insurance design

For the range \([n_0, e_2]\), we can discuss similar to the range \([0, e_1]\). In more detail, we take \([n_0, e_1]\), \([e_1, d_2]\) and \([d_2, e_2]\) as \([0, e_0]\), \([e_0, d_1]\) and \([d_1, e_2]\) respectively. We can obtain \(K_m(z)\) on \([n_0, n_1]\) similar to \(K_m(Z)\) on \([0, n_0]\). Generally, we can achieve \(K_m(z)\) on \([n_{i-1}, n_i]\) similar to \(K_m(Z)\) on \([0, n_0]\) \((i = 1, 2, \ldots, N - 1 \text{ or } N - 2)\).

Because we note that \(k(z)\) must be the left-continuous, we are particularly concerned with whether the value of \(K_m(z)\) equals \(K_m(z)\) which directly decides whether \(k(z)\) is the left-continuous. That is, does the following the equality

\[K_m(n_0) = K_m(n_0)\]  
(3.3)

hold?

In the first four cases, we can find \(F^{-1}_x(n_0) \geq J(n_0)\). So, \(K(n_0) = J(n_0)\) and \(K_m(n_0) = J(n_0) = K_m(n_0)\). In the last case, we have \(F^{-1}_x(n_0) \leq J(n_0)\). So, \(K(n_0) = F^{-1}_x(n_0)\) and \(K_m(n_0) = F^{-1}_x(n_0) = K(n_0) = K_m(n_0)\).

Hence, the function \(K_m(n_0) = K_m(n) = K_m(z)\) on \([0, n_1]\) is non-decreasing and left-continuous. Generally, the range \([n_{i-1}, n_i]\) is researched similarly. Ultimately, noticing (3.3), we can indicate the non-decreasing and left-continuous function \(K(z)\) on \([0, 1]\) as follows:

\[
K(z) = \begin{cases} 
K_m(n_0)I_{n_0 \leq z \leq n_0} + K_{m_1}(z)I_{n_0 < z \leq n_1} + \ldots + K_{m_{N-1}}(z)I_{n_{N-2} < z \leq 1} & \text{if } M = N \\
K_m(n_0)I_{n_0 \leq z \leq n_0} + \ldots + K_{m_{N-2}}(z)I_{n_{N-3} < z \leq n_{N-2}} + K(z)I_{n_{N-2} < z \leq 1} & \text{if } M < N.
\end{cases}
\]

(3.4)

Now, we can safely come to the important conclusion that

**Theorem 3.2.** For and feasible solution \(k(z)\) of Model 2.9, \(K(z)\) satisfies

i) \(U_\lambda(k(z), z) \leq U_\lambda(K(z), z), 0 < z < 1;\)

ii) The equality holds if and only if \(k(z) = K(z), 0 < z < 1.\)

The proof is seen in Appendix 7.3.

From Theorem 3.2, we can reduce Model 2.9 to the auxiliary problem with Lagrange dual method:

**Model 3.3.**

\[
\begin{align*}
\max_{k(z)} U_\lambda(k(z), z) &= \int_0^1 u(w_0 - Q - k(z))G'(z) + \lambda k(z) \\
&- \lambda C(F^{-1}_x(z) - k(z))dz + \lambda(Q - Ex),
\end{align*}
\]

where \(k(z)\) is satisfied to (3.4).

Now, we desire to solve the original optimal problem through \(U_\lambda(K(z), z).\) We can attain the key proposition as follows.
Proposition 3.4. The optimal solution of Model 2.1 is as follows:

\[ I^*(x) = x - K(n_i)I_{n_{i+1} > z > n_i}, F_X^{-1}(n_{i+1}) > x > F_X^{-1}(n_i), \]

where

\[ K(n_i)I_{n_{i+1} > z > n_i} = K_{n_i}(z) = K_{m_{i+1}}(z) \]

The proof is seen in Appendix 7.4.

Remark 3.1. Comparing the conclusion with the result in the substance of Bernard and Zhou’s paper, we can not help asking: why does our conclusion seem simpler than Zhou and Bernard’s one, while our fixed cost of handling an indemnity is more general than Bernard’s research? It is because we subtly make \( J(z) \) become segments, \([n_i, n_{i+1}]\) before solving this optimal problem which delicately simplifies the form of solution.

4 Numerical Analysis

In this section, we provide an insight into the optimal result with a numerical simulation. We suppose the loss \( x \) satisfies truncated Pareto distribution on \([0, w_0]\). That is, the density function is

\[
f(x) = \begin{cases} 
\frac{1}{1 - 2^{-\alpha}} \alpha \frac{w_0}{2w_0 - x}^{\alpha+1} & 0 \leq x \leq w_0 \\
0 & \text{others.}
\end{cases}
\]

Accordingly, its distribution function is

\[
F(x) = \int_0^x f(t) dt \\
= \int_0^x \frac{1}{1 - 2^{-\alpha}} \frac{\alpha w_0}{2w_0 - t}^{\alpha+1} dt \\
= \frac{1}{1 - 2^{-\alpha}} w_0^{\alpha} (2w_0 - t)^{-\alpha} \bigg|_0^x \\
= \frac{1}{1 - 2^{-\alpha}} w_0^{\alpha} [(2w_0 - x)^{-\alpha} - (2w_0)^{-\alpha}]
\]

As for the insured’s utility function, we discuss two different kinds of utility functions. They are separately

\[ u(w) = \frac{1}{1 - \beta} w^{1-\beta} \]

and

\[ u(w) = \log w. \]
We let the fixed cost function of handling an indemnity be
\[ C(I) = \frac{a}{2}I^2 + bI + c, \quad a, b \geq 0. \]
The probability weighting function (see [28]) is
\[ G(z) = \frac{z^\gamma}{(z^\gamma + (1 - z)^\gamma)^{1/\gamma}}, \quad \gamma \approx 0.61. \]
Then,
\[ G'(z) = \frac{\gamma z^{\gamma-1}(z^\gamma + (1 - z)^\gamma)^{1/\gamma} - z^{\gamma/2}(z^\gamma + (1 - z)^\gamma)^{1/\gamma} - (z^\gamma - \gamma(1 - z)^{\gamma-1})}{(z^\gamma + (1 - z)^\gamma)^{2/\gamma}}, \]
\[ \gamma \approx 0.61. \]

In the body of this section, we mainly focus on the optimal solutions’ sensitivity to the fixed cost function of handling an indemnity and the initial wealth of the insured. Here, we choose the utility function of the insured \( u(w) = \log w \).

Firstly, we consider that different parameters in the fixed cost function of an indemnity impact on an optimal insurance contract. We fix the initial wealth of the insured \( w_0 = 7 \), the premium \( Q = 0.3w_0 = 2.1 \) and \( b = 1 \). By setting \( a = 0.1, 0.5, 2.0, 15.0 \) (See Figure 3), we display that the optimal indemnity depends on the values of \( a \) since the change of \( a \) effects on curvature of \( J(z) \). Especially, as for large losses and small losses, larger \( a \) becomes, higher the limit of the indemnity becomes and smaller the deductible becomes. Meanwhile, as for medium losses, when the loss is larger than the fixed threshold value (here the fixed threshold value is about 1.0), the insurer will full pay for the losses. When the loss is smaller than this fixed threshold value, there is partial reimbursement. With increasing the value of \( a \), the proportion of the partial payment becomes higher. Until \( a = 15.0 \), the proportion of the partial indemnity almost reaches 1, that is, the partial reimbursement becomes the full payment. Moreover, when \( a = 15.0 \), the limit of the indemnity nearly runs up to 3.0 and the deductible almost disappears. In a word, the raise of the parameter \( a \) makes the insurance contract more beneficial for the insured. In particular, when \( a = 15.0 \), the insured can get full reimbursement under the limit of the indemnity without the deductible.

Sequently, we also carry out the sensitivity of an optimal solutions to \( b \). Similar to the above, we fix the initial wealth of the insured \( w_0 = 7 \), the premium \( Q = 0.3w_0 = 2.1 \) and \( a = 0.5 \) first. Then, let \( b = 0.1, 0.3, 1.0, 2.0 \) (See Figure 4). Figure 4 describes that \( b \) dominates the values of \( J(z) \) rather than the shape of \( J(z) \) which leads to the change of optimal solutions. Particularly, for large losses, with increasing the value of \( b \), the limit of recovery rises. This
Figure 3: Different parameters of the cost $a$
effect of $b$ is similar to $a$, but the impact of $b$ is not more obvious than $a$. Simultaneously, as for small losses, the raise of $b$ also decreases the deductible. Of course, we also notice medium losses. With our best endeavors, it is not difficult to show that when the loss is larger than a threshold value, the insured will get full reimbursement. When the loss is smaller than this threshold value, there is a partial payment. It is necessary to make a special note that this threshold is distinguished from the threshold value in the case of the parameter $a$. This threshold increases with the value of $b$ raising. At the same time, the proportion of the partial indemnity doesn’t depend on the parameter $b$, which is different from the case of the parameter $a$. Until $b = 2.0$, the deductible almost reaches 0 and the limit of the indemnity nearly runs up to the maximal value that is about 3.0. In short, the raise of $b$ is more advantageous for the insured. In particular, when $b = 2.0$, the insured can get full reimbursement under the limit of the indemnity without the deductible.

Now, we are interested in the influence of the insured’s initial wealth on
the optimal solution. Similar to the above parameters, we fix the premium 
\( Q = 0.3w_0, a = 0.5 \) and \( b = 1.0 \) and set \( w_0 = 4.5, 5.2, 8.0, 20 \). It is 
surprise to find the different relationship between \( J(z) \) and \( F_{-1}^{-1}(z) \) (See Figure 5) 
with different values of the initial wealth of insured \( w_0 \). It is not difficult to obtain 
the optimal solutions, through the above discussion of five different cases in 
Section 2 (See Figure 2). The computation of the optimal solution identifies 
the independent threshold value which plays the pivotal role in relationship 
between \( J(z) \) and \( F_{-1}^{-1}(z) \) and the choice of the optimal contract. This thresh-
old is \( w_0 = 5.2 \) in Figures 5. Especially, when \( w_0 = 5.2, F_{-1}^{-1}(z) \) tangents to 
\( J(z) \). When \( w_0 < 5.2, J(z) \) and \( F_{-1}^{-1}(z) \) are disjoined. When \( w_0 > 5.2, F_{-1}^{-1}(z) \) 
intersects \( J(z) \). The various relationships between \( J(x) \) and \( F_{-1}^{-1}(z) \) make the 
optimal solution different shapes. Specifically, when \( w_0 = 4.5, 5.2, \) three dif-
ferent segments constitute the optimal solution, which respectively represent 
the limit of indemnity, deductible and the full payment for medium losses. 
It is valuably noticed that comparing \( w_0 = 4.5 \) and \( w_0 = 5.2 \), the limit of 
recovery is higher in the case of \( w_0 = 5.2 \) than one in the case of \( w_0 = 4.5 \). 
Another important point, as we think, is that all of the optimal solutions when 
\( w_0 < 5.2 \) are the same. The optimal solutions consist of four segments, when 
\( w_0 = 7.0, w_0 = 8.0 \) and \( w_0 = 15.0 \). These cases are more complicated than the 
cases of \( w_0 = 4.5 \) and \( w_0 = 5.2 \), since there are other threshold values between 
full and partial indemnity for medium losses. Meanwhile, we emphasize that 
with increasing the value of \( w_0 \), the threshold value for medium losses become 
smaller and smaller. Further, this rise of the initial wealth of the insured \( w_0 \) 
makes the deductible become smaller. Until \( w_0 = 20 \), both the threshold for 
medium losses and the deductible scarcely exit. Simultaneously, the limit of 
the recovery nearly reaches to the maximal value which is about 10.0. Namely, 
all of the losses under 10 can be full reimbursed at this time.

After testing the sensitivity of the optimal insurance policy, we compare 
the different utility functions of insured in terms of the influence of the optimal 
solution. We fix necessary parameters, the initial wealth of the insured \( w_0 = 
4.5 \), the premium \( Q = 0.3w_0 = 2.1 \), \( a = 0.5 \) and \( b = 1.0 \). Figure 6 indicates 
that the various utility functions do not change the shapes of \( J(z) \) expect 
for shifting up. This shift brings the optimal insurance contract some tiny 
distinctions. When 

\[
u(w) = \frac{1}{1-\beta}w^{1-\beta},
\]

the numerical analysis reveals that the limit of the recovery is about 2.0 and the 
insurer must fully pay for losses under the limit of the indemnity. Meanwhile, 
when 

\[
u(w) = \log w.
\]

there are a small deductible for small losses and a partial payment for medium 
losses.
Figure 5: Different initial wealth of insured
5 Conclusion

In contrast with Bernard et al’s work, our main contributions are main two aspects. On the one hand, we generalize the fix cost functions. Besides, through the mathematically sophisticated and complicated derivation of solution, we state the considerably surprising and subtle solution in explicit form for both the linear cost functions and the nonlinear cost functions. The shortcoming of prospect theory is that a conventional well-posed problem becomes an ill-posed problem with prospect theory. The nonexistence of an explicit solution impedes the progress in the application and impact of prospect theory. Hence, the generally novel results are meaningful and significant. On the other hand, the adverse selection and moral hazard are considered by us while Bernard et al don’t involve them. We use bonus-malus system and the empirical test to decide the premium $Q$. Based on the fixed $Q$, we further explore the optimal insurance design under Rand-Dependent Expected Utility. Because we focus on the optimal insurance contract for the general cost functions, we only provide a brief thought of how to determine the premium $Q$ and not carefully study it. However, this significant idea not only makes close relationship between these critical issues and our research but also offers the basic framework for the further research.

Compared with the Ben Dhiab’s research, our novelty is research from the different perspective. Particularly, he researches the optimal insurance contract from a insured’s (or agent’s) point of view, so he only needs to maximize the RDEU of the insured without considering the utility function of a insurer and relative restrictive conditions. However, we study the optimal insurance policy from a insruer’s point of view. Furthermore, I reveal the accurately quantified relationship between the indemnity function $I(x)$ and the random
loss $x$ while he doesn’t present the quantified relationship between the indemnity function $I(x)$ and the random loss $x$ in the cases of partial insurance and over-insurance. On the technological detail, we employ the S-shape probability weighting function and suppose the the loss $x$ is a random variable on $[0, w_0]$, whereas Ben Dhiab only assumes the probability weighting function is always concave or convex and the loss $x$ with probability $p$ and no-loss with probability $1 - p$. Sufficient evidences state that our assumptions are more practical than his.

In the numerical aspect, we tested the sensitivity of an optimal insurance contract for the fixed cost function of handling the indemnity, the initial wealth of the insured and two different utility functions of the insured in the numerical means. For the parameter $a$, raising the value of $a$ makes the insurance policy more beneficial for the insured, because the raising the parameter $a$ can increase the limit of recovery and the proportion of indemnity for medium losses. Meanwhile, raising $a$ can decrease the deductible do so. Similar to the parameter $a$, increasing the parameter $b$ also brings the insured more profit. It is the main reason that increasing parameter $b$ can increase the limit of reimbursement and reduce the deductible while the larger parameter $b$ makes the threshold value between full and partial payments become smaller and the partial reimbursement be almost replaced by full indemnity.

6 Further research

Although we have achieved notable and novel findings, this research is not perfect. For example, we only involve two kinds of utility functions of the insured and do not discuss others. As for different utility functions, we believe, the sensitivity of optimal insurance design for the fixed cost of handling the indemnity and the initial wealth of the insured is different. But, we only emphasize the sensitivity of utility function $u(w) = \log(w)$. Besides, the alterable parameters in our problem are far more than the illustrated parameters $a$, $b$ and $w_0$ by us. Other parameters’ variety, we believe, will impact the optimal solution.

In the optimal insurance contract, there are some important game relationships, which are the relationship between principal-agent, agent-agent, principal-principal. In the latter research, we will pay attention to these game relationships. Particularly, based on Sappington’s frameworks, we will combine the monitoring and competition with the optimal insurance contract under Rank-Dependent Expected Utility. Besides, considering the critical issues, adverse selection and moral hazard in the optimal insurance, we will apply bonus-malus system and empirical tests to decide the premium $Q$ and further research the dynamic optimal insurance design relying on the fixed the premium $Q$. 


7 Appendix

7.1 The proof of Lemma 2.6

Proof. Suppose the probability density function of $A - g(x)$ is $H(\cdot)$. Noticing that

$$f(t) = f(g(x) = t) = f(A - g(x) = A - t) = H(A - t),$$

we have sound reason to state that

$$\int_{-\infty}^{A-t} H(y)dy = 1 - \int_{-\infty}^{t} f(y)dy.$$ (7.1)

With the definition of the cumulative distribution function, it is easy to see that

$$F_{g(x)}(t) = \int_{-\infty}^{t} f(x)dx$$

and

$$F_{a-g(x)}(A - t) = \int_{-\infty}^{A-t} H(x)dx.$$

Let

$$z = F_{a-g(x)}(A - t) = \int_{-\infty}^{A-t} H(x)dx.$$

With (7.1), we have

$$1 - z = F_{g(x)}(t) = \int_{-\infty}^{t} f(x)dx.$$

Hence,

$$A - t = F_{A-g(x)}^{-1}(z)$$

and

$$t = F_{g(x)}^{-1}(1 - z).$$

So,

$$F_{A-g(x)}^{-1}(z) = A - F_{g(x)}^{-1}(1 - z).$$

$\Box$
7.2 The proof of Lemma 3.1

Proof. To simply write, we set

\[ h(k(z), z) = u(w_0 - Q - k(z))G'(z) + \lambda k(z) - \lambda C(F^{-1}_X(z) - k(z)). \]

Then,

\[
\begin{align*}
U_\lambda(k(z), z) &= \int_0^{m_0} u(w_0 - Q - k(z))G'(z) + \lambda k(z) - \lambda C(F^{-1}_X(z) - k(z))dz + \lambda(Q - Ex) \\
&= \int_0^{m_0} h(k(z), z)dz + \lambda(Q - Ex) \\
&= \int_0^{m_0} h(k(z), z)dz + \int_{m_0}^{l_0} h(k(z), z)dz + \int_{l_0}^{m_0} h(z)(k(z), z)dz + \lambda(Q - Ex) \\
&= I_1 + I_2 + I_3 + \lambda(Q - Ex) \quad (7.2)
\end{align*}
\]

It is easy to see that \( K(z) \) is the unique maximal value of \( h(k(z), z) \) on \([0, F^{-1}_X(z)]\) for each fixed \( z \in (0, m_0) \). So, \( h(k(z), z)(x) \) is strictly increasing on \([0, K(z)]\) and strictly decreasing on \([K(z), F^{-1}_x(z)]\).

We firstly discuss \( I_1 \). On \([0, m_0]\), if \( k(z) \geq K(z) \), then \( k(z) \in [K(z), F^{-1}_x(z)] \). Since \( h(k(z), z) \) is strictly decreasing on \([K(z), F^{-1}_x(z)]\),

\[ h(k(z), z) \leq h(K(z), z) = h(K_{m_0}(z), z); \]

if \( k(z) \leq K(z) \), then \( k(z) \in [0, K(z)] \). Since \( h(k(z), z) \) is strictly increasing on \([0, K(z)]\),

\[ h(k(z), z) \leq h(K(z), z) = h(K_{m_0}(z), z). \]

Therefor, we have

\[ I_1 = \int_0^{m_0} h(k(z), z)dz \leq \int_0^{m_0} h(K(z), z)dz = \int_0^{m_0} h(K_{m_0}(z), z)dz. \]

In the next, we focus on \( I_2 \). On \([m_0, l_0]\),

\[ k(z) \leq K_{m_0}(z) = K(m_0) \leq K(z). \]

\( k(z) \in [0, K(Z)] \) and \( K_{m_0}(z) \in [0, K(Z)] \), hence, \( h(k(z), z) \leq h(K_{m_0}(z), z) \). That is,

\[ I_2 = \int_{m_0}^{l_0} h(k(z), z)dz \leq \int_{m_0}^{l_0} h(K_{m_0}(z), z)dz. \]

As for \( I_2 \), noticing under Case 2, Case 4 and Case 5 \( n_0 = l_0 \), we have

\[ I_3 = \int_{n_0}^{l_0} h(z)(k(z), z)dz = \int_{n_0}^{l_0} h(z)(K_{m_0}(z), z)dz = 0 \]

26
Under **Case 1 and Case 3**, we have sound reason to state that

\[ K(z) \leq K_{m_0}(z) = K(m_0) \leq k(z) \]

on \([l_0, n_0]\).

Since \(h(k(z), z)\) is strictly decreasing on \([K(z), F_x^{-1}(z)]\),

\[ h(k(z), z) \leq h(K(z), z) = h(K_{m_0}(z), z). \]

So,

\[
I_3 = \int_{l_0}^{n_0} h(k(z), z)dz \leq \int_{l_0}^{n_0} h(K_{m_0}(z), z)dz.
\]

Now, we come to (7.2), it is easy to see that

\[
U_\lambda(k(z), z) = I_1 + I_2 + I_3 + \lambda(Q - Ex)
\leq \int_{0}^{m_0} h(K_{m_0}(z), z)dz + \int_{l_0}^{n_0} h(K_{m_0}(z), z)dz
+ \int_{l_0}^{n_0} h(K_{m_0}(z), z)dz + \lambda(Q - Ex)
= \int_{0}^{n_0} h(K_{m_0}(z), z)dz + \lambda(Q - Ex)
= U_\lambda(K_{m_0}(z), z)
\]

and the equality holds if and only if \(k(z) = K_{m_0}(z), 0 < z < n_0.\)

\[\square\]

### 7.3 The proof of Theorem 3.2

**Proof.** Recalling Lemma 3.1, we can achieve a more general result.

For feasible solution \(k(z)\) of Model 2.9, \(K_{m_i}(z)\) satisfies

i) \(U_\lambda(k(z), z) \leq U_\lambda(K_{m_i}(z), z), n_{i-1} < z < n_i;\)

ii) The equality holds if and only if \(k(z) = K_{m_0}(z), n_{i-1} < z < n_i.\)
Therefore, when \( N = M \), we have

\[
U_\lambda(k(z), z) = \int_0^1 h(k(z), z)dz + \lambda(Q - Ex)
\]

\[
= \int_0^{n_0} h(k(z), z)dz + \int_{n_0}^{n_1} h(k(z), z)dz + \lambda(Q - Ex)
\]

\[
+ \int_{n_3}^{n_2} h(z)(k(z), z)dz + \int_{n_2}^{n_1} h(z)(k(z), z)dz + \lambda(Q - Ex)
\]

\[
\leq \int_0^{n_0} h(K_{m_0}(z)I_{0 \leq z \leq n_0}, z)dz + \int_{n_0}^{n_1} h(K_{m_1}(z)I_{n_0 < z \leq n_1}, z)dz + \lambda(Q - Ex)
\]

\[
+ \int_{n_3}^{n_2} h(z)(K_{m_2}(z)I_{n_3 < z \leq n_2}, z)dz + \lambda(Q - Ex)
\]

\[
+ \int_{n_2}^{n_1} h(z)(K_{m_1}(z)I_{n_2 < z \leq 1}, z)dz + \lambda(Q - Ex)
\]

\[
= \int_0^1 h(\bar{K}(z), z)dz + \lambda(Q - Ex)
\]

\[
= U_\lambda(\bar{K}(z), z)
\]

and the equality holds if and only if \( k(z) = \bar{K}(z), 0 < z < 1 \).

When \( N \neq M \), we slip the detailed proof, since the result is derived in a similar means to the above case of \( N = M \). \( \square \)
7.4 The proof of Proposition 3.4

\textit{Proof.} Due to $\bar{K}(z)$ being piecewise function, we take $K_{m_{i+1}}(z)$ on $[n_i, n_{i+1}]$ into account without loss of generality.

\[
U_\lambda(K_{m_{i+1}}(z), z) = \int_{n_i}^{n_{i+1}} u(w_0 - Q - K_{m_{i+1}}(z))G'(z) dz \\
+ \lambda(K_{m_{i+1}}(z) - C(F_X^{-1}(z) - K_{m_{i+1}}(z)))dz + \lambda(Q - Ex) \\
= \int_{n_i}^{m_{i+1}} u(w_0 - Q - K_{m_{i+1}}(z))G'(z) dz \\
+ \lambda(K_{m_{i+1}}(z) - C(F_X^{-1}(z) - K_{m_{i+1}}(z)))dz \\
+ \lambda(K_{m_{i+1}}(z) - C(F_X^{-1}(z) - K_{m_{i+1}}(z)))dz + \lambda(Q - Ex) \\
= \int_{n_i}^{m_{i+1}} u(w_0 - Q - K(z))G'(z) dz \\
+ \lambda(K(z) - C(F_X^{-1}(z) - K(z)))dz \\
+ \lambda(K(m_{i+1}) - C(F_X^{-1}(z) - K(m_{i+1})))dz + \lambda(Q - Ex) \\
= \int_{n_i}^{m_{i+1}} u(w_0 - Q - K(z))G'(z)dz \\
+ \int_{n_{i+1}}^{m_{i+1}} \lambda(K(z))dz - \int_{n_i}^{m_{i+1}} C(F_X^{-1}(z) - K(z)))dz \\
+ \int_{n_{i+1}}^{m_{i+1}} u(w_0 - Q - K(m_{i+1}))G'(z)dz \\
+ \int_{n_{i+1}}^{m_{i+1}} \lambda(K(m_{i+1}))dz - \int_{n_{i+1}}^{m_{i+1}} C(F_X^{-1}(z) - K(m_{i+1})))dz + \lambda(Q - Ex)
\]
Let

\[ I = \int_{n_i}^{m_{i+1}} u(w_0 - Q - K(z))G'(z)dz, \]

\[ II = \int_{n_i}^{m_{i+1}} \lambda(K(z))dz, \]

\[ III = -\int_{n_i}^{m_{i+1}} C(F_X^{-1}(z) - K(z))dz, \]

\[ IV = \int_{n_{i+1}}^{n_i} u(w_0 - Q - K(m_{i+1}))G'(z)dz, \]

\[ V = \int_{m_{i+1}}^{m_{i+1}} \lambda K(m_{i+1})dz, \]

\[ VI = -\int_{m_{i+1}}^{n_{i+1}} C(F_X^{-1}(z) - K(m_{i+1}))dz. \]

Then,

\[ U_\lambda(K_{m_{i+1}}(z), z) = I + II + III + IV + V + VI + \lambda(Q - Ex). \]

\[ I = \int_{n_i}^{m_{i+1}} u(w_0 - Q - K(z))G'(z)dz \]

\[ = -\int_{n_i}^{m_{i+1}} u(w_0 - Q - K(z))d(G(n_{i+1}) - G(z)) \]

\[ = \int_{n_i}^{m_{i+1}} (G(n_{i+1}) - G(z))du(w_0 - Q - K(z)) - u(w_0 - Q - K(z))(G(n_{i+1}) - G(z)) \bigg|_{n_i}^{m_{i+1}} \]

\[ = -\int_{n_i}^{m_{i+1}} (G(n_{i+1}) - G(z))u'(w_0 - Q - K(z))dK(z) \]

\[ - u(w_0 - Q - K(m_{i+1}))(G(n_{i+1}) - G(m_{i+1})) \]

\[ + u(w_0 - Q - K(n_i))(G(n_{i+1}) - G(n_i)). \]

\[ IV = \int_{m_{i+1}}^{n_{i+1}} u(w_0 - Q - K(m_{i+1}))G'(z)dz \]

\[ = \int_{m_{i+1}}^{n_{i+1}} u(w_0 - Q - K(m_{i+1}))dG(z) \]

\[ = u(w_0 - Q - K(m_{i+1}))(G(n_{i+1}) - G(m_{i+1})). \]

So,

\[ I + IV = -\int_{n_i}^{m_{i+1}} (G(n_{i+1}) - G(z))u'(w_0 - Q - K(z))dK(z) \]

\[ + u(w_0 - Q - K(n_i))(G(n_{i+1}) - G(n_i)). \]

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Similarly,

\[ II = \int_{n_i}^{m_i+1} \lambda K(z)dz \]
\[ = \lambda K(z)z \bigg|_{n_i}^{m_i+1} - \int_{n_i}^{m_i+1} \lambda zdK(z) \]
\[ = \lambda K(m_i+1)m_i+1 - \lambda K(n_i)n_i - \int_{n_i}^{m_i+1} \lambda zdK(z). \]

\[ V = \int_{m_i+1}^{n_i+1} \lambda K(m_i+1)dz \]
\[ = \lambda K(m_i+1)z \bigg|_{m_i+1}^{n_i+1} \]
\[ = \lambda K(m_i+1)(n_i+1 - m_i+1). \]

Hence,

\[ II + V = -\lambda K(n_i)n_i - \int_{n_i}^{m_i+1} \lambda zdK(z) + \lambda K(m_i+1)n_i+1. \]

\[ III = -\int_{n_i}^{m_i+1} C(F^{-1}_X(z) - K(z))dz \]
\[ = \int_{n_i}^{m_i+1} zdC(F^{-1}_X(z) - K(z)) - C(F^{-1}_X(z) - K(z))z \bigg|_{n_i}^{m_i+1} \]
\[ = \int_{n_i}^{m_i+1} zC'(F^{-1}_X(z) - K(z))dF^{-1}_X(z) - \int_{n_i}^{m_i+1} zC'(F^{-1}_X(z) - K(z))dK(z) \]
\[ - C(F^{-1}_X(m_i+1) - K(m_i+1))m_i+1 + C(F^{-1}_X(n_i) - K(n_i))n_i. \]

\[ VI = -\int_{m_i+1}^{n_i+1} C(F^{-1}_X(z) - K(m_i+1))dz \]
\[ = \int_{m_i+1}^{n_i+1} zdC(F^{-1}_X(z) - K(m_i+1)) - C(F^{-1}_X(z) - K(m_i+1))z \bigg|_{m_i+1}^{n_i+1} \]
\[ = \int_{m_i+1}^{n_i+1} zC'(F^{-1}_X(z) - K(m_i+1))dF^{-1}_X(z) - C(F^{-1}_X(n_i+1) - K(m_i+1))n_i+1 \]
\[ + C(F^{-1}_X(m_i+1) - K(m_i+1))m_i+1 \]

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Thus,

\[ III + VI = \int_{n_i}^{m_{i+1}} zC'(F_X^{-1}(z) - K(z))dF_X^{-1}(z) - \int_{n_i}^{m_{i+1}} zC'(F_X^{-1}(z) - K(z))dK(z) + C(F_X^{-1}(n_i) - K(n_i))n_i + \int_{m_{i+1}}^{n_{i+1}} zC'(F_X^{-1}(z) - K(m_{i+1}))dF_X^{-1}(z) - C(F_X^{-1}(n_{i+1}) - K(m_{i+1}))n_{i+1} \]

Then,

\[ U_\lambda(K_{m_{i+1}}(z), z) = -\int_{n_i}^{m_{i+1}} (G(n_{i+1}) - G(z))u'(w_0 - Q - K(z)) + \lambda z + zC'(F_X^{-1}(z) - K(z))dK(z) + \int_{n_i}^{m_{i+1}} zC'(F_X^{-1}(z) - K(z))dF_X^{-1}(z) + \int_{m_{i+1}}^{n_{i+1}} zC'(F_X^{-1}(z) - K(m_{i+1}))dF_X^{-1}(z) - C(F_X^{-1}(n_{i+1}) - K(m_{i+1}))n_{i+1} + u(w_0 - Q - K(n_i))(G(n_{i+1}) - G(n_i)) - \lambda K(n_i)n_i + \lambda K(m_{i+1})n_{i+1} + C(F_X^{-1}(n_i) - K(n_i))n_i + \lambda(Q - Ex). \]

We plan to find \( m_{i+1}^* \) such that

\[ U_\lambda(K_{m_{i+1}^*}(z), z) = \max_{n_i < m_{i+1} \leq n_{i+1}} U_\lambda(K_{m_{i+1}}(z), z) \]

in order to obtain the global optimal solution.

It is easy to get

\[
\frac{\partial U_\lambda(K_{m_{i+1}}(z), z)}{\partial m_{i+1}} = -(G(n_{i+1}) - G(m_{i+1}))u'(w_0 - Q - K(m_{i+1})) - \lambda m_{i+1} - m_{i+1}C'(F_X^{-1}(m_{i+1}) - K(m_{i+1})) + m_{i+1}C'(F_X^{-1}(m_{i+1}) - K(m_{i+1})) - m_{i+1}C'(F_X^{-1}(m_{i+1}) - K(m_{i+1})) = -(G(n_{i+1}) - G(m_{i+1}))u'(w_0 - Q - K(m_{i+1})) - \lambda m_{i+1} - m_{i+1}C'(F_X^{-1}(m_{i+1}) - K(m_{i+1})).
\]

Noticing \( u'(\cdot) \geq 0 \) and \( C'(\cdot) \geq 0 \), we can show \( \frac{\partial U_\lambda(K_{m_{i+1}}(z), z)}{\partial m_{i+1}} \leq 0 \). In the other words, \( U_\lambda(K_{m_{i+1}}(z), z) \) decreases on \([n_i, n_{i+1}]\). So, when \( m_{i+1} = n_i \), we can get
the maximal value of $U_{\lambda}(K_{m+1}(z), z)$. Therefore,

$$K_{m+1}(z) = K_n(z) = K(n_i)I_{n+1 > z > n_i}$$

It is not difficult to come to the conclusion that the optimal solution of Model 2.1 is as follows:

$$I^*(x) = x - K(n_i)I_{n+1 > z > n_i}, F_{X^{-1}}(n_i+1) > x > F_{X^{-1}}(n_i).$$

References


