A study of decision process in MCDM problems with large number of criteria

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Received 20 March 2013; received in revised form 11 December 2013; accepted 1 May 2014

Abstract

In this paper, an effective decision process method is proposed to address the challenge in a multiple criteria decision-making (MCDM) problem because of large number of criteria. This method is based on the criteria reduction, tolerance relation, and prospect theory (PT). By building a discernibility matrix for tolerance relation (DMTR) in an MCDM problem with numerical values or interval numbers, this method first allows us to recognize a set of critical criteria from a large criteria pool, and ignore the other criteria. Next, it establishes the criteria weights through the DMTR as they are usually not indicated in the data. Then, the method ranks all the choices and selects the most desirable choice according to the weighted majority advantage value (WMAV). Here two risk-preference assumptions are proposed based on the PT and tolerance degree to select the WMAVs of different interval numbers with the same expectation. Using different risk-preference assumptions, we separately build WMAVs for different types of DMs. Finally, we presented two voting examples to demonstrate the applicability and effectiveness of the proposed method.

Keywords: decision process; criteria reduction; tolerance relation; prospect theory; risk preferences; weighted majority advantage value

1. Introduction

In today’s information age, one important requirement for decision makers (DMs) is relatively short time for making a wise decision while dealing with massive data. With massive data in the decision table given, they need to resort to a scientific methodology to screen the data and obtain the most relevant data in order to make a right decision effectively. Such a challenge in the information explosion age has motivated much research in the area of multicriteria decision making (MCDM).
It was first proposed by Wallenius et al. (2008) and has now become one of the most popular research topics (Danan, 2010; Feit et al., 2010; Ghemawat and Levinthal, 2008; Podinovski, 2010). Massive data MCDM problems can be classified into three categories: (a) large number of criteria, (b) large number of alternatives, and (c) large number of both criteria and alternatives. This paper primarily focuses on the problems in which large number of criteria exist in the decision table. Thus, it is important to know how to select critical criteria in order to make a wise decision. The aim of this paper is to propose a new decision process method to address the challenge in an MCDM problem having large number of criteria. Based on tolerance relation and rough set theory, we apply the method of criteria reduction to large number of criteria MCDM problems with numerical values or interval numbers. Therefore, we will change the traditional procedure for MCDM problems.

With the advent of information explosion, extracting useful information from large quantity of uncertain problems has become an important research area in computer science (Pawlak, 1997; Ziarko, 1993). The rough set theory has been recognized as one of the most powerful techniques to deal with uncertain problems since its appearance in 1982 (Lee and Wu, 2009; Mönch et al., 2006). The original rough set approach has been proved to be very useful in dealing with discrete problems. Criteria reduction in rough sets is based on the concept of quality of approximation, which captures useful information taking advantage of attribute reduction (Miao et al., 2009; Pawlak, 1982, 1985; Pawlak and Skowron, 2007; Zhang and Qiu, 2005). With the development of society and fuzziness of human thinking, it is inevitable to describe decision-making information with interval numbers most of the time. The original rough set approach can only deal with the problems with numerical values. Because an interval number has different characteristics than a numerical value, it becomes a challenge as how we can find useful information for interval numbers in MCDM problems. In this paper, we propose a new method of criteria reduction for interval numbers to resolve this problem.

Obtaining criteria weights is another important research topic in MCDM. For problems with uncertain criteria weights, there are several methods to obtain them, such as subjective obtaining criteria weights approach (Saaty and Vargas, 1987; Saen, 2009; Vansnick, 1986), objective obtaining criteria weights approach (Jahanshahloo et al., 2006; Zhang et al., 2005), subjective and objective obtaining criteria weights approach (Ma et al., 1999), and the weighted correlation coefficient method (Ye, 2010). Similar to the uncertain criteria weight problems, most approaches obtain criteria weights according to their standard deviation of criteria values to facilitate the ranking of alternatives. Generally speaking, the larger the standard deviations are, the greater the weights will be assigned (Xu, 2004). In reality, we find that some criteria values change more than other values, but these criteria only have little influence on the result, while some tiny changes of a few criteria would lead to totally different consequences for some MCDM problems.

According to the traditional procedure of MCDM problems, we need several steps such as standardizing criteria and obtaining criteria weights, and information aggregation as well as ranking, and selecting the most desirable alternative(s) (Qian et al., 2008; Sayadi et al., 2009; Wang and Luo, 2010; Xu, 2004). There are several existing techniques to rank alternatives in MCDM, such as the Gower Plots and Decision Balls method (Fernandez and Navarro, 2011), and the THESEUS method (Ma and Li, 2011). However, both of them need to compare expectations of weighted criteria values (EWCV; Guo and Tanaka, 2010; Nikolaev and Jacobson, 2010). We cannot filter off the absolute disparity through standardizing criteria. So, different criteria may be incomparable.
even if we combine them into the same meaning. Some errors would be produced in the processing of uniform criteria. In order to filter off these errors, we compare the weighted majority advantage value (WMAV). According to the existing methods, DMs still do not know which one is better about different interval numbers with the same expectation, such as \( E(\tilde{a}) = E(\tilde{b}) \) but \( \tilde{a} \neq \tilde{b} \). Therefore, we propose two prospect theory (PT) assumptions for interval numbers in MCDM problems. With the prospect assumptions, we can solve this problem easily.

The rest of this paper is organized as follows. The related work is discussed in Section 2. In Section 3, we propose two criteria reduction techniques based on tolerance relation (Guan and Wang, 2006) and rough set theory to find critical criteria in an MCDM problem with numerical values and interval numbers. Section 4 presents a new method to obtain criteria weights. It uses the discernibility matrix for tolerance relation (DMTR) and WMAV instead of alternative ranking. We also propose two risk assumptions to solve the challenge of different interval numbers with the same expectation. Using different risk-preference assumptions, we will achieve different WMAVs for different types of DMs. In Sections 5 and 6, we demonstrate and validate the applicability and effectiveness of the proposed method for MCDM problems with numerical values or interval numbers, respectively.

2. Related work

PT has successfully been used as behavioral model for decision making under risks mainly in economics and finance (Dhami and Al-Nowaihi, 2007; Gurevich et al., 2009). Unfortunately, few research works have reported the application of PT in MCDM problems. Korhonen et al. (1990) presented an explanatory use of PT. The validation of the theory by Korhonen et al. (1990) took advantage of linear piecewise marginal value functions, which means that these three authors used a linear approximation to PT. TODIM (an acronym in Portuguese of interactive and MCDM) is one of the first MCDM methods based on PT developed by Gomes and Lima (1992).

Nobre et al. (1999) presented an MCDM approach to support public health decision making that takes into account the fuzziness of the decision goals and the behavioral aspect of the DM. The method, known as TODIM, relies on evaluating alternatives with a set of decision criteria assessed using an ordinal scale. In this paper, the authors considered an MCDM using PT, in place of utility function, for describing the DM evaluation of each criterion. Gomes et al. (2009) approached the problem of analyzing and selecting the best option for the destination of the natural gas using the TODIM method. The TODIM method is a multicriteria method that should be very popular because of its theoretical base, and, without doubt, for the practicality of its application. The limitation of this method, which occurred during the execution of this study, relates to accessibility of primary data. It is impossible to access all the decision agents during data collection via questionnaires and validation interviews. It is also a discrete multicriteria method based on PT.

Gomes and Rangel (2009) defined a reference value for the rents of these properties using the TODIM method of multicriteria decision aiding. The research related to the behavior of executives in the decision-making process, mainly concerning questions of motivation and personal wishes, may contribute decisively to a better understanding of the recommendations and choices defined. In this paper, the evaluation of the alternatives in relation to all the criteria produces the matrix
of evaluation, where the values are all numerical. Moshkovich et al. (2011) presented an integrated approach to the problem based on the MCDM framework. The process is carried out through three phases using the multicriteria method TODIM. In this paper, the drawbacks of using the TODIM method are associated with the possibility of ranking reversals and absence of a logical way to evaluate properties with only one “close” value. Gomes et al. (2013) developed a multicriteria decision aiding model based on nonlinear cumulative PT. The model can solve MCDM problems with crisp values, interval data, and fuzzy triangular number. It also can be used for taking dependencies between criteria into consideration. But the authors only consider the DMs belonged to risk-aversion situation in this paper.

Chen et al. (2010) presented a new method for multiple attribute decision-making (MADM) problems with interval numbers. This method is based on loss aversion that considers that DMs will exhibit risk-seeking behavior in loss situations and risk-adverse behavior in gain situations, and are more sensitive to losses than to gains. In this method, the behavioral and cognitive factors of DMs are considered. However, this method did not consider other types of DMs’ risk preferences such as risk-neutral and risk-seeking. The original TODIM has some shortcomings due to its inherent inability to deal with uncertainty and imprecision in the process of decision making. Krohling and Souza (2012) developed a fuzzy extension of TODIM, for short, F-TODIM method, which is able to address the risk and uncertainty in MCDM problems by prospect function and triangular or trapezoidal fuzzy numbers, respectively. Through a case of oil spill in the sea they observe different behaviors by changing the uncertainty to the data or the factor of attenuation of losses. Fan et al. (2013) proposed an extended TODIM method to solve the hybrid MADM problem with three forms of attribute values (crisp numbers, interval numbers, and fuzzy numbers) when considering DM’s behavior. According to the concept of the classical TODIM method, the gain and loss matrices concerning each attribute are constructed by calculating the gain and loss of each alternative relative to the others. By calculating the dominance degree of each alternative over the others, the overall value of each alternative can be obtained to rank the alternatives. The extended TODIM method uses all the criteria to make a decision without considering how to find the critical criteria.

Unfortunately, few research works have applied criteria reduction to MCDM problems. Liu et al. (2013) first proposed a new method in MCDM with intuitionistic fuzzy sets based on risk preferences and criteria reduction. In their paper, four aspects have been discussed: (a) building three different advantage relationship models based on different types of risk preferences, (b) using these models to find useful criteria, (c) obtaining different criteria weights for different types of DMs, and (d) using different advantage relationship models to obtain different WCA V to rank and select the best alternative(s). In this paper, our research primarily focuses on decision process method for criteria reduction to address the challenge in MCDM problems with large number of criteria.

3. Large number criteria MCDM problems and criteria reduction

In this section, we present two types of MCDM problems that involve large number of criteria in a decision table. In one type, criteria values would be given by numerical values, and for the other is that its criteria would be given by interval numbers.

In this paper, our research primarily focuses on the “large decision table” (e.g. large number of criteria) challenge in MCDM problems with numerical values and interval numbers.
Table 1
The result of “yes” answers from 100 experts

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>...</th>
<th>$C_m$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>55</td>
<td>50</td>
<td></td>
<td>58</td>
<td></td>
</tr>
<tr>
<td>$A_2$</td>
<td>61</td>
<td>62</td>
<td></td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>$A_3$</td>
<td>55</td>
<td>55</td>
<td></td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>$A_4$</td>
<td>65</td>
<td>65</td>
<td></td>
<td>65</td>
<td></td>
</tr>
<tr>
<td>$A_5$</td>
<td>89</td>
<td>85</td>
<td></td>
<td>55</td>
<td></td>
</tr>
</tbody>
</table>

For MCDM problems with numerical values, we will discuss three aspects: (a) finding useful criteria based on DMTR, (b) obtaining criteria weights based on DMTR, and (c) using WMAVs to rank and select the best alternative.

For MCDM problems with interval numbers, four aspects will be discussed: (a) finding useful criteria based on the tolerance degree of interval numbers and DMTR, (b) obtaining criteria weights based on the tolerance degree of interval numbers and DMTR, (c) building two different relationship assumptions based on risk preferences of DMs and tolerance degree of interval numbers, and (d) ranking and selecting the alternatives by WMAVs built for preferences of DMs separately.

Then, a new decision process method based on the criteria reduction is proposed for MCDM problems with numerical values and interval numbers, respectively. The problems we are concerned in this section are how to find the critical criteria for numerical values and interval numbers MCDM problems, respectively.

3.1. Addressing numerical values MCDM problems

Example MCDM problem (supplier choice; Liu et al., 2013): The Commercial Aircraft Corporation of China, Ltd. (CACC) builds huge commercial aircrafts to serve commercial airlines in China. To build aircrafts, the company needs to buy and use some key parts from international or domestic suppliers. Therefore, the CACC must make a scientific decision to choose the most desirable supplier that relates to the success of commercial aircraft program. There are many complicated factors that affect decision of DMs as they need to combine and analyze all information for every supplier and analyze them as well as select the most desirable supplier(s).

Suppose that there are five international suppliers in the first round competing for the CACC demand of some key parts of the huge commercial aircrafts, and these five suppliers are represented by $A = \{A_1, A_2, A_3, A_4, A_5\}$. Suppose we invite 100 experts to make judgments in order to obtain the degrees to which alternative $A_i$ satisfies and does not satisfy criterion $C_j$ ($i = 1, 2, 3, 4, 5; j = 1, 2, \ldots, m$). There are two kinds of poll results “yes” or “no” to the question whether alternative $A_i$ satisfies criterion $C_j$. Which is the most desirable choice according to the results in Table 1?

3.1.1. The challenge of numerical values MCDM problems

Obviously, there are large number of criteria in decision Table 1. The first problem is how to make a wise decision within limited time when DMs face large number of criteria? To address this problem, we have to find critical criteria that really affect the decision results.
3.1.2. The principles and methods of criteria reduction for numerical values MCDM problems

Our idea regarding how to find useful information. We often doubt whether we can remove some criteria from an information table while preserving its basic properties, that is, whether a table contains redundant criteria. Through criteria reduction we want to find useful information that really affects the decision-making process. This applies rough set theory (Miao et al., 2009; Pawlak, 1982, 1985; Pawlak and Skowron, 2007; Zhang and Qiu, 2005) for criteria reduction. In addition, the rough set theory includes many attributes reduction (Pawlak, 1997; Ziarko, 1993) techniques, such as the reduction of attributes based on similarity relation (Mönch et al., 2006; Zhang and Qiu, 2005) or tolerance relation (Guan and Wang, 2006) or advantage relation (Lee and Wu 2009; Liu et al., 2013; Zhang and Qiu, 2005), etc. In this paper, we apply rough set theory to finding critical criteria from the decision table by using criteria reduction based on tolerance relation.

Now, let us use the example shown in Table 1 to illustrate our idea in using tolerance relation and rough set theory to find critical criteria. Two criteria values of alternatives $A_1$ and $A_3$ on criterion $C_1$ are $f(A_1, C_1)$ and $f(A_3, C_1)$. Obviously, both of them have the same criteria value on $C_1$, that is, $f(A_1, C_1) = f(A_3, C_1)$ in Table 1. Thus, $C_1$ is a useless criterion when we compare these two alternatives $A_1$ and $A_3$. We do not need to consider those criteria, on which two or more alternatives have the same value. So, we can remove them. Though we should consider those criteria on which two or more alternatives show different values. It is obvious that the criteria values of alternative $A_1$ and $A_2$ on criterion $C_1$ are different, that is $f(A_1, C_1) \neq f(A_2, C_1)$. Thus, if we want to compare alternatives between $A_1$ and $A_2$, criterion $C_1$ is needed. In short, criterion $C_1$ is useless to comparing alternatives $A_1$ and $A_3$, but it is needed when we want to know which one is better, alternative $A_1$ or alternative $A_2$. That means the same criteria may play different roles in the same decision table. Thus, if we want to find critical criteria that we need in decision tables, we need to make a comparison on “every” pair of alternatives.

According to attribute reduction based on “tolerance relation” (Definition 2) and rough set theory, if two alternatives have the same criteria value on the same criteria, we can say that these two alternatives have a tolerance relation for numerical value MCDM problems. When two alternatives have different values on the same criterion, it means that a tolerance relation does not exist. Furthermore, the tolerance degree is either 1 or 0 for numerical value MCDM problems. If there is a tolerance relation, the degree will be one (1); otherwise, the degree will be zero (0). Definitely, for numerical value MCDM problems, when the tolerance degree goes to 1, we can use it as the threshold to find critical criteria. To check “every” pair of alternatives separately, we need to construct a discernibility matrix (Guan and Wang, 2006; Zhang and Qiu, 2005) and find critical criteria for all the alternatives from the decision table. The criteria in the discernibility matrix indicate a set of criteria that must be considered when we want to compare two corresponding alternatives. In this paper, we propose a method of criteria reduction for MCDM problems using the tolerance relation.

3.2. Addressing interval numbers MCDM problems

Example MCDM problem (supplier choice). Take the case in Section 3.2 as an example, because of time pressure, shortage of knowledge or criteria, and the limited expertise, experts are expected to answer “yes” or “no” or “I do not know” to the question of whether alternative $A_i$ satisfies criterion
Table 2
The result of “yes” and “no” answers from 100 experts

<table>
<thead>
<tr>
<th>U</th>
<th>C_1</th>
<th>C_2</th>
<th>\ldots</th>
<th>C_m</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1</td>
<td>(75, 15)</td>
<td>(73, 10)</td>
<td>\ldots</td>
<td>(18, 82)</td>
<td>\ldots</td>
</tr>
<tr>
<td>A_2</td>
<td>(76, 5)</td>
<td>(85, 0)</td>
<td>\ldots</td>
<td>(12, 15)</td>
<td>\ldots</td>
</tr>
<tr>
<td>A_3</td>
<td>(88, 0)</td>
<td>(85, 25)</td>
<td>\ldots</td>
<td>(25, 30)</td>
<td>\ldots</td>
</tr>
<tr>
<td>A_4</td>
<td>(75, 14)</td>
<td>(77, 10)</td>
<td>\ldots</td>
<td>(22, 78)</td>
<td>\ldots</td>
</tr>
<tr>
<td>A_5</td>
<td>(89, 0)</td>
<td>(85, 0)</td>
<td>\ldots</td>
<td>(25, 75)</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

Table 3
The final result of “yes” answers from 100 experts

<table>
<thead>
<tr>
<th>U</th>
<th>C_1</th>
<th>C_2</th>
<th>\ldots</th>
<th>C_m</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1</td>
<td>[75, 85]</td>
<td>[73, 90]</td>
<td>\ldots</td>
<td>[18, 18]</td>
<td>\ldots</td>
</tr>
<tr>
<td>A_2</td>
<td>[76, 95]</td>
<td>[85, 100]</td>
<td>\ldots</td>
<td>[12, 85]</td>
<td>\ldots</td>
</tr>
<tr>
<td>A_3</td>
<td>[88, 100]</td>
<td>[68, 75]</td>
<td>\ldots</td>
<td>[25, 70]</td>
<td>\ldots</td>
</tr>
<tr>
<td>A_4</td>
<td>[75, 86]</td>
<td>[77, 90]</td>
<td>\ldots</td>
<td>[22, 22]</td>
<td>\ldots</td>
</tr>
<tr>
<td>A_5</td>
<td>[89, 100]</td>
<td>[85, 100]</td>
<td>\ldots</td>
<td>[25, 25]</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

Then we choose the most desirable choice according to the results from Table 2. In Table 2, the total counts of “yes” and “no” answers represent the number of membership and nonmembership, respectively. The total counts of “I do not know” answers are derived from using the number of the experts (100) minus the total counts of “yes” and “no” answers. Subtracting the total number of experts from the total counts of “no” answers, we will get the highest total counts of “yes” answers in theory. The final total counts of “yes” answers should be between the initial total counts and highest total counts of “yes” answers in theory. We will illustrate the final probability total counts of “yes” answers in Table 3. From it we know all the criteria are expressed by interval numbers (Guo and Tanaka, 2010; Xu, 2004).

3.2.1. The challenge of interval numbers MCDM problems

Obviously, there are also large number of criteria in Table 3. In addition, all the criteria values are expressed by interval numbers. The original rough set theory is validated to be a very useful tool in dealing with numerical value problems; however attribute reduction or criteria reduction techniques do not exist for interval numbers at present. How can we find the critical criteria for interval number MCDM problems? To address this problem, we need to know the basic characteristics of interval numbers.

3.2.2. Definition and characteristics for interval numbers

Definition 1. (Interval number) Suppose that \( \tilde{a} = [a^L, a^U] = \{x | a^L \leq x \leq a^U, a^L, a^U \in \mathbb{R} \} \), \( \tilde{a} \) is an “interval number” (Guo and Tanaka, 2010; Xu, 2004). Obviously, if and only if \( a^L = a^U \), then \( \tilde{a} \) becomes a numerical value. \( l_{\tilde{a}} = a^U - a^L \) indicates the length of range of interval number \( \tilde{a} \); moreover, if \( l_{\tilde{a}} = 0 \), \( \tilde{a} \) also becomes a numerical value.

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Undoubtedly, every numerical value can be indicated by an interval number. In this paper, we suppose that the (numerical) values of the domain of criteria have the uniform distribution. For interval numbers, several universal identification characteristics are defined as follows.

Suppose that $\tilde{a} = [a^L, a^U]$, $\tilde{b} = [b^L, b^U]$ are two interval numbers, and $k \geq 0$, then the characteristics of interval numbers about $\tilde{a}$ and $\tilde{b}$ are defined as follows:

1. $\tilde{a} + \tilde{b} = [a^L + b^L, a^U + b^U]$.
2. $\tilde{a} = \tilde{b}$ if and if $a^L = b^L$ and $a^U = b^U$.
3. If and only if $a^U \geq b^L$ and $b^U \geq a^L$, $\tilde{a} \cap \tilde{b} = [\min\{a^L, b^L\}, \min\{a^U, b^U\}]$.
4. If and only if $a^U \geq b^L$ and $b^U \geq a^L$, $\tilde{a} \cap \tilde{b} = [\min\{a^L, b^L\}, \max\{a^U, b^U\}]$.
5. $E(\tilde{a}) = \frac{a^L + a^U}{2}$, $E(\tilde{b}) = \frac{b^L + b^U}{2}$.

**Definition 2.** (Tolerance degree) Two interval numbers of $\tilde{a} = [a^L, a^U]$ and $\tilde{b} = [b^L, b^U]$, if $a^U \geq b^L$ and $b^U \geq a^L$, then $\tilde{a} \cap \tilde{b} = [\max\{a^L, b^L\}, \min\{a^U, b^U\}]$, $\tilde{a} \cup \tilde{b} = [\min\{a^L, b^L\}, \max\{a^U, b^U\}]$, the “tolerance degree” for two interval numbers of $\tilde{a}$ and $\tilde{b}$, denoted $T(\tilde{a}, \tilde{b})$ is defined as follows:

$$
T(\tilde{a}, \tilde{b}) = \begin{cases} 
0 & b^U < a^L \text{ or } a^U < b^L \\
1 & \tilde{a} = \tilde{b} \\
\frac{l_{\tilde{a} \cap \tilde{b}}}{l_{\tilde{a} \cup \tilde{b}}} & b^U \geq a^L \text{ or } a^U \geq b^L
\end{cases}
$$

The tolerance degree is between 0 and 1 for two interval numbers, that is, $0 \leq T(\tilde{a}, \tilde{b}) \leq 1$. The ratio between $l_{\tilde{a} \cap \tilde{b}}$ and $l_{\tilde{a} \cup \tilde{b}}$ indicates $T(\tilde{a}, \tilde{b})$. Obviously, there is $T(\tilde{a}, \tilde{b}) = T(\tilde{b}, \tilde{a})$.

### 3.2.3. The principles and methods of criteria reduction for interval numbers MCDM problems

**Our idea regarding how to find useful information.** In the discussion in Section 3.1, we use criteria reduction method based on tolerance relation to find critical criteria that really affect the decision. For interval number MCDM problems, we still want to use the same method to find the critical criteria. But the original rough set approach can only deal with the numerical value problems. As the interval numbers have different characteristics from those of numerical values, to resolve this problem, we need to propose a new method of criteria reduction for interval numbers.

Now, let us use the example shown in Table 3 to illustrate our idea in using tolerance relation and rough set theory to find critical criteria. [88, 100] and [89, 100] are two criteria values of alternatives $A_3$ and $A_5$ on criterion $C_1$, respectively. Obviously, they are two different interval numbers. According to Definition 2, we know the tolerance degree of them goes to about 0.9167 (11/12). That means the value ranges’ overlap percentage goes to $91.67\%$ of alternatives $A_3$ and $A_5$ on criterion $C_1$. Undoubtedly, when DMs want to make a decision especially if there are large number of criteria in the decision table, they will think these two criteria values reflect and convey the same meaning or the same information. Thus, $C_1$ is a useless criterion when DMs want to compare the alternatives between $A_3$ and $A_5$ in Table 3. Furthermore, if the tolerance degree of interval numbers on the same criteria goes to an expected number (threshold), such as 80% or 90%, the DMs may think these two criteria values reflect the same meaning for them to make a decision. Thus, different DMs will
choose different thresholds to find critical criteria. We also need to construct a discernibility matrix and find critical criteria for all alternatives from the decision table.

4. The decision process method for large number of criteria MCDM problems

According to the traditional procedure for MCDM problems, we need several steps, such as unifying criteria, obtaining criteria weights, information fusion, and ranking and selecting the most desirable alternative(s) (Wang and Luo, 2010; Xu, 2004). In this paper, we apply the method of criteria reduction based on tolerance relation and rough set theory to large number of criteria MCDM problems. So, we will change the traditional procedure for MCDM problems.

4.1. Resolution procedure for the large number of criteria MCDM problems

To solve the problems above, a new resolution procedure is proposed, as shown in Fig. 1. A brief description of the resolution procedure is given below.

First, using Eqs. (1)–(3), we construct a discernibility matrix based on tolerance relation to find the critical criteria for numerical value and interval number of MCDM problems, respectively. Second, by using Eqs. (4) and (5), we find a way to obtain criteria weights for critical criteria and all criteria in the decision table, respectively. Third, by using Eqs. (6)–(13), we find a new algorithm to rank alternatives of numerical value and interval number of MCDM problems. Finally, we validate this new method. Equations (2)–(13) and details (Sections 5.2 and 6.2) will be provided in the following sections.
4.2. How to find critical criteria for numerical value MCDM problems?

As proposed in Section 3.2, we need to construct a DMTR in order to find critical criteria from the decision table. The discernibility matrix for numerical value MCDM problems is defined as follows.

**Definition 3.** (DMTR for numerical value MCDM problems) Suppose that \( \{A_1, A_2, \ldots, A_n\} \) indicates a set of \( n \) alternatives, and \( \{C_1, C_2, \ldots, C_m\} \) is a set of criteria for numerical value MCDM problems. \( f(A_i, C_j) \) and \( f(A_k, C_j) \) indicate the possible outcomes of alternatives \( A_i \) and \( A_k \) on criterion \( C_j \). “DMTR for numerical value MCDM problems” is defined as follows:

\[
DMTR = \begin{bmatrix}
    m_{11} & \cdots & m_{1n} \\
    \vdots & \ddots & \vdots \\
    m_{n1} & \cdots & m_{nn}
\end{bmatrix},
\]

where

\[
m_{ik} = \begin{cases} 
    \{C_j \in C : f(A_i, C_j) \neq f(A_k, C_j)\} & \text{if } i \neq k \\
    \phi & \text{otherwise}
\end{cases}
\]

\( m_{ik} \) denotes a criteria set of two alternatives \( A_i \) and \( A_k \) with different criteria values on that criteria in set \( C \). Obviously, there is \( m_{ik} = m_{ki} \) in the discernibility matrix as \( T(A_i, A_k) = T(A_k, A_i) \). Thus, the discernibility matrix is a symmetric matrix.

Using a relative discernibility function of discernibility matrix via Boolean reasoning techniques (Greco et al., 2001; Guan and Wang, 2006; Skowron, 1993, 1995; Skowron and Rauszer, 1992), we can get the critical criteria for all the alternatives.

4.3. How to find critical criteria for interval number MCDM problems?

According to the previous discussion, we need to construct a discernibility matrix. We propose the tolerance degree of interval numbers in order to solve the interval numbers MCDM problems. By using the discernibility matrix, we can find the critical criteria for interval numbers MCDM problems. We construct the discernibility matrix for interval numbers MCDM problems as follows.

**Definition 4.** (DMTR for interval number MCDM problems) Suppose that \( \{A_1, A_2, \ldots, A_n\} \) indicates a set of \( n \) alternatives for interval number MCDM problems, \( \{C_1, C_2, \ldots, C_m\} \) is a set of \( m \) criteria. \( \tilde{f}(A_i, C_j) \) and \( \tilde{f}(A_k, C_j) \) are two criteria values for \( A_i \) and \( A_k \) \( \{i, k \in 1, 2, \ldots, n\} \) on criterion \( C_j \) \( \{j \in 1, 2, \ldots, m\} \), where \( \tilde{f}(A_i, C_j) \) and \( \tilde{f}(A_k, C_j) \) are two interval numbers. Different criteria have different meanings. Thus, DMs should use different thresholds to find critical criteria when considering different criteria. In other words, when the tolerance degree goes to \( \alpha_j \) \( \{j \in 1, 2, \ldots, m\} \) on criterion \( C_j \) \( \{j \in 1, 2, \ldots, m\} \), it reflects the same meaning and transfers the same information for DMs. “DMTR for interval number MCDM problems” is defined as follows:
\[ \alpha_j = A_1 \ldots A_n \]

\[
\text{DMTR}_{\alpha_j} = \begin{bmatrix}
  m_{11} & \ldots & m_{1n} \\
  \vdots & \ddots & \vdots \\
  m_{n1} & \ldots & m_{nn}
\end{bmatrix},
\]

where

\[
m_{ik} = \begin{cases}
  \{C_j \in C : T(\tilde{f}(A_i, C_j), \tilde{f}(A_k, C_j)) < \alpha_j \} & \text{if } \phi \\
  \phi & \text{otherwise}
\end{cases}
\]

\(m_{ik}\) is a set of criteria containing the criteria that the tolerance degree below the threshold of DMs are considered as alternatives \(A_i\) and \(A_k\) in set \(C\). It is obvious that \(m_{ik} = m_{ki}\) in the discernibility matrix.

Similar to numerical value MCDM problems, we still use the relative discernibility function of discernibility matrix via Boolean reasoning techniques (Greco et al., 2001; Guan and Wang, 2006; Skowron, 1993, 1995; Skowron and Rauszer, 1992). We can get critical criteria for all the alternatives.

4.4. Obtaining criteria weights of the critical criteria

The challenge of obtaining criteria weights: There are several methods to obtain the criteria weights. Traditionally, researchers think the most popular existing method to obtain the criteria weights is the one based on the deviation of criteria values (Wang and Luo, 2010; Xu, 2004). But we think this method has two issues that need further consideration.

First, for some MCDM problems, some criteria values change larger than others and these criteria only have little influence on the result, while some tiny changes of few criteria would lead to different consequences.

Second, finding out critical criteria depending on whether the criteria values are the same or not instead of the deviation of criteria values.

How to find a scientific and reasonable method and obtain the criteria weights has become a very important research topic in MCMD problem. Based on the discussion above, we use the method that makes use of DMTR (Liu et al., 2013) to obtain the criteria weights of MCMD problems.

The method based on DMTR to obtain criteria weights: In DMTR, \(m_{ik}(i \in 1, 2, \ldots, n; k \in 1, 2, \ldots, n)\) indicates a set of critical criteria containing the criteria that two alternatives \(A_i\) and \(A_k\) have different criteria values. In other words, \(m_{ik}(i \in 1, 2, \ldots, n; k \in 1, 2, \ldots, n)\) contains all the criteria that we must compare if we want to know which one is better, \(A_i\) or \(A_k\) in Table 4. How many times the criteria appear in the DMTR means how many times we need to consider it. Thus, the criteria weights would have a proportional relationship with the times it appears in DMTR. So, \(\omega_j(j = 1, 2, \ldots, m')\) for critical criteria are listed as follows:

\[
\omega_j = \frac{||C_j||}{\sum_{j=1}^{m'} C_j},
\]

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Table 4
The result of “yes” from 100 experts

<table>
<thead>
<tr>
<th>$U$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$C_6$</th>
<th>$C_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>45</td>
<td>50</td>
<td>75</td>
<td>20</td>
<td>50</td>
<td>40</td>
<td>48</td>
</tr>
<tr>
<td>$A_2$</td>
<td>61</td>
<td>62</td>
<td>65</td>
<td>54</td>
<td>45</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>$A_3$</td>
<td>45</td>
<td>55</td>
<td>30</td>
<td>54</td>
<td>45</td>
<td>45</td>
<td>70</td>
</tr>
<tr>
<td>$A_4$</td>
<td>65</td>
<td>65</td>
<td>30</td>
<td>65</td>
<td>70</td>
<td>45</td>
<td>65</td>
</tr>
<tr>
<td>$A_5$</td>
<td>89</td>
<td>85</td>
<td>65</td>
<td>65</td>
<td>65</td>
<td>50</td>
<td>50</td>
</tr>
</tbody>
</table>

where $||C_j||$ indicates the times that criteria $C_j$ appear in the DMTR, $||\sum_{j=1}^{m'} C_j||$ indicates the total times all critical criteria appear in the DMTR, and $m'$ indicates how many critical criteria we need in Table 4.

Likewise, we get $\omega_j$ ($j = 1, 2, \ldots, m$) for all criteria are listed as follows:

$$\omega_j = \frac{||C_j||}{\sum_{j=1}^{m} ||C_j||},$$

(5)

where $||C_j||$ indicates the times criteria $C_j$ appear in the DMTR, $||\sum_{j=1}^{m} C_j||$ indicates the total times all criteria appear in the DMTR, and $m$ indicates how many criteria we need in Table 4.

For interval numbers MCDM problems, we use the same method to obtain criteria weights.

4.5. Ranking and selecting the most desirable alternative(s) for numerical value MCDM problems

The challenge of ranking alternatives: According to the existing methods for MCDM problems, while ordering criteria and alternatives (as both are ordered in one way or another in the presented approach), the authors assume a rather simple preference model, one which only allows linear preordering. These simple ordering models result in utilizing simple summing (or arithmetic averaging) throughout the method. But it would produce errors in this step. Unitig criteria and comparing the EWCVs will lead to another problem. We cannot filter off the absolute disparity through unifying criteria. Thus, different criteria may be incomparable even if we unify them into the same meaning. We think it is better that the criteria values are compared on the same criteria.

Our idea of how to rank and select alternatives: In this paper, via two steps of criteria reduction and obtaining criteria weights, we have not considered the arithmetical difference between two criteria values, we only consider whether they are the same or not and find the critical criteria as well as obtain the criteria weights. In this paper, we compare the WMAV to rank alternatives. In this way, we do not need to consider the deviation of criteria values. At the same time, we can also filter off some errors in the decision-making process.

Generally speaking, there are two types of criteria.

- The criteria of cost type (the smaller the better of criteria values).
• The criteria of benefit type (the larger the better of criteria values).

Obviously, all the criteria belong to benefit type in this paper.

For benefit type criteria, criteria values of two alternatives \( A_1 \) and \( A_2 \) on criterion \( C_1 \) are \( f(A_1, C_1) \) and \( f(A_2, C_1) \). If \( f(A_1, C_1) \) is larger than \( f(A_2, C_1) \), it means that on criterion \( C_1 \) the alternative \( A_1 \) is preferred to the alternative \( A_2 \). In other words, alternative \( A_1 \) is better than \( A_2 \) on criterion \( C_1 \). In this paper, we use \( A_1 \succ A_2/C_1 \) to indicate the relation. If there is \( A_1 \prec A_2/C_1 \), it means that on criterion \( C_1 \) the alternative \( A_2 \) is preferred to the alternative \( A_1 \). In other words, \( f(A_1, C_1) < f(A_2, C_1) \). If two alternatives have the same criteria values, it means that they are in the same position on this criterion regarding decision making.

Suppose that we use \( AV_{A_1 \succ A_2/C_1} \) to express an advantage value (AV) between decision alternative \( A_1 \) and \( A_2 \) on criterion \( C_1 \). So we get the AV as follows:

\[
AV_{A_1 \succ A_2/C_1} = \begin{cases} 
0 & A_1 < A_2/C_1 \\
0.5 & A_1 \equiv A_2/C_1 \\
1 & A_1 > A_2/C_1 
\end{cases}
\]

(6)

where the AV of alternatives \( A_1 \) and \( A_2 \) on criterion \( C_1 \) \( (AV_{A_1 \succ A_2/C_1} = 1) \) indicates on criterion \( C_1 \) that the alternative \( A_1 \) is preferred to the alternative \( A_2 \). If \( AV_{A_1 \succ A_2/C_1} = 0.5 \), it means that two alternatives \( A_1 \) and \( A_2 \) locate at the same position on \( C_1 \). \( AV_{A_1 \succ A_2/C_1} = 0 \) indicates on criterion \( C_1 \) the alternative \( A_2 \) is preferred to the alternative \( A_1 \).

Suppose \( \{A_1, A_2, \ldots, A_n\} \) is a set of \( n \) alternatives of MCDM problems, \( \{C_1, C_2, \ldots, C_{m'}\} \) indicates a set of \( m' \) critical criteria of MCDM problems, and \( \{\omega_1, \omega_2, \ldots, \omega_{m'}\} \) is a set weights corresponding to critical criteria. \( WAV_{A_1 \succ A_2} \) represents a WAV between \( A_1 \) and \( A_2 \) for critical criteria as follows:

\[
WAV_{A_1 \succ A_2} = AV_{A_1 \succ A_2/C_1} \cdot \omega_1 + AV_{A_1 \succ A_2/C_2} \cdot \omega_2 + \cdots + AV_{A_1 \succ A_2/C_{m'}} \cdot \omega_{m'}.
\]

(7)

Suppose that \( \{A_1, A_2, \ldots, A_n\} \) is a set of \( n \) alternatives of MCDM problems, \( \{C_1, C_2, \ldots, C_m\} \) indicates a set of \( m \) criteria of MCDM problems, and \( \{\omega_1, \omega_2, \ldots, \omega_m\} \) is a set weights corresponding to all criteria. \( WAV_{A_1 \succ A_2} \) represents a WAV between \( A_1 \) and \( A_2 \) for all criteria as follows:

\[
WAV_{A_1 \succ A_2} = AV_{A_1 \succ A_2/C_1} \cdot \omega_1 + AV_{A_1 \succ A_2/C_2} \cdot \omega_2 + \cdots + AV_{A_1 \succ A_2/C_m} \cdot \omega_m.
\]

(8)

For the WAV, there are following characteristics:

1. \( WAV_{A_1 \succ A_2} + WAV_{A_2 \succ A_1} = 1 \),
2. \( WAV_{A_1 \succ A_2} = 0.5 \iff A_1 \equiv A_2 \),
3. \( WAV_{A_1 \succ A_2} > 0.5 \iff A_1 > A_2 \),

where \( WAV_{A_1 \succ A_2} > 0.5 \), it means that alternative \( A_1 \) is better than \( A_2 \) in the decision table.
Table 5
The poll result of “yes” and “no” from 100 experts

<table>
<thead>
<tr>
<th>C</th>
<th>C_1</th>
<th>C_2</th>
<th>C_3</th>
<th>C_4</th>
<th>C_5</th>
<th>C_6</th>
<th>C_7</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1</td>
<td>[50, 80]</td>
<td>[55, 75]</td>
<td>[65, 95]</td>
<td>[20, 70]</td>
<td>[50, 90]</td>
<td>[40, 80]</td>
<td>[50, 70]</td>
</tr>
<tr>
<td>A_2</td>
<td>[65, 75]</td>
<td>[60, 85]</td>
<td>[60, 90]</td>
<td>[55, 75]</td>
<td>[50, 87]</td>
<td>[40, 80]</td>
<td>[50, 75]</td>
</tr>
<tr>
<td>A_3</td>
<td>[70, 90]</td>
<td>[65, 85]</td>
<td>[35, 55]</td>
<td>[65, 80]</td>
<td>[45, 90]</td>
<td>[40, 80]</td>
<td>[65, 85]</td>
</tr>
<tr>
<td>A_4</td>
<td>[65, 86]</td>
<td>[65, 80]</td>
<td>[35, 60]</td>
<td>[65, 65]</td>
<td>[70, 90]</td>
<td>[65, 90]</td>
<td>[65, 85]</td>
</tr>
<tr>
<td>A_5</td>
<td>[71, 92]</td>
<td>[86, 90]</td>
<td>[70, 91]</td>
<td>[65, 65]</td>
<td>[65, 80]</td>
<td>[50, 70]</td>
<td>[50, 72]</td>
</tr>
</tbody>
</table>

Comparing every pair of all alternatives, we construct the advantage relation matrix (ARM). Then, we get the ARM for all the alternatives of MCDM problems as follows:

$$ARM = \begin{bmatrix}
WAV_{A_1>A_1} & WAV_{A_1>A_2} & \cdots & WAV_{A_1>A_n} \\
WAV_{A_2>A_1} & WAV_{A_2>A_2} & \cdots & WAV_{A_2>A_n} \\
\vdots & \vdots & \ddots & \vdots \\
WAV_{A_n>A_1} & WAV_{A_n>A_2} & \cdots & WAV_{A_n>A_n}
\end{bmatrix}.$$  \hspace{1cm} (9)

$WMAV_{A_k}^{>}$ represents a WMAV of alternative $A_k$ in decision tables as follows:

$$WMAV_{A_k}^{>} = \frac{1}{n-1} \sum_{i \neq k} WAV_{A_k>A_i}.$$ \hspace{1cm} (10)

The larger the WMAV of $A_k$ is, the better alternative $A_k$ is. Therefore, all the alternatives can be ranked according to the WMAV. In this way, the best alternative can be selected.

4.6. Ranking and selecting the most desirable alternative(s) for interval number MCDM problems

In order to reduce the error of decision making, we propose a new method to rank alternatives in Section 4.5. In this section, we will use this method to rank and select the best alternative(s). For interval number MCDM problems, there are also two types of criteria, cost type criteria and benefit type criteria. Obviously, all criteria belong to benefit type in this paper.

For numerical value MCDM problems, we propose the advantage degree depending on whether the criteria values are larger or smaller. According to the interval number MCDM problems, we need to compare the expectations of criteria values and determine which one is better (Guo and Tanaka, 2010; Nikolaev and Jacobson, 2010). Thus, we get the equation as follows:

If $E(\tilde{a}) > E(\tilde{b})$, then $\tilde{a} \succ \tilde{b}.$ \hspace{1cm} (11)

The challenge of comparing interval numbers: Suppose $E(A_i/C_j)$ indicates the expectation of the criterion value of alternative $A_i$ on criterion $C_j$. Criteria values of alternatives $A_2$ and $A_4$ on criteria $C_2$ are [60, 85] and [65, 80] in Table 5, respectively. Obviously, they are two different interval numbers, but they have the same expectation that is $E(A_2/C_2) = E(A_4/C_2) = 72.5$. According to the existing method, DMs do not know which is better between [60, 85] and [65, 80]. Thus, how
can we compare the different interval numbers that has the same expectation, such as \( E(\tilde{a}) = E(\tilde{b}) \) but \( \tilde{a} \neq \tilde{b} \)?

In fact it is a behavioral problem for decision making. It was recognized by Kahneman, the award owner of the 2002 Nobel Prize in Economics. Kahneman and Tversky proposed “PT” (Tversky and Kahneman, 1981) for decision making under uncertainty according to a lot of empirical surveys. The “PT” tells us that when DMs face several different economy decision-making behaviors with the same expectation result, most DMs will choose the behavior with least risk. In other words, people are distinctively more sensitive to losses than to gains (the latter is a behavior called “loss aversion”; He and Zhou, 2011; Tversky and Fox, 1995; Tversky and Kahneman, 1992). We think that the problem of different interval numbers with the same expectation is similar to the problem in the behavioral field.

Our idea to compare the interval numbers: From the PT, we know most DMs belong to the risk aversion. According to the risk preferences based on PT, we divide DMs into two types, risk aversion and risk preference. To solve the problem mentioned above, we propose two new risk assumptions for interval number MCDM problems listed as follows. The justification of these two risk assumptions is provided in Appendix A.

**Assumption of risk aversion 1.** (Assumption of risk aversion) Suppose \( \tilde{a} = [a_L, a_U] \) and \( \tilde{b} = [b_L, b_U] \) are two interval numbers that indicate two criteria values of MCDM problems in decision tables, and \( \tilde{c}^{-*} \) is a negative ideal point for both of them. The ideal point indicates the ideal decision-making criteria values in the decision table. We get the “assumption of risk aversion” for the benefit criteria as follows:

\[
\tilde{a} > \tilde{b} \iff \begin{cases} 
E(\tilde{a}) > E(\tilde{b}) \\
E(\tilde{a}) = E(\tilde{b}) \text{ and } T(\tilde{c}^{-*}, \tilde{a}) < T(\tilde{c}^{-*}, \tilde{b})
\end{cases} \tag{12}
\]

**Assumption of risk preference 2.** (Assumption of risk preference) Suppose that two interval numbers \( \tilde{a} = [a_L, a_U] \) and \( \tilde{b} = [b_L, b_U] \) indicate two criteria values of MCDM problems in decision tables, and \( \tilde{c}^{+*} \) is a positive ideal point for both of them. The ideal point indicates the ideal decision-making criteria values in the decision table. We get the “assumption of risk preference” for the benefit criteria as follows:

\[
\tilde{a} > \tilde{b} \iff \begin{cases} 
E(\tilde{a}) > E(\tilde{b}) \\
E(\tilde{a}) = E(\tilde{b}) \text{ and } T(\tilde{c}^{+*}, \tilde{a}) > T(\tilde{c}^{+*}, \tilde{b})
\end{cases} \tag{13}
\]

In these two risk assumptions, we proposed the ideal point, and we will define it in Definition 5.

**Definition 5.** Suppose \( \tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \ldots, \tilde{x}_m^*) \) is a group of ideal alternative of MCDM problems in the decision table, where

\[
\tilde{x}_j^{+*} = [x_j^{+*L}, x_j^{+*U}] = [\max(x_{ij}^L), \max(x_{ij}^U)], \; j = (1, 2, \ldots, m): \text{ the positive ideal point (the larger, the better)}
\]

\[
\tilde{x}_j^{-*} = [x_j^{-*L}, x_j^{-*U}] = [\min(x_{ij}^L), \min(x_{ij}^U)], \; j = (1, 2, \ldots, m): \text{ the negative ideal point (the smaller, the better)}
\]

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The ideal alternative comprises the ideal points (Wang and Luo, 2009; Xu, 2004) of MCDM problems. If it comprised the positive ideal points, we call it a positive ideal alternative of MCDM problems. If it comprised the negative ideal points, it is a negative alternative of MCDM problems.

For the cost criteria, we still use the same risk assumption to rank different interval numbers with the same expectation. Like the previous discussion, we know that we need to use opposite method to determine the positive and negative ideal points for the benefit type. In other words, a positive ideal point of benefit criteria is a negative ideal for cost criteria; and a negative point of benefit criteria is a positive ideal point for cost criteria.

5. Applying the method to numerical value MCDM problems

In this section, an example of numerical value MCDM problem is used to illustrate the feasibility and validity of the proposed method. In order to demonstrate the method of criteria reduction based on tolerance relation and rough set theory, we proposed an effective tool for MCDM problems. Suppose that the DMs only have seven criteria \( C = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7\} \), such as quality \( C_1 \), competitive \( C_2 \), price \( C_3 \), design plan \( C_4 \), delivery time \( C_5 \), safety index \( C_6 \), and sale service \( C_7 \). Suppose we invite 100 experts to make judgment and vote for all the competitors based on the seven criteria, there are two types of poll results “yes” or “no.”

5.1. The procedures for numerical value MCDM problems

As denoted in Fig. 1, we need several steps to rank and select the most desirable alternative(s).

**Step 1.** Find the critical criteria. According to criteria of Table 4 using Eq. (2), we construct the DMTR and find the critical criteria as follows:

\[
DMTR = \begin{bmatrix}
\phi & C & C_2C_4C_5C_6C_7 & C & C \\
\phi & C_1C_2C_3C_5C_7 & C & C_1C_2C_4C_5 \\
\phi & C_1C_2C_3C_5C_7 & C & C \\
\phi & C_1C_2C_3C_5C_6C_7 & C & C_1C_2C_4C_5 \\
\end{bmatrix}.
\]

From previous discussion, we know DMTR is a symmetric matrix. In this paper, we only give out the upper triangular matrix.

If we want to compare alternatives \( A_1 \) and \( A_2 \), we must compare them on the criteria contained in \( m_{12} \). As proposed in Section 3, \( C \) is a set of criteria that contains all the criteria in Table 4. Thus, we need to compare all criteria in Table 4 if we want to know which is better between alternatives \( A_1 \) and \( A_2 \).

We use a relative discernibility function of discernibility matrix by using Boolean reasoning techniques (Greco et al., 2001; Guan and Wang, 2006; Skowron, 1993, 1995; Skowron and Rauszer, 2014 The Authors. International Transactions in Operational Research © 2014 International Federation of Operational Research Societies
We get \( \{C_1, C_2, C_4, C_5\} \) a group of critical criteria. If we want to compare alternatives these four criteria are needed.

**Step 2.** Obtain criteria weights of the critical criteria. Using Eq. (4) and the critical criteria of \( \{C_1, C_2, C_4, C_5\} \), we can get the criteria weights as follows:

\[
\omega_1 = \frac{1}{4}, \omega_2 = \frac{5}{18}, \omega_4 = \frac{2}{9}, \omega_5 = \frac{1}{4}.
\]

As we have denoted in Fig. 1, we need to rank and select the most desirable alternative(s) in the next step.

**Step 3.** Rank and select the most desirable alternative(s). Using Eqs. (7) and (9) as well as the critical criteria of \( \{C_1, C_2, C_4, C_5\} \), we construct the ARM as follows:

\[
ARM = \begin{bmatrix}
0.5 & 0.25 & 0.375 & 0 & 0 \\
0.75 & 0.5 & 0.764 & 0 & 0 \\
0.625 & 0.236 & 0.5 & 0 & 0 \\
1 & 1 & 1 & 0.5 & 0.361 \\
1 & 1 & 1 & 0.639 & 0.5 \\
\end{bmatrix}.
\]

Using Eq. (10), we get the WMAV for all alternatives as follows:

\[
WMAV_{A_1}^> = 0.156, \quad WMAV_{A_2}^> = 0.379, \quad WMAV_{A_3}^> = 0.215, \quad WMAV_{A_4}^> = 0.840,
\]

\[
WMAV_{A_5}^> = 0.910.
\]

Therefore, the ranking order of all the alternatives is \( A_5 \succ A_3 \succ A_2 \succ A_1 \).

Thus, the alternative \( A_5 \) is the best choice.

5.2. Validating the method using all criteria for numerical value MCDM problems

In Section 5.1, we got the alternatives order for numerical value MCDM problems of CACC using the new method. In this section, we will validate this method showing that we proposed an effective and useful tool for MCDM problems. We are going to use all the criteria in Table 4 to make a decision. If we get the same ranking order and the most desirable alternative as we have got in Section 5.1 by using the critical criteria, it means that our method is correct.

Using all criteria in Table 4 of MCDM problems to make a decision, using the Eq. (5), we obtain the criterion weights as follows:

\[
\omega_{C_1} = \frac{9}{61}, \quad \omega_{C_2} = \frac{10}{61}, \quad \omega_{C_3} = \frac{8}{61}, \quad \omega_{C_4} = \frac{8}{61}, \quad \omega_{C_5} = \frac{9}{61}, \quad \omega_{C_6} = \frac{8}{61}, \quad \omega_{C_7} = \frac{9}{61}.
\]
Using Eqs. (8) and (9) as well as all the criteria in Table 4, we get the ARM as follows:

\[
ARM = \begin{bmatrix}
0.5 & 0.279 & 0.352 & 0.131 & 0.131 \\
0.721 & 0.5 & 0.730 & 0.131 & 0.131 \\
0.648 & 0.270 & 0.5 & 0.279 & 0.148 \\
0.869 & 0.869 & 0.721 & 0.5 & 0.361 \\
0.869 & 0.869 & 0.852 & 0.639 & 0.5
\end{bmatrix}.
\]

Using Eq. (10), we get the WMAV for all alternatives as follows:

\[
WMAV_{A_1} = 0.223, \; WMAV_{A_2} = 0.428, \; WMAV_{A_3} = 0.336, \; WMAV_{A_4} = 0.705,
\]
\[
WMAV_{A_5} = 0.807.
\]

Therefore, the ranking order of all the alternatives is \(A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1\).

Obviously, we get the same ranking order and the most desirable choice in two different situations. So, finding out and using the critical criteria to make a decision is an effective and useful tool for numerical value MCDM problems, especially, when there are large number of criteria in decision tables. Thus, we find a useful method to make a decision when DMs face large number of criteria of numerical value MCDM problems.

5.3. Validating the model using maximizing deviation method for numerical value MCDM problems

In this section, we compare the results from our method with the one based on the “maximum deviation method” to obtain the criteria weights. Using the “maximizing deviation method” for interval numbers proposed by Wang and Luo (2010) and Xu (2004), we need to take all the criteria into consideration in Table 4. If we use all the criteria in Table 4 to make a decision, the ranking order of all the alternatives is also \(A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1\).

Alternative \(A_5\) is the best choice. We get the same best choice and ranking order of all alternatives.

6. Applying the method to interval number MCDM problems

In Section 5, we validated the method of criteria reduction as an effective tool for numerical value MCDM problems. In this section, we will apply this method to interval number MCDM problems, in order to demonstrate that the criteria reduction method for interval numbers MCDM problems is a feasible tool. For the same problem, suppose we invite 100 experts to vote for all the competitors on every criteria, but there are three kinds of poll results “yes” or “no” or “I do not know.” Which is the best one(s)?

As noted in Section 4, we have given a new resolution procedure for numerical value MCDM problems, when there are large number of criteria in the decision table. For the interval number MCDM problems, we still use the same resolution procedure to make a decision.
6.1. The procedure for interval number MCDM problems

**Step 1.** Find the critical criteria. In this paper, suppose that DMs take all criteria into consideration with the same threshold that is 85% of tolerance degree in Table 5. Using Eq. (3), we can construct a DMTR and find the critical criteria as follows:

\[
DMTR_{0.85} = \begin{bmatrix}
\phi & C_1C_2C_3C_4C_6C_7 & C_1C_2C_3C_4C_7 & C_1C_2C_3C_4C_6 \\
\phi & C_1C_2C_3C_4C_6C_7 & C_1C_2C_3C_4C_6 & C_1C_2C_3C_4C_6 \\
\phi & C_1C_2C_3C_4C_6C_7 & C_1C_2C_3C_4C_6 & C_1C_2C_3C_4C_6 \\
\phi & C_1C_2C_3C_4C_6C_7 & C_1C_2C_3C_4C_6 & C_1C_2C_3C_4C_6 \\
\end{bmatrix}.
\]

We take advantage of a relative discernibility function of discernibility matrix, by using Boolean reasoning techniques (Greco et al., 2001; Guan and Wang, 2006; Skowron, 1993, 1995; Skowron and Rauszer, 1992). We get \{C_1, C_2, C_3, C_4, C_7\} as a set of critical criterion, on which we rely to rank alternatives.

**Step 2.** Obtain criteria weights of the critical criteria. By using Eq. (4) and critical criteria of \{C_1, C_2, C_3, C_4, C_7\}, we get criteria weights as follows:

\[
\omega_1 = \frac{1}{5}, \omega_2 = \frac{1}{5}, \omega_3 = \frac{2}{9}, \omega_4 = \frac{1}{5}, \omega_7 = \frac{8}{45}.
\]

**Step 3.** Rank and select the most desirable alternative(s). By using these two prospect theorem models based on Eqs. (11) and (12), we can obtain the advantage relation and advantage degree that two different interval numbers have the same expectation.

For AD, ARM, and WMAV of interval numbers MCDM problems, we still use the same algorithm, such as the Eqs. (7)–(10). Comparing the WMAV to rank all alternatives and select the most desirable alternative(s).

6.1.1. The DMs belong to risk-preference type

Using Eqs. (7), (9), and (12) as well as critical criteria of \{C_1, C_2, C_3, C_4, C_7\}, we get the ARM as follows:

\[
ARM = \begin{bmatrix}
0.5 & 0.222 & 0.222 & 0.222 & 0.00 \\
0.778 & 0.5 & 0.222 & 0.622 & 0.378 \\
0.778 & 0.778 & 0.5 & 0.778 & 0.378 \\
0.778 & 0.778 & 0.222 & 0.5 & 0.278 \\
0.00 & 0.822 & 0.622 & 0.722 & 0.5 \\
\end{bmatrix}.
\]
Using Eq. (10), we can obtain the WMAV for all alternatives as follows:

\[ WMAV_{A_1}^> = 0.167, \quad WMAV_{A_2}^> = 0.5, \quad WMAV_{A_3}^> = 0.678, \quad WMAV_{A_4}^> = 0.414, \]
\[ WMAV_{A_5}^> = 0.742. \]

Therefore, the ranking order of all the alternatives is \( A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1 \). Thus, the alternative \( A_5 \) is the best choice in Table 5.

### 6.1.2. The DMs belong to risk-aversion type

Using Eqs. (7), (9), and (13), and critical criteria of \( \{C_1, C_2, C_3, C_4, C_7\} \), we get the ARM as follows:

\[
ARM = \begin{bmatrix}
0.5 & 0.222 & 0.222 & 0.222 & 0 \\
0.778 & 0.5 & 0.222 & 0.222 & 0.178 \\
0.778 & 0.778 & 0.5 & 0.6 & 0.378 \\
0.778 & 0.778 & 0.4 & 0.5 & 0.278 \\
0 & 0.822 & 0.622 & 0.722 & 0.5
\end{bmatrix}.
\]

Using Eq. (10), we can obtain the WMAV for all alternatives as follows:

\[ WMAV_{A_1}^> = 0.167, \quad WMAV_{A_2}^> = 0.35, \quad WMAV_{A_3}^> = 0.634, \quad WMAV_{A_4}^> = 0.559, \]
\[ WMAV_{A_5}^> = 0.792. \]

Therefore, the ranking order of all the alternatives is \( A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1 \). Thus, the alternative \( A_5 \) is the most desirable alternative in Table 5. Obviously, for DMs of different types, it is reasonable that we get different ranking orders.

### 6.2. Validating the method using all criteria for interval number MCDM problems

To solve this problem, we still use the same method as in Section 5.1 to demonstrate that our method is a scientific and effective tool for interval numbers MCDM problems. First, we use all the criteria to make decision in Table 5. If we can get the same ranking order and the most desirable result as Section 6.1, which means it is a useful tool. Thus, criteria reduction method based on tolerance relation and rough sets theory is an effective tool of MCDM problems especially when there are large number of criteria.

Using Eq. (5), we can obtain the criterion weights as follows:

\[ \omega_{C_1} = 9/62, \omega_{C_2} = 9/62, \omega_{C_3} = 10/62, \omega_{C_4} = 9/62, \omega_{C_5} = 8/62, \omega_{C_6} = 9/62, \omega_{C_7} = 8/62. \]
6.2.1. The DMs belong to risk-preference type

Using Eqs. (8), (9) and (13) as well as critical criteria in Table 5, we construct the ARM as follows:

\[
\begin{bmatrix}
0.5 & 0.435 & 0.363 & 0.161 & 0.145 \\
0.565 & 0.5 & 0.290 & 0.452 & 0.274 \\
0.637 & 0.710 & 0.5 & 0.565 & 0.419 \\
0.839 & 0.839 & 0.435 & 0.5 & 0.476 \\
0.855 & 0.726 & 0.581 & 0.524 & 0.5
\end{bmatrix}
\]

Using Eq. (10), we can obtain the WMAV for all alternatives as follows:

\[
WMAV_{A_1}^{\succ} = 0.276, \quad WMAV_{A_2}^{\succ} = 0.395, \quad WMAV_{A_3}^{\succ} = 0.583, \quad WMAV_{A_4}^{\succ} = 0.575,
\]
\[
WMAV_{A_5}^{\succ} = 0.672.
\]

Therefore, the ranking order of all alternatives is \(A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1\).

Thus, the \(A_5\) is still the most desirable alternative.

For the risk-preference type DMs, we get different ranking orders, but we get the same best one as Section 6.1.1.

6.2.2. The DMs belong to risk-aversion type

Using Eqs. (8), (9) and (12) as well as all the criteria in Table 5, we can get the ARM as follows:

\[
\begin{bmatrix}
0.5 & 0.435 & 0.363 & 0.161 & 0.145 \\
0.565 & 0.5 & 0.290 & 0.452 & 0.274 \\
0.637 & 0.710 & 0.5 & 0.565 & 0.419 \\
0.839 & 0.839 & 0.565 & 0.5 & 0.476 \\
0.855 & 0.871 & 0.581 & 0.524 & 0.5
\end{bmatrix}
\]

Using Eq. (10), we get the WMAV of all alternatives as follows:

\[
WMAV_{A_1}^{\succ} = 0.276, \quad WMAV_{A_2}^{\succ} = 0.286, \quad WMAV_{A_3}^{\succ} = 0.550, \quad WMAV_{A_4}^{\succ} = 0.680,
\]
\[
WMAV_{A_5}^{\succ} = 0.715.
\]

Therefore, the ranking order of all the alternatives is \(A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1\). Thus, the alternative \(A_5\) is the most desirable alternative. Obviously, the ranking order is different to Section 6.1.2, but we get the same best choice.

If we can get the same ranking order, it means that this method is a useful tool. In fact, if we get the same most desirable alternative, we can say this method is still a useful tool for interval number MCDM problems. We will explain why we get different ranking orders when DMs have the same mentality by using critical criteria and all the criteria in the next section.

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6.3. Validating the model using maximizing deviation method for interval number MCDM problems

In this section, we compare the results from our method with the one based on the “maximum deviation method” to obtain the criteria weights. Using the “maximizing deviation method” for interval numbers proposed by Wang and Luo (2010) and Xu (2004), we need to take all the criteria into consideration in Table 5.

According to the result mentioned in Section 6.2, if we use all the criteria in Table 5, for “risk revision” DMs, the ranking order of all the alternatives is listed as follows:

\[ A_5 > A_4 > A_3 > A_2 > A_1. \]

For “risk-preference” type DMs, the ranking order of all the alternatives is listed as follows:

\[ A_5 > A_3 > A_4 > A_2 > A_1. \]

Obviously, the ranking of all alternatives are different with these two methods, but we get the same best choice. In the next section, we explain why we get different ranking orders using critical criteria and all the criteria separately in Table 5, when DMs have the same risk preferences.

6.4. Analysis of validated results

For interval number MCDM problems, there are different criteria values with the same expectation. We proposed two assumptions according to DMs’ preference of risk (generally speaking there are two types: risk version and risk preference). It is reasonable that different decision results are made by DMs of different types. If we still get different ranking orders when DMs belong to the same type in two situations, we think there are at least two reasons as follows.

First, alternatives are incomparable. Any alternative located in the advantageous position on all criteria does not exist. Most criteria of an alternative locate in a more advantageous position than others, but only a few criteria locate in the disadvantageous position. We think this alternative does not locate in a more advantageous position in MCDM problems.

Second, the threshold we used could be higher than it should be. We tried different thresholds to obtain better.

As previously stated, we got a new decision process method to address “larger decision table” (e.g. large number of criteria) in this paper. The following is a comparison of “our new decision process method” and the “traditional procedure” with large number of criteria MCDM problems (Figs. 2 and 3).

7. Discussion and conclusion

This paper mainly discusses how to make a wise decision within limited time when DMs face large number of criteria of MCDM problems (“numerical values and interval numbers,” respectively).
Large decision table (e.g. large number of criteria)

Finding useful criteria

Obtaining criteria weights for useful criteria

Information aggregation using useful criteria

Ranking and selecting alternatives via WMAV

Fig. 2. Our new decision process method to address the large decision table.

Large decision table (e.g. large number of criteria)

Standardizing criteria

Obtaining criteria weights for all criteria

Information aggregation using all criteria

Ranking and selecting alternatives via EWCV

Fig. 3. The traditional procedure to address the large decision table.

For MCDM problems with numerical values: It contributes to MCDM problems from three aspects. (a) Finding useful criteria based on DMTR; (b) obtaining criteria weights based on DMTR; and (c) using WMAVs to rank and select the best alternative.

For MCDM problems with interval numbers: It contributes to MCDM problems from four aspects. (a) Finding useful criteria based on the tolerance degree of interval numbers and DMTR; (b) obtaining criteria weights based on the tolerance degree of interval numbers and DMTR; (c) building two different relationship assumptions (risk aversion and risk-preference assumption, respectively) based on risk preferences of DMs and tolerance degree of interval numbers; and (d) ranking and selecting the alternatives by WMAVs built for preferences of DMs separately.
At the end of the paper, we validated the method of finding out the critical criteria through criteria reduction. It is demonstrated to be an applicable and effective method for MCDM problems. Then, a new decision process method based on the criteria reduction is proposed for MCDM problems with numerical values and interval numbers, respectively.

In the future, we will improve the method to find the critical criteria based on decision-theoretic rough sets. Also, we will study the case of MCDM problems when criteria are dependent. In addition, we will study the decision process and decision strategy for MCDM problems with large numbers of alternatives.

Acknowledgments

This research is supported by the National Natural Science Foundation of China (no. 71301075), National Natural Science Foundation of Jiangsu Province, China (no. BK20130770), China Post-doctoral Science Foundation funded project (no. 2013M530261), Postdoctoral Science Foundation funded project of Jiangsu Province, China (no. 1301108C), Research Fund for the Doctoral Program of Higher Education (no. 20123219120032), Fundamental Research Funds for the Central Universities, China (no. 30920130132014), Young Teachers Research Foundation in School of Economics and Management, Nanjing University of Science and Technology, China (no. JGQN1401), and Zijin Intelligent Program, Nanjing University of Science and Technology, China. We would like to thank the editor and reviewers for their detailed and constructive comments.

References


Appendix: Risk assumption justification

As discussed in Section 3.2.3, the tolerance degree of alternatives $A_3$ and $A_5$ on criterion $C_1$ indicates the percentage of the overlap of value ranges on $C_1$. Then, when the tolerance degree of an alternative’s criteria values and positive ideal point is larger than others, it means that this alternative overlaps more than others with positive ideal point. This alternative is located at a more advantageous position than others. Similarly, if the tolerance degree is the smallest between this criterion value and negative ideal point, it means that this criterion value is also the best one on this criterion.

According to Definition 1, different interval numbers have different value ranges even if they have the same expectation. In addition, different interval numbers with the same ideal point will have different tolerance degrees although their expectations are the same. According to Definition 5, the ideal point is confirmed and fixed that comprised two criteria values in decision tables. So, different interval numbers with the same ideal point will have different tolerance degree. Thus, we can rank and select these interval numbers with the same expectation by comparing the tolerance degrees among them with the ideal point.

Since different DMs have different risk preferences, so different DMs will choose different ideal alternatives as reference alternatives. The goal is to choose a reasonable reference alternative to count the tolerance degree when we want to validate these two theories. So, the experts who answer “I do not know” become quite important under uncertain circumstances. If DMs predict that these experts would answer “yes,” they choose the positive ideal alternative as the reference alternative. If the DMs predict that these experts would answer “no,” they choose the negative ideal alternative as
the reference alternative. No matter these experts would answer “yes” or “no,” it is only a prediction of DMs. If the DMs belong to the risk aversion, they will pay more attention to the worst situation. If the DMs belong to the risk preference, they will consider all the advantages of every situation. DMs of the risk-aversion type pay their attention to the present. They are pessimistic about future. Thus, the risk aversion of DMs will choose the negative ideal alternative as the reference alternative. On the other hand, DMs of the risk-preference focus more on the future, and they hold positive attitude toward the future. So, DMs of the risk preference will choose the positive alternative as the reference alternative.

Two criteria values of alternatives $A_2$ and $A_4$ have the same expectation on criterion $C_2$ in Table 5, they are $[60, 85]$ and $[65, 80]$, respectively. We get the positive ideal and negative ideal points, which are $x^{++} = [65, 85]$ and $x^{--} = [60, 80]$ according to Definition 5.

For DMs of the risk-aversion type, DMs will choose the negative ideal point of $x^{--} = [60, 80]$ as the reference point. The tolerance degree between the criteria value of alternatives $A_2$ and $A_4$ on criteria $C_2$ and the negative ideal point is $x^{--} = [60, 80]$, we get them as follows:

Using Eq. (1), we get the tolerance degrees $T(\tilde{f}(A_2, C_2), x^{--})$ and $T(\tilde{f}(A_4, C_2), x^{--})$ as follows:

$$ T(\tilde{f}(A_2, C_2), x^{--}) = \frac{4}{5}, \quad T(\tilde{f}(A_4, C_2), x^{--}) = \frac{3}{4}. $$

Obviously,

$$ T(\tilde{f}(A_2, C_2), x^{--}) > T(\tilde{f}(A_4, C_2), x^{--}). $$

Thus, the advantage order is $A_2 \prec A_4/C_2$.

That is to say the alternative $A_4$ is better than $A_2$ on criterion $C_2$.

In other words, in Table 5, $A_4$ has 65 affirmative votes and $A_2$ has 60 affirmative votes. And, $A_4$ may get 80 affirmative votes and $A_2$ may get 85 affirmative votes in the future. For DMs of the risk aversion, they think $A_4$ is better than $A_2$ on criterion $C_2$. For DMs of the risk aversion, they pay more attention to the present. They are pessimistic about the future. Similarly, the DMs can directly use the lower bound as the rule to judge alternatives in risk-aversion situation.

For DMs of the risk-preference type, they will choose the positive ideal point of $x^{++} = [65, 85]$ as the reference point. The tolerance degree between the criteria values of alternatives $A_2$ and $A_4$ on criterion $C_2$ with the positive ideal point is $x^{++} = [65, 85]$, we get them as follows:

Using Eq. (1), we get the tolerance degrees $T(\tilde{f}(A_2, C_2), x^{++})$ and $T(\tilde{f}(A_4, C_2), x^{++})$ as follows:

$$ T(\tilde{f}(A_2, C_2), x^{++}) = \frac{4}{5}, \quad T(\tilde{f}(A_4, C_2), x^{++}) = \frac{3}{4}. $$

Obviously,

$$ T(\tilde{f}(A_2, C_2), x^{++}) > T(\tilde{f}(A_4, C_2), x^{++}). $$

Thus, the advantage order is $A_2 \succ A_4/C_2$.

Therefore, the alternative $A_2$ is better than $A_4$ on criteria $C_2$.

In other words, in Table 5, $A_4$ has 65 affirmative votes and $A_2$ has 60 affirmative votes. In future, $A_4$ may get 80 affirmative votes and $A_2$ may get 85 affirmative votes. For DMs of the risk preference, they think $A_4$ is better than $A_2$ on criterion $C_2$. Because DMs of risk-preference focus on the future...
and they hold positive attitude toward the future. Similarly, the DMs can directly use the upper bound as the rule to judge alternatives in risk-preference situation.

Alternative $A_4$ has 65 and $A_2$ has 60 affirmative votes on criterion $C_2$ in Table 5, respectively. $A_4$ is better than $A_2$ according to the number of affirmative votes. On the other hand, $A_4$ get 20 and $A_2$ get 15 negative votes, respectively. Through comparing the number of negative votes, we can find that $A_2$ is better than $A_4$ on criterion $C_2$. The difference comes from the fact that some experts answer “I do not know” in the voting process. If those experts finally answer “yes,” it means that $A_4$ and $A_2$ will get 80 and 85 affirmative votes, respectively, at last. Thus $A_2$ is better than $A_4$. If those experts answer “no” finally, it indicates $A_4$ and $A_2$ will get 65 and 60 affirmative votes, respectively, at last. Thus, $A_4$ is better than $A_2$.

No matter which one is considered as reference alternatives for DMs, it is reasonable from the perspective of the affirmative votes or negative votes. But it makes great difference in these two situations. By using different ideal alternatives as reference alternatives to calculate the tolerance degrees and compare them, we will get different ranking orders. How rank different interval numbers with the same expectation depends on the risk preferences of DMs and tolerance degree of interval numbers? All in all, we validate these two risk assumptions as a scientific and useful tool for ranking interval numbers.