The impact of consumer returns policies on consignment contracts with inventory control

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\textbf{Abstract}

We consider a consignment contract with consumer non-defective returns behavior. In our model, an upstream vendor contracts with a downstream retailer. The vendor decides his consignment price charged to the retailer for each unit sold and his refund price for each returned item, and then the retailer sets her retail price for selling the product. The vendor gets paid based on net sold units and salvages unsold units as well as returned items in a secondary market. Under the framework, we study and compare two different consignment arrangements: the retailer/vendor manages consignment inventory (RMCI) programs. To study the impact of return policy, we discuss a consignment contract without return policy as a benchmark. We show that whether or not the vendor offers a return policy, it is always beneficial for the channel to delegate the inventory decision to the vendor. We find that the vendor's return policy depends crucially on the salvage value of returns. If the product has no salvage value, the vendor's optimal decision is not to offer a return policy; otherwise, the vendor can gain more profit by offering a return policy when the salvage value turns out to be positive.

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1. Introduction

With the emergence of virtual marketplaces, consignment selling has been thriving at an unprecedented pace. Under such a consignment contract, the vendor retains ownership of the inventory and gets paid from the retailer based on the net sold units. The arrangement is especially popular in online marketplaces, such as Amazon.com, Alibaba.com, and eBay.com (Li et al., 2009).

The online channel provides consumers with only a virtual description of the product, using text, graphics, or symbols in a paper or web page catalog. This eliminates the use of touch, taste, smell and may cause evaluation mistakes by shoppers. After purchasing, when a customer further experiences the product, he/she may not like the product as much as anticipated, and will then return the product to the retailer for a refund. So the saliency of returns is unquestionable in such items as toys, Christmas decorations, books, seasonal/fashion items and the like. In consignment selling, the consumer returns are sent to the vendor through the retailer. At the end of the selling season, the retailer returns the unsold units together with consumer returns to the vendor. In order to differentiate consumer-retailer returns from retailer-vendor returns, we use the term “consumer returns” to refer to the consumer-vendor agreement, and “channel return” to refer specially to agreements between the vendor and its retailer. The vendor can salvage all the returned items in secondary and global markets.

The practice of returns policy has been reported widely in both research literature and business, see Bose and Anand (2007). The format of returns policy varies in and across industries. The most generous policy promises to refund the full price for all returned products, while less generous policies only provide partial credits or refund. Under the consignment contract considering consumer returns, the upstream vendor decides his consignment price charged to the retailer for each unit sold and the refund price for returned item. The retailer then chooses her retail price for selling the product to the market. The retailer pays the vendor based on the net sale (total sales minus total returns).

With consignment, the decision about how much inventory to hold in a period can be operated in one of two ways: the traditional way, in which the downstream retailer decides the inventory level, or the new way in which inventory-keeping responsibility and stock level decision are switched to the upstream vendor. The latter arrangement is called Vendor Managed Consignment Inventory (VMCI), and we label the former as Retailer Managed Consignment Inventory (RMCI). Both arrangements take place in practice, however VMCI seems to be a trend. In particular, many big retailers, such as Wal-Mart, Target, Ahold USA, and Meijer Stores, have...
implemented or are considering the implementation of a VMCI arrangement \cite{Lee2005, Rungtusanatham2007}. There have been some continued debates among practitioners as to who should be responsible for the level of consigned inventory in the supply chain. Ru and Wang \cite{Ru2010} study who should control the inventory under a consignment contract without considering possible product returns.

In this paper, we analyze a return policy and inventory control problem under a consignment contract, in which the market demand is uncertain and consumers' post-purchase valuation of the product is also uncertain. The upstream vendor decides his consignment price charged to the retailer for each sold item and his refund price for each returned item before any uncertainty is resolved. The retailer then chooses her response retail price for selling the product to market. Under this general framework, we consider two scenarios regarding who should make a decision about the supply chain inventory. The first scenario is RMCI, in which the retailer controls the inventory; the second is VMCI where the vendor controls the inventory.

To benchmark the channel and individual firm's performance under a consignment contract with return policy, we also consider the channel managed under an alternative contract without return policy. Especially, we are interested in: (i) who should control the inventory under a consignment contract with/without return policy, further, (ii) what is the impact of return policy on consignment contacts with inventory control, and (iii) should the upstream vendor offer a return policy to the unsatisfied customers or not.

The above inventory control issue has previously been tackled in a similar framework by Ru and Wang \cite{Ru2010} (henceforth referred to as R&W). They assume the leftovers have no salvage value and they do not focus on the prevalent consumer return phenomenon because of unsatisfying purchase. R&W then find that it is always beneficial for both parties to delegate the inventory decision to the vendor rather than to the retailer. The non-negative salvage value and the customers' uncertain post-purchase valuation in our analysis, however, cover a much wider range and randomness compared to R&W. Consequently, we are able to generate analytical results with greater scope and application. Moreover, we show that it is profitable to both parties if the vendor takes charge of the inventory decision. We also get that if the salvage value is zero, then the vendor's optimal decision is not to offer a return policy; otherwise, the vendor can gain more profit by offering a return policy. In this sense, our primary contributions are in unifying previous results and in deriving more general results within this area.

Specifically, we consider a supply chain consisting of a vendor and a retailer, in which the vendor contracts with the retailer to sell products through a consignment contract during a single selling season, and the vendor and the retailer each incurs a linear cost for producing and handling the product. The market demand is price-sensitive and uncertain. When the customer further experiences the product after the purchase, if he/she is not satisfied with the product, he/she will return the product back to the vendor through the retailer for a refund. Before the demand uncertainty is resolved, the vendor offers a consignment price charged to the retailer for each unit of product sold and the refund price for each returned unit, and then the retailer chooses a retail price for selling the product to the market. After the demand uncertainty is realized, a portion of consumers will return the product, the proportion depending on the refund price. At the end of the selling season, the retailer pays the vendor based on the net selling units, and the vendor can salvage both the consumer returned items and unsold items in a secondary market. Under the framework of consignment contract with consumer returns, we compare two scenarios regarding who makes the decision about the supply chain inventory. The first scenario is an RMCI setting where the retailer controls the inventory and determines the retail price; the second is VMCI in which the vendor controls the inventory level, and determines consignment price as well as the refund price.

With the multiplicative exponential demand function, we derive the unique equilibrium solutions for both settings with/without return policy. We show that both RMCI and VMCI always lead to a 50–50 or equal split of the achieved channel profit between the vendor and the retailer. We also find that with/without return policy, both parties should prefer VMCI over RMCI. Furthermore, whether the vendor should offer a return policy depends crucially on the salvage value of the product. If the salvage value turns out to be zero, the vendor’s best choice is not to provide a return policy; otherwise, the vendor can earn more profit with return policy compared to without return policy. To sum up, motivated by consignment contracts that have been popularly applied in online marketplace and large quantity of product returns (the readers can refer to http://www.w7collective.com/ and http://www.ehow.com for details of specific examples about the agreement between vendor and retailer), our works include: (1) constructing a game model of a consignment contract that involves the common phenomenon of consumer returns and salvage value; (2) deriving the equilibrium solution of the model and showing the uniqueness of the solution; (3) proposing an optimal inventory control strategy; and (4) showing that the vendor’s return policy depends crucially on the salvage value.

The paper proceeds as follows: in Section 2, we provide a brief review of related literature. The model assumption is presented in Section 3. Sections 4 and 5 offer analysis of both RMCI and VMCI with/without return policy. A comparison between no return policy and return policy is provided in Section 6. Finally, concluding remarks are presented in Section 7.

2. Related literature

The model setting we consider is a combination of two distinctive features: (1) consignment contract with inventory decision; and (2) the emerging area of research on consumer return behavior because of misfit. In the following, we provide a brief review of papers that relate to these model features.

The first stream of research is relevant to consignment contracts with inventory ownership. Wang et al. \cite{Wang2004} consider a pure consignment contract, in which the vendor retains full ownership of the product and bears all the risk associated with overstocking. Lee and Chu \cite{Lee2005} address the issue of who should control the supply chain inventory. In their model, the consignment price and retail price are exogenously given parameters, different from decision variables in our model. Based on Lee and Chu’s \cite{Lee2005} work, R&W investigate the similar model, in which the consignment price, retail price as well as the inventory ownership are decisions, and they derive that it is always beneficial for the system to delegate the inventory decision to the vendor. All the above-mentioned papers assume no salvage value and do not include consumer returns. Our research extends R&W’s model by assuming non-negative salvage value, and the incorporation of returns into R&W’s model with salvage value of leftovers is a new contribution to the literature.

The second relevant literature concerns research on the consumer return problem. A large portion of consumer returns are non-defective, but are returned only because of not fitting customer’s taste or expectation. Sciarrotta \cite{Sciarrotta2003} reports that the non-defective returns rate was very high. The impact of those returns on the bottom line was significant, amounting to tens of millions of dollars in losses. Lawton \cite{Lawton2008} points out that only about 5% of consumer returns were truly defective. A customer buy a product that performs effectively and properly, but the product may not match his/her taste and expectation, and thus the customer will return it back to the retailer (Davis et al., 1995; Anderson et al., 2009).
Cohen et al. (1980) study a model where a fixed fraction of demand is returned. Vlachos and Dekker (2003) assume that consumer returns are a fixed proportion of quantity sold. Mostard and Teunter (2006) address the return problem, by restricting attention to a fixed retail price and to a one-period newsvendor problem with consumer returns that arrive back before the end of the selling season undamaged and could be resold. They also assume a fixed return rate. Chiu et al. (2011) consider a policy that combines the use of wholesale price, channel rebate, and returns can coordinate a channel with both additive and multiplicative price-dependent demands. Chiu et al. (2009) illustrate the use of service level can prevent a drop in price and profit. Anderson et al. (2006) conduct a field test for customers buying from a mail-order catalog of women’s fashion clothing and find that the number of returns has a strong positive linear relationship with the quantity sold. Mukhopadhyay and Setoputro (2005), Su (2009) and Liu et al. (2012) model a higher refund price that induces a higher return rate. Chen and Bell (2009) focus on full refund policy and examine cases where the quantity of returned product is a function of both the quantity sold and the refund price. Matsui (2010) investigates how economic outcomes differ for the introduction of a model under contracts with full buyback and no buyback. Chen and Bell (2010) examine how consumer returns influence the retailer’s ordering decision when the retail price is determined exogenously and the customer’s post-purchase valuation is a random variable.

Research highlights: our main contributions can be summarized as follows: (1) we investigate the consignment contract and consumer return behavior problem in two different supply chain inventory mechanisms: VMCI and RMCI. Models and results in our paper are broader in scope than those aforementioned studies which do not take consumer returns into consideration; (2) our model incorporates post-purchase valuation uncertainty as well as demand uncertainty simultaneously. Moreover, we also consider consumer returns that depend on decision variables; (3) we derive the unique solutions under VMCI and RMCI for both with and without return policy, and more importantly, and (4) by comparing performances in both scenarios with and without return policy, we provide managerial insights for practical application. Specifically, when we extend R&W’s model by only assuming a non-negative salvage value, our results turn out to be the same as that in R&W, which has both the vendor and retailer prefer VMCI over RMCI. When we contribute to R&W’s model by considering the retailer’s returns with salvage value of leftovers. We also find that it is always beneficial for both parties to let the vendor control the inventory decision. Furthermore, we investigate that whether the upstream vendor should offer a return policy to the unsatisfied customer or not depends on the salvage value. If the salvage value turns out to be zero, the vendor’s best choice is not to provide a return policy; otherwise, the vendor will offer a return policy.

3. Model assumptions

We consider a single-period supply chain model in which a vendor contracts with a retailer. The upstream vendor (he) produces a product and sells it through the downstream retailer (she). The vendor decides his consignment price charged to the retailer for each unit sold and the unit refund price for returned items, and the retailer chooses her retail price for selling the product to consumers. The vendor’s unit production cost is $c_v$ and the retailer’s unit handling cost is $c_r$. Define $c = c_v + c_r$ as the unit total cost for the channel, and $s = c/c_r$ as the share of the channel cost that is incurred by the retailer. The market demand is price-dependent and uncertain. We use the general multiplicative function to model the demand

$$d(p) = y(p)/e,$$

where $p$ is the selling price, $e$ represents randomness of the demand, and $F(\cdot)$ and $f(\cdot)$ are its CDF and PDF, respectively. $y(p)$ is a deterministic and decreasing function of $p$. As in R&W, we assume that $y(p)$ takes the form of

$$y(p) = ae^{-bp} \text{ with } a, b > 0. \quad (2)$$

The empirical studies (Anderson et al., 2009; Hess and Mayhew, 1997) show that the quantity sold has a strong positive linear relationship with the number of returns. Making use of these results in the literature, we also consider customer behavior in modeling the consumer returns function. We assume that $r$ is the post-purchase valuation of the product as perceived by a customer, which is assumed to be a random variable with cumulative function $G$. After the purchase, when the customer further experiences the product, if the value is higher than the refund price, the customer will keep the product; otherwise, he/she will return the product to the retailer for a refund.

At the beginning of the selling season, the vendor offers a consignment price $w$ charged to the retailer for each unit of product sold and a refund price $r$ for returned items. The retailer then determines a retail price $p$ for selling the product to the market. After demand uncertainty is realized, consumers may return the product to the vendor through the retailer after purchasing. As in Chen and Bell (2010), we consider consumer returns that depend on the vendor’s refund $r$. If the customer’s post-purchase valuation is lower than the refund, he/she will return the product; otherwise, he/she will keep the product. So the probability that a consumer returns the purchased product is then defined as $G(r) = \text{Prob}(v < r)$, which is interpreted as the consumer returns rate, and is increasing in the price $r$. At the end of the selling season, the channel returns occur. As we concentrate on non-defective returns, we assume the vendor salvages all leftover units at a same salvage value $s$, where $s < (1 - s)c$. The salvage value could be realized in a number of ways including a mark-down on the product (typical with records, and books), transfer of the product into an alternative use, sale of the product for scrap or recycling, or donation of the product to charity (Pasternack, 1985).

Under this general framework, we focus on and compare two settings regarding who should make the decision about the supply chain inventory. The first is RMC, in which the retailer chooses the inventory, the second is VMCI where the vendor chooses the inventory level for the supply chain. Under RMCI, decisions are made in following steps:

**Step 1:** the vendor specifies the consignment price $w$ to determine the amount of payment that he will get from the retailer for each net sold unit and refund price $r$ for each sold but returned unit.

**Step 2:** the retailer decides the quantity $Q$ for the vendor to deliver and the retail price $p$ for selling the product to the market.

**Step 3:** the demand realization is given by (1). After purchase, when consumers further experience the product, some of them may return the product, and the return rate is $G(r)$.

**Step 4:** the retailer returns the unsold items and consumer returns back to the vendor. The vendor gets paid from the retailer based on the net sale and salvages all the leftovers in a secondary market.

VMCI differs from RMCI only in the inventory decision, where the quantity decision of the retailer in Step 2 is shifted to the vendor in Step 1. So under a VMCI situation, in Step 1, the vendor chooses the consignment price $w$, the refund price $r$, and the quantity $Q$ to produce and deliver to the retailer. In Step 2, the retailer only decides the retail price $p$. 
4. The model with no consumer returns policy

In order to benchmark the total channel and individual firm’s performance under a consignment contract with consumer returns, we also consider the inventory control problem under a consignment contract without consumer returns. The latter modifies the former only by setting \( r = 0 \), namely, “no returns allowed”. To differentiate both the scenarios with return policy and no return policy, we add “n” in superscript to denote the “no returns allowed” case. Compared to R&W, the model proposed in this section considers the salvage value of leftovers.

We first characterize the solutions of a centralized channel, which will serve as a benchmark for comparing the performance of decentralized channels. For a centralized channel, the decision is to simultaneously choose the selling price \( p \) and the downstream retailer chooses her re-

\[
\Pi^c_R(p, Q) = pE\min\{D, Q\} + s(Q - E\min\{D, Q\}) - cQ.
\]  

(3)

Following Petruzzi and Dada (1999), we define \( z = Q(y(p), \text{ and then maximizing } \Pi^c_R(p, Q) \) over \( p, Q \) is equivalent to that over \( p, z \). Actually, each value of \( z \) corresponds to a unique \( Q \), and choosing a value for \( z \) is equivalent to setting up a customer service level for the system. So, for convenience in our later discussions, we will designate the customer service level as \( z \).

Now substituting \( Q = y(p)z \) into (3), the objective function can be rewritten as

\[
\Pi^c_R(p, z) = y(p)\{pE\min\{c, z\} + s(z - E\min\{c, z\}) - cz\} = y(p)\{p[z - A(z)] + s(A(z) - cz)\}.
\]  

(4)

where \( A(z) = \int_0^z (z - x)f(x)dx \).

The next theorem characterizes the system’s optimal decision \((p^c_R, z^c_R)\).

**Theorem 1.** For the centralized system without a return policy, for any given \( z \), the unique optimal retail price \( p^c_R(z) \) is given by

\[
p^c_R = \frac{1}{b} - \frac{s}{b} \frac{A(z^c_R)}{z^c_R - A(z^c_R)} + \frac{c}{b} \frac{z^c_R}{z^c_R - A(z^c_R)}.
\]  

(6)

Moreover, if the probability distribution function \( f(.) \) satisfies the property of increasing failure rate (IFR), then the optimal service level \( z^c_R \) is determined by

\[
1 - F(z^c_R) = \frac{z^c_R}{b} \frac{A(z^c_R)}{z^c_R - A(z^c_R)} = \frac{1}{b(c - s)}
\]  

(7)

Note that failure rate \( h(x) = f(x)[1 - F(x)] \) is increasing in \( x \), as required by Theorem 1. It is a relatively weak condition, which can be satisfied by most commonly used probability distributions, such as Uniform, Normal, and exponential (please see Barlow and Proschan, 1965 for details). Substituting (6) and (7) into (4), we derive the total optimal channel profit as

\[
\Pi^c_T = ae\left\{z^c_R - A(z^c_R)\right\} - \frac{1}{b(c - s)}
\]  

(8)

4.1 Decentralized channel decisions and performance under RMI

In the decentralized channel under RMI, the vendor sets his consignment price, and the downstream retailer chooses her response retail price and quantity level. We begin our analysis by focusing on the retailer’s decision.

4.1.1. The retailer’s retail price and order quantity

For given consignment price \( w \) in the first step, the retailer’s problem is to simultaneously choose the retail price \( p \) and order quantity \( Q \) in the second step to maximize her own expected profit

\[
\Pi^R(p, Q|w) = (p - w)E\min\{D, Q\} - czQ,
\]  

(9)

where the superscript \( R \) refers to the retailer, the subscript \( R \) refers to the setting under RMI, and other notations introduced later indicate the similar meaning. If we substituting \( Q = y(p)z \) in (9), then choosing \((p, Q)\) is equivalent to choosing \((p, z)\) for the retailer. Hence, we can rewrite the above profit function as

\[
\Pi^R(p, z|w) = y(p)\{(p - w)[z - A(z)] - caz\}.
\]  

(10)

From R&W, if the demand distribution satisfies the property of increasing failure rate, the optimal retail price and quantity decision are given by

\[
p^R_F(w) = \frac{1}{1 - F(z^R)} - \frac{z^R}{b} - \frac{A(z^R)}{b} + 1.
\]  

(11)

\[
1 - F(z^R) = \frac{z^R}{b} \frac{A(z^R)}{z^R - A(z^R)} = \frac{1}{b(c - s)}
\]  

(12)

From (11), we know that the retailer’s optimal selling price consists of the amount \( w \) paid to the vendor and a margin for herself. (12) implies that when the retailer chooses the supply chain inventory, her optimal order quantity is independent of the vendor’s consignment price \( w \) and salvage price \( s \), but is uniquely determined by the demand function and system parameters.

4.1.2. The vendor’s consignment price

Knowing that the retailer chooses \((p^R_F, z^R)\) according to (11) and (12), the vendor then aims to set the consignment price \( w \) to maximize his own expected profit, which is expressed as

\[
\Pi^R(w) = wE\min\{D, Q^w\} + s\{Q^w - E\min\{D, Q^w\}\} - c(1 - z)Q^w
\]  

\[
y(p^R_w)\{w[z^R - A(z^R)]\} + sA(z^R) - c(1 - z)z^R
\]  

\[
= ae\left\{W^\text{w} - \frac{A(z^R)}{b} + c(1 - z)\frac{z^R}{b} - A(z^R)\right\}.
\]  

(13)

where \( Q^w = y(p^R_w)z^R \).

**Theorem 2.** For RMI without a return policy, the vendor’s optimal consignment price \( w^R \) is given by

\[
w^R = \frac{1}{b} - \frac{s}{b} \frac{A(z^R)}{z^R - A(z^R)} + \frac{c}{b} \frac{z^R}{z^R - A(z^R)}.
\]  

(14)

Substituting Eq. (14) in (11), we get

\[
p^R_w = \frac{2}{b} - \frac{s}{b} \frac{A(z^R)}{z^R - A(z^R)} + \frac{c}{b} \frac{z^R}{z^R - A(z^R)}.
\]  

(15)

Substituting all the optimal decisions back into the vendor’s and retailer’s profit functions, we derive the vendor’s and retailer’s profit functions as

\[
\Pi^R_{w} = ae\left\{z^R - A(z^R)\right\} - \frac{1}{b(c - s)}
\]  

(16)

\[
\Pi^R_{p} = ae\left\{z^R - A(z^R)\right\} - \frac{1}{b(c - s)}
\]  

(17)

From the above two equations, we find that the vendor and the retailer each obtain 50% of the total channel profit, which is

\[
\Pi^R_{w} = \Pi^R_{p} = 2ae\left\{z^R - A(z^R)\right\} - \frac{1}{b(c - s)}.
\]  

(18)

4.2. Decentralized channel decisions and performance under VMCI

Under VMCI, the vendor chooses the consignment price \( w \), together with the quantity decision \( Q \) in the first step, and the
retailer only decides the retail price \( p \) in the second step. Following the same procedure as in the former subsection, we begin our investigation by discussing the retailer’s decision.

### 4.2.1. The retailer’s retail price

In the second step, for given \( w \) and \( z \) chosen by the vendor in the first step, the retailer determines \( p \) appropriately to maximize her own expected profit,

\[
\Pi_v^w(p|w, z) = y(p)\{p - w|z - A(z)| - c\, z\}.
\]

(19)

Similar to the derivation of (11), for any given \( w \) and \( z \), the optimal retail price is

\[
p_v^w(w, z) = w + \frac{cz\, z}{z - A(z)} + \frac{1}{b}.
\]

(20)

As in the RMCI scenario, (20) indicates that the retailer’s optimal selling price consists of two parts: the amount she has to pay to the vendor for each unit sold and a mark-up for herself.

### 4.2.2. The vendor’s decision

In the first step, knowing that the retailer’s best response is (20), the vendor decides the consignment price \( w \) and service level \( z \) to maximize his expected profit, which is given by

\[
\Pi_v^w(w, z) = ae^{-b\, p_v^w(w, z)} \{w|z - A(z)| + s\, A(z) - c(1 - z)\, z\}
\]

\[
= ae^{-b\, w + \frac{cz\, z}{z - A(z)}} \{w|z - A(z)| + s\, A(z) - c(1 - z)\, z\}.
\]

(21)

The vendor’s optimal solution can be characterized by the following theorem.

**Theorem 3.** For VMCI without a return policy, for any given \( z \), the vendor’s optimal wholesale price is

\[
w_v^w(z) = \frac{1}{b} - s\frac{A(z)}{z - A(z)} + c\frac{(1 - z)\, z}{z - A(z)}.
\]

(22)

Moreover, if the probability distribution \( f(z) \) satisfies the property of increasing failure rate (IFR), his optimal service level \( z \) is determined by

\[
\frac{1}{1 - F(z_v^w)} - \frac{z_v^w}{z_v^w - A(z_v^w)} = b(c - s).
\]

(23)

Recall that under RMCI, the quantity decision made by the retailer is independent of the vendor’s decisions and salvage value, while under VMCI, the retailer’s order quantity decision depends on the vendor’s salvage value \( s \) and \( z_v^w = z_v^r \).

Note that the expression of the consignment price here in (22) is the same as that of (14) under RMCI. However, the quantity decision here is different from the counterpart under RMCI, so the consignment price is not the same for the two scenarios.

Substituting (22) and (23) back into (20), we can get

\[
p_v^w = \frac{2}{b} - s\frac{A(z_v^w)}{z_v^w - A(z_v^w)} + c\frac{z_v^w}{z_v^w - A(z_v^w)}.
\]

(24)

Correspondingly, we can present the vendor’s and retailer’s profit functions as:

\[
\Pi_v^{w,z} = ae^{-b\, \left(\frac{z_v^w - A(z_v^w)}{z_v^w - A(z_v^w)}\right)} \left(\frac{z_v^w - A(z_v^w)}{b}\right).
\]

(25)

\[
\Pi_v^{w,z} = ae^{-b\, \left(\frac{z_v^w - A(z_v^w)}{z_v^w - A(z_v^w)}\right)} \left(\frac{z_v^w - A(z_v^w)}{b}\right).
\]

(26)

From the above two equations, we find that the vendor and the retailer each obtain 50% of the total channel profit, which is

\[
\Pi_v = \Pi_v^{w,z} + \Pi_v^{w,z} = 2ae^{-b\, \left(\frac{z_v^w - A(z_v^w)}{z_v^w - A(z_v^w)}\right)} \left(\frac{z_v^w - A(z_v^w)}{b}\right).
\]

(27)

### 4.3. Channel performance

Based on the results obtained in the former subsections, we can make a comparison of the two decentralized channel decisions with their centralized counterparts in this section. To that end, we need the following lemma about the property of any general distribution.

**Lemma 1.** Let \( K(z) = \frac{1}{1 - F(z)} - \frac{z}{z - A(z)} \) with \( A(z) = \int f(x)\, dx \). If the failure rate function \( h(x) = \frac{f(x)}{1 - F(x)} \) is increasing in \( x \), then \( K(z) \) is increasing in \( z \).

With this lemma, we can obtain the following lemma.

Let

\[
\lambda(z) = e^{-b\, \left(\frac{1}{1 - F(z_v^w)} - \frac{z_v^w}{z_v^w - A(z_v^w)}\right)} \left(\frac{z_v^w - A(z_v^w)}{b}\right).
\]

(28)

Then, we can rewrite \( \Pi_v^{w,z} \), \( \Pi_v^{w,z} \), and \( \Pi_v^{w,z} \) as

\[
\Pi_v^{w,z} = ae^{-b\, \left(\frac{z_v^w - A(z_v^w)}{z_v^w - A(z_v^w)}\right)} \left(\frac{z_v^w - A(z_v^w)}{b}\right).
\]

**Lemma 2.** \( \lambda(z) \) is a unimodal function that reaches its maximum at \( z_v^w \).

**Proposition 1.** \( \frac{\Pi_v^{w,z} - \Pi_v^{w,z}}{\Pi_v^{w,z}} \) is decreasing in \( z \), and reaches its minimum

\[
1 - \frac{e}{2} \approx 26.4\% \quad \text{when} \quad z = 1; \quad \frac{\Pi_v^{w,z} - \Pi_v^{w,z}}{\Pi_v^{w,z}} = 1 - \frac{e}{3} \approx 26.4\%.
\]

Under the two decentralized channels, the profit loss is always larger than 26.4% under RMCI, while under VMCI, the channel incurs a profit loss of 26.4%. Furthermore, since the channel profit is split evenly between the vendor and retailer, both parties should prefer VMCI over RMCI. The result here is the same as that in R8&W.

As \( s \in (1 - z)c \), from 7, 12, and 23, we have \( z_v^w = z_v^r \). So under VMCI, the service level chosen by the vendor optimizes the total channel profit, while under RMCI, the service level chosen by the retailer is too high for the channel. Anticipating a higher service level under RMCI, the vendor requires a higher consignment price from the retailer than he does under VMCI. Accordingly, the retailer charges a higher retail price to her customers under RMCI than under VMCI. Consequently, the inventory decision made by the downstream retailer leads to a more severe double marginalization effect which hurts channel profit. Moreover, since the vendor and retailer each gain half of the total channel profit, both parties should prefer VMCI over RMCI.

### 5. The model with consumer returns policy

Now we focus on the scenario, in which the upstream vendor offers the unsatisfied customer a return policy. For the rest of this section we derive the optimal solutions for a centralized channel, which will be a benchmark for comparing the performance of decentralized channels. For the centralized channel, the decision is to choose retail price \( p \), refund price \( r \), and quantity \( Q \) simultaneously to maximize the total expected profit,

\[
\Pi_v(p, Q, r) = pC(r)E\min(D, Q) + (p - r + s)G(r)E\min(D, Q) - \frac{net\ sale}{net\ sold} - \frac{\text{returned}}{\text{net\ sold}} - cQ.
\]

(29)

In Eq. (29), we see that each unit that is sold and kept by consumers yields revenue \( p \), each returned unit yields \( p - r \) from consumers
and $s$ from salvaging it, and each unsold unit yields only the salvage value $s$. The last term is the production or procurement cost. If we use $z = Q(y(p))$, then to maximize $\Pi_c(p, Q, r)$ over $(p, Q, r)$ is equivalent to maximizing that over $(p, z, r)$.

Now substituting $Q = y(p)z$ into (29), the objective function can be rewritten as

$$
\Pi_c(p, z, r) = y(p)\{pG(r)E\min(e, z) + (p + r + s)G(r)E\min(e, z) + s(z - E\min(e, z) - cz)\}y(p)\{p - A(z)\} + (p + r + s)G(r)(z - A(z)) + sA(z) - cz).$

(30)

Under this formulation, the refund price is separated from the retail price and quantity decision. The next theorem characterizes the system’s optimal decision $(p^*_c, z^*_c, r^*_c)$.

**Theorem 4.** For the centralized system with return policy, the refund $r^*_c$ is given by

$$
r^*_c = \arg\max_x (x + s)G(r).

(31)

For any given $z$, the unique optimal retail price $p^*_c(z, r^*_c)$ is given by

$$
p^*_c(z, r^*_c) = \frac{1}{b} \{ - (r^*_c + s)G(r) - s A(z) - A(z) + e z - A(z) \}.

(32)

Moreover, if the probability distribution function $f(\cdot)$ satisfies the property of increasing failure rate (IFR), the optimal service level $z^*_c$ is uniquely determined by

$$
\frac{1}{1 - F(z^*_c)} - \frac{z^*_c - A(z^*_c)}{b} = \frac{1}{e - s}.

(33)

Substituting (31)–(33) into (30), we derive the total optimal channel profit as

$$
\Pi^*_c = ae^{-b} \left\{ \left( \frac{- (r^*_c + s)G(r) - s A(z^*_c) - A(z^*_c) + e z - A(z^*_c)}{b} \right) \right\}.

(34)

From (33), we find the inventory decision is independent of the refund price. Moreover, together with (7), we can get that the inventory decisions are the same for the upstream vendor no matter whether or not to offer a return policy to the unsatisfied customer. While the retail price depends on the refund price, so is the profit.

### 5.1.2. The vendor's consignment price and refund price

Knowing that the retailer chooses $(p^*_c, z^*_c)$ according to (37) and (38), the vendor then aims to set the consignment price $w$ and refund price $r^*_c$ to maximize his own expected profit, which is expressed as

$$
\Pi^*_V(w, r) = wcG(r)E\min(D, Q_w) + (w - r + s)G(r)E\min(D, Q_w) + s(Q_w - E\min(D, Q_w)) - c(1 - x)Q_w.

(39)

where $Q_w = y(p^*_c)z^*_c$, and after plugging $D = y(p^*_c)z^*_c$ into (39), we can further have

$$
\Pi^*_V(w, r) = y(p^*_c)\{wz^*_c - A(z^*_c)\} + (w - r + s)G(r)z^*_c - A(z^*_c) + sA(z^*_c)

(40)

From the above two equations, we find that the vendor and the retailer each obtain 50% of the total channel profit, which is

$$
\Pi^*_V = \Pi^*_V + \Pi^*_C.

(41)

$$
\Pi^*_V = 2ae^{-b} \left\{ \left( \frac{- (r^*_c + s)G(r) - s A(z^*_c) - A(z^*_c) + e z - A(z^*_c)}{b} \right) \right\}.

(42)

Theorem 5. For RMCI with return policy, the vendor's unique optimal consignment price $w^*_v$ and refund price $r^*_v$ are given by

$$
w^*_v = \frac{1}{b} \{ - (r^*_v + s)G(r^*_v) - s A(z^*_v) - A(z^*_v) + e z^*_v - A(z^*_v) \}.

(43)

Plugging (42) into (37), we have

$$
p^*_c(z^*_c) = \frac{1}{b} \{ - (r^*_c + s)G(r^*_c) - s A(z^*_c) - A(z^*_c) + e z^*_v - A(z^*_v) \}.

(44)

Substituting all the optimal decisions back into the vendor’s and retailer’s profit functions, we derive

$$
\Pi^*_V = \Pi^*_V + \Pi^*_C = 2ae^{-b} \left\{ \left( \frac{- (r^*_c + s)G(r) - s A(z^*_c) - A(z^*_c) + e z^*_v - A(z^*_v)}{b} \right) \right\}.

(45)

For the decentralized channel under RMCI, the equilibrium solutions turn out to be: the vendor decides the refund price and consignment price according to (41) and (42), respectively; and the retailer chooses the corresponding retail price and quantity level as given by (37) and (38).

When the retailer makes the inventory decision, we can infer this decision is irrelevant to the refund price and salvage value from (38). Her retail price consists of the consignment price paid to the vendor and a margin. From (42), the vendor's consignment price depends crucially on his refund price and salvage value, so is the retail price. Then these pricing strategies are interrelated, which, however have no influence on quantity decision. All these decisions entail that the total channel profit of (46) will be shared equivalently between the retailer and the vendor.

Reminding of (16)–(18), and together with (44)–(46), we find the vendor and the retailer equally share the total profit under RMCI. Note that these results are solely due to the specific
exponential multiplicative demand function. However, considering other demand functions may cause much difficulties in deriving these elegant results, which will be discussed in our future works.

5.2. Decentralized channel decisions and performance under VMCI

Under VMCI, the vendor chooses the consignment price \( p \), the refund price \( r \) together with the quantity decision \( Q \) in the first step, the retailer only decides the retail price \( p \) in the second step. Following the same procedure as in the former subsection, we begin our investigation by discussing the retailer’s decision.

5.2.1. The retailer’s retail price

In the second step, for any given \( w \), \( r \), and \( z \) chosen by the vendor in the first step, the retailer determines \( p \) appropriately to maximize her own expected profit,

\[
\Pi_V^w(p, w, r, z) = \gamma(p)\{\mu - w[z - A(z)] - czz\}. \tag{47}
\]

Similar to the derivation of (37), for any given \( w \), \( r \), and \( z \), the optimal retail price is

\[
p_V^w(w, r, z) = w + \frac{czz}{z - A(z)} + \frac{1}{b}. \tag{48}
\]

As in the RMCI scenario, (48) indicates that the retailer’s optimal selling price consists of two parts: the amount she has to pay to the vendor for each unit sold and a mark-up for herself.

5.2.2. The vendor’s decision

In the first step, knowing that the retailer’s best response is (48), the vendor decides the consignment price \( w \), refund price \( r \), and service level \( z \) to maximize her expected profit, which is given by

\[
\Pi_V^w(w, r, z) = ae^{-bw}\{w[z - A(z)] + (-r + s)G(r)[z - A(z)]
+ sl(z) - c(1 - z)z\} = ae^{-bw}\{w[z - A(z)]
+ (-r + s)G(r)[z - A(z)] + sl(z) - c(1 - z)z\}. \tag{49}
\]

The vendor’s optimal solution can be characterized by the following theorem.

**Theorem 6.** For VMCI with return policy, the optimal refund price of the vendor is given by \( r_V = \arg \max_r (-r + s)G(r) \).

For any given \( z \), the optimal wholesale price is

\[
w_V^z(r_V, z) = \frac{1}{b} - (-r_V + s)G(r_V) - s\frac{A(z)}{z - A(z)} + c(1 - z)\frac{z}{z - A(z)}. \tag{51}
\]

Moreover, if the probability distribution \( F(.) \) satisfies the property of increasing failure rate (IFR), his optimal service level \( z \) is determined by

\[
1 - F(z_V^* - 1) = \frac{1}{b(c - s)}. \tag{52}
\]

Together with (33), we can find \( z^*_V = z_V^* \).

Substituting all the optimal decisions back into the vendor’s and retailer’s profit functions, we derive

\[
\Pi_V^* = \Pi_V^{w^*} = \Pi_V^{r^*} = 2ae^{-bw}\left\{\frac{1}{b} - \frac{w}{z - A(z)} - \frac{s}{z - A(z)}\right\} \frac{z_V^* - A(z_V^*)}{b} \tag{53}
\]

5.3. Channel performance

Based on the results obtained in the former subsections, we continue to study the performance of the two decentralized channels. Recall that \( \lambda(z) \) is given by (28), and thus we can rewrite \( \Pi_V^* \) and \( \Pi_V^c \) as

\[
\Pi_V^c = ae^{-bw}\left\{\frac{1}{b} - \frac{w}{z - A(z)} - \frac{s}{z - A(z)}\right\} \lambda(z_V^*), \tag{54}
\]

\[
\Pi_V^c = 2ae^{-bw}\left\{\frac{1}{b} - \frac{w}{z - A(z)} - \frac{s}{z - A(z)}\right\} \lambda(z_V^*), \tag{55}
\]

**Proposition 2.** \( \Pi_V^c - \Pi_V^c \) is decreasing in \( z \), and reaches its minimum \( 1 - \frac{z}{b} \approx 26.4\% \) when \( z = 1; \Pi_V^c - \Pi_V^c \approx 26.4\% \).

When combining with Proposition 1, we find that whether the vendor offers a return policy or not, it is always beneficial for both the vendor and retailer when delegating the inventory decision to the vendor rather than to the retailer.

6. Comparison between no return policy and return policy

Above all, from the former sections we know that whether we take the return policy into consideration or not, it is always optimal for the system to choose to delegate the inventory decision to the vendor, i.e., both parties prefer VMCI over RMCI. So now we focus on VMCI, and continue to discuss whether the upstream vendor should offer a return policy to the unsatisfied customer or not.

**Theorem 7.** If \( s = 0 \), then \( r_V = 0 \).

In R&W, it is assumed that the salvage value is zero, and there is no return behavior in their paper. They find that it is profitable for both parties if the upstream vendor controls the inventory decision. In our model, we assume that the salvage value is non-negative, and the upstream vendor may offer a return policy because of consumers’ uncertain post-purchase valuation, and we also derive that both parties should prefer VMCI over RMCI. So our framework is broader in scope.

This theorem shows that if unsold goods and returned items have no scrap value for the vendor, then his optimal decision is not to offer return policy. It degenerates to the case in R&W, i.e., no salvage value, no return policy. In this sense, our results can unify previous results in R&W.

Next, we make a list of the comparison between without return policy and with return policy under VMCI in Table 1 and we can draw the following results.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) ( s = 0 )</td>
<td>No return policy</td>
</tr>
<tr>
<td>(ii) ( z_V^* = z_V^* )</td>
<td>( w_V^V = w_V^V ) and ( p_V^V = p_V^V )</td>
</tr>
<tr>
<td>(iii) ( w_V^V = w_V^V ) and ( p_V^V = p_V^V )</td>
<td>( \Pi_V^V \geq \Pi_V^V ), ( \Pi_V^V \geq \Pi_V^V ) and ( \Pi_V^V \geq \Pi_V^V )</td>
</tr>
</tbody>
</table>

This theorem clearly shows that the vendor’s inventory decisions are the same between the scenarios with and without return policy. When the salvage value is non-negative, if the system takes the consumer’s valuation uncertainty into consideration, both par-
ties can gain more profit from the vendor’s return policy. The reason is that if the vendor’s optimal refund price is less than the salvage value, he can gain a profit from the returned items. As the vendor and the retailer equally share the total profit, the retailer can also benefit from the return policy. Besides, the consignment price and retail price are lower under return policy compared to without the return policy. Because the system can earn profit from returned items, it is optimal to set a lower retail price which will induce more demand, and hence more returned items.

Totally, the main results in our paper are: (i) it is always up to the upstream vendor to control the inventory decision under a consignment contract with and without return policy. (ii) when the product’s salvage value is zero, the vendor’s optimal choice is not to offer a return policy and (iii) when the product’s salvage value is positive, both parties can gain more profit from the vendor’s return policy.

7. Conclusions

Consignment contracts have been widely applied in many industries and are especially popular in on-line marketplaces. Returns of products from customers to retailers are a common feature of on-line markets. Most consumers return products that perform well but do not meet their expectations or tastes. We consider a consignment contract with consumer misfit returns behavior. Under the demand uncertainty and post-purchase valuation uncertainty, the upstream vendor offers a consignment price charged to the downstream retailer for each unit of the product sold and the refund price for each unit returned from customers, then the retailer sets a retail price for selling the product to the market. After all the uncertainties are realized, the retailer pays the vendor based on the net selling units, and the vendor can salvage both the returned items and unsold items in a secondary market. Based on the general framework, we analyze consignment selling with return policy, and we also consider consignment selling without return policy as a benchmark. Furthermore, we consider two different inventory regimes, labeled as RMCI and VMCI, under consignment contracts with and without return policy, respectively. One of our objectives is to resolve the debate among practitioners who should control the inventory, by comparing the performance under RMCI and VMCI for two scenarios with and without return policy. We also focus on whether it is profitable for the upstream vendor to offer a return policy to the unsatisfied customer.

Using a multiplicative demand model, we fully characterize the decentralized decisions and derive closed-form performance measures. Our model indicates that under both RMCI and VMCI, the vendor would evenly split the net channel profit with the retailer. Moreover, it is always beneficial to both parties if the vendor takes charge of the inventory decision under the consignment contract with and without return policy. Whether the upstream vendor should offer a return policy or not turns out to depend critically on the salvage value. Especially, if the salvage value is zero, the vendor’s optimal decision is not to offer a return policy; otherwise, the vendor can make more money by offering a return policy.

We would like to reiterate our contributions compared to R&W. R&W’s model setting and research objectives resemble ours, and they prove/conjecture a number of our results. However, their model does not incorporate consumers’ return behavior and they assume that the salvage value is zero. Evidently, our framework is broader in scope. Furthermore, we derive that the vendor’s optimal return policy depends crucially on the salvage value, which is not available in R&W. Our results also suggest that when the salvage value is zero, it is optimal for the vendor to control the inventory decision and the vendor should not offer a return policy. So our conclusions cover the results in R&W. Overall, we believe that this paper generalizes previous insights regarding inventory control under consignment contract and provides more managerial insights for practical application.

For further research, our model can be used to study the reverse supply chain fully by incorporating processing costs such as inspection, distribution, and remanufacturing. Incorporating disposal cost into the model proposed in Section 5 can be a future scope. We can also consider a retailer who exerts sales effort, which will promote the selling and this will cast new light on how decisions should be redesigned. Many vendors permit consumer returns up to a certain time limit, especially for short-life products. It is interesting to understand how the duration of return policies should be set. We believe that all the above extensions present fruitful opportunities for future research. Finally, the elegance of the results depends on the specific demand function. Actually, we derive closed-form solutions by using the exponential multiplicative demand function. Considering alternative demand function forms, like a linear additive form, may cause difficulties for obtaining an analytical solution for the model. On the other hand, Wang et al. (2004) demonstrate numerically that for the linear demand function, qualitative properties and conclusions derived based on a specific demand function turn out to be robust.

Acknowledgments

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Appendix A. Mathematical proofs

Proof of Theorem 1. The proof is a simple case of Theorem 4. Please see the proof of Theorem 4 for details. □

Proof of Theorem 2. The proof is a simple case of Theorem 5. Please see the proof of Theorem 5 for details. □

Proof of Theorem 3. The proof is a simple case of Theorem 6. Please see the proof of Theorem 6 for details. □


Proof of Lemma 2. Taking first derivative with z of ´(z), we get

d(´(z)) = e^{\frac{\text{b}(c-s)}{\text{b}(z-s)-c}} \left[ 1 - F(z) \right] (c-s) - \frac{1}{b(c-s)} \left[ 1 - F(z) - z - A(z) \right].

The first three terms are all positive, and from (7) we can get d(´(z)) = 0. We combine with Lemma 1 and we know that when z < z^*, d(´(z)) > 0 and when z > z^*, d(´(z)) < 0, then ´(z) is a unimodal function, which reaches its maximum at z^*. □

Proof of Proposition 1. From Lemma 2, we can get d(´(z^*)) = \frac{d}{d(z^*)} and \frac{d}{d(z^*)} = \frac{d}{d(z^*)}. Hence,

\frac{\Pi'_r - \Pi'^*}{\Pi'_r} = \frac{ae^{-\lambda(z^*)}}{ae^{-\lambda(z^*)}} = \frac{\lambda(z^*)}{\lambda(z^*)} = 1 - 2 \frac{\lambda(z^*)}{\lambda(z^*)}.

Next we go on to show \frac{\Pi'_r - \Pi'^*}{\Pi'_r} is decreasing in x. Recall the expression of \lambda(z), we can have

\frac{\Pi'_r - \Pi'^*}{\Pi'_r} = 1 - 2 \frac{e^{\frac{b(c-s)}{b(z-s)-c}}}{e^{\frac{b(c-s)}{b(z-s)-c}}} \left[ z^* - A(z^*) \right].

From (7), we know z^* is independent of x, so

d(\Pi'_r - \Pi'^*) = \frac{d\Pi'_r - \Pi'^*}{d(z^*)} dz^*.

from (12) and Lemma 1, taking derivative with implicit function, we can easily get dz^*/dz^* < 0, and

we can get [1 - F(z^*)] [z^* - A(z^*)] = b(c-s)F(z^*) - A(z^*)] from (12), as well as x < (1 - z)c, then \frac{d\Pi'_r - \Pi'^*}{d(z^*)} is increasing in z^*.

Above all, \frac{d\Pi'_r - \Pi'^*}{d(z^*)} is decreasing in x and reaches its maximum at x = 1. We can find z^*_1 = z^* when x = 1. Furthermore,

\frac{\Pi'_r - \Pi'^*}{\Pi'_r} = 1 - 2 \frac{e^{\frac{b(c-s)}{b(z-s)-c}}}{e^{\frac{b(c-s)}{b(z-s)-c}}} \left[ z^* - A(z^*) \right] = 1 - 2 \frac{z^*}{z^*} = 29.4%.

Proof of Theorem 4. Because the refund price r enters only in the term (–r + s)G(r), the optimal r^* should maximize this term. We then take a sequential procedure to maximize \Pi_r(z, p, r^*_z) over (p, z). First, for any given z where A ≤ z ≤ B, we take the partial derivative of \Pi_r(z, p, r^*_z) with respect to p as

\frac{\partial\Pi_r(p, z, r^*_z)}{\partial p} = ae^{-b} \left[ -bp(z - A(z)) + [z - A(z)] - b(-r^*_z + s) G(r^*_z) \right]

[x - A(z) - bsA(z) + bcz].

Since ae^{-b} > 0, \frac{\partial\Pi_r(p, z, r^*_z)}{\partial p} = 0 implies (32). For any given z, \frac{\partial\Pi_r(p, z, r^*_z)}{\partial p} > 0 for p < p^*_z(z, r^*_z) and \frac{\partial\Pi_r(p, z, r^*_z)}{\partial p} < 0 for p > p^*_z(z, r^*_z). Then p^*_z(z, r^*_z) maximizes \Pi_r(z, p, r^*_z). Next we want to find z^*. By the chain rule, we have

\frac{d\Pi_r(p^*_z(z, r^*_z), z, r^*_z)}{dz} = \frac{\partial\Pi_r(p^*_z(z, r^*_z), z, r^*_z)}{\partial z} + \frac{\partial\Pi_r(p^*_z(z, r^*_z), z, r^*_z)}{\partial p}.

Due to the optimality of p^*_z(z, r^*_z), we have \frac{\partial\Pi_r(p^*_z(z, r^*_z), z, r^*_z)}{\partial p} = 0. Thus we get

\frac{d\Pi_r(p^*_z(z, r^*_z), z, r^*_z)}{dz} = \frac{ae^{-b}G(r^*_z)}{b(z - A(z))} / \Pi_r.

where (z - A(z)) = (1 - F(z)) - b(c - s)zF(z) - A(z).

Since the first factor in the above expression is always positive, the first-order condition requires that z^* satisfies l(z) = 0, which brings out (33). To verify this, we have l(\lambda) = z - A(z) > 0, l(B) = z = (-b(c - s)z < 0. Define h(z) = f(z)[1 - F(z)] as the failure rate of the demand distribution. Then

\frac{d\Pi_r(p^*_z(z, r^*_z), z, r^*_z)}{dz} = \frac{ae^{-b}G(r^*_z)}{b(z - A(z))} / \Pi_r.

If h(z) > 0, then l'(z) < 0 at the point z that satisfies l(z) = 0, which implies l(z) is a unimodal function. Combining with l(\lambda) > 0 and l(B) < 0, guarantees the uniqueness of z^* ∈ (A, B). □

Proof of Theorem 5. Only the term (–r + s)G(r) is interrelated with r in (40), the optimal r^*_z should maximize this term. After substituting r^*_z back into (40) and taking the first derivative with w, we have

\frac{d\Pi'_w(w, r^*_w)}{dw} = ae^{-b} \left[ -bw(z - A(z)) + [z - A(z)] - b(-r^*_w + s) G(r^*_w) \right]

- b(-r^*_w + s)G(r^*_w) z^*_w - A(z^*_w)] - bsA(z^*_w) + bc(1 - z^*_w]z^*_w].

Since ae^{-b} > 0, \frac{d\Pi'_w(w, r^*_w)}{dw} > 0 implies (42). For any given z, \frac{d\Pi'_w(w, r^*_w)}{dw} > 0 for w < w^* and \frac{d\Pi'_w(w, r^*_w)}{dw} < 0 for w > w^*, which implies \Pi'_w(w, r^*_w) is first increasing in w for w < w^* and then decreasing in w for w > w^*. So \Pi'_w(w, r^*_w) is unimodal in w and reaches its maximum at w^*. □
Proof of Theorem 6. Because the refund price \( r \) only exists in the term \((-r+s)G(z)\), the optimal \( r_u \) should maximize this term. We then take a sequential procedure to maximize \( \Pi^g_v(w, r_u, z) \) over \((w, z)\). First, for any given \( z \) where \( A < z \leq B \), we take the partial derivative of \( \Pi^g_v(w, r_u, z) \) with respect to \( w \) as

\[
\frac{\partial \Pi^g_v(w, r_u, z)}{\partial w} = ae^{-b\psi_1} \left\{-b[w + (-r_u + s)G(z)] - [z - A(z)] \right\} + [z - A(z)] - bsA(z) + bc(1 - z)A(z).
\]

Since \( ae^{-b\psi_1} > 0, \partial \Pi^g_v(w, r_u, z)/\partial w = 0 \) implies (51). For any given \( z, \partial \Pi^g_v(w, r_u, z)/\partial w > 0 \) for \( w < w^*_v(r_u, z) \) and \( \partial \Pi^g_v(w, r_u, z)/\partial w < 0 \) for \( w > w^*_v(r_u, z) \). Then \( w^*_v(r_u, z) \) maximizes \( \Pi^g_v(w, r_u, z) \).

Plugging \( r_u^* \) and \( w^*_v(r_u, z) \) into (49), by the chain rule, we get \( z^*_v \) as follows,

\[
diP^g_v(w^*_v(r_u, z), r_u, z) \frac{dz}{dz} = \frac{\partial \Pi^g_v(w^*_v(r_u, z), r_u, z)}{\partial w} \frac{dw^*_v(r_u, z)}{dz}.
\]

Due to the optimality of \( w^*_v(r_u, z) \), we have \( \partial \Pi^g_v(w^*_v(r_u, z), r_u, z)/\partial w = 0 \). Thus we get

\[
\frac{dI^g_v(w^*_v(r_u, z), r_u, z)}{dz} = ae^{b\psi_1} \left\{ [w^*_v + (-r_u + s)G(z)] - [1 - F(z)] \right\} + sF(z) - c(1 - z) + ae^{-b\psi_1} \left\{ [w^*_v + (-r_u + s)G(z)] - [z - A(z)] \right\} + sA(z) - c(1 - z) + \frac{cz - [z - A(z)]}{b[A(z)]} + \frac{b(1 - z)}{b[A(z)]}.
\]

The expression of \( m(z) \) is exactly same as that of \( l(z) \) defined in the proof of Theorem 4. Consequently, following the same procedures in the proof of Theorem 4, we can get that \( z^*_v \) derived by setting \( m(z) = 0 \) is the unique maximizer of \( \Pi^g_v(w^*_v(r_u, z), r_u, z) \). The proof is complete. \( \square \)

Proof of Proposition 2. The proof is quite similar to that of Proposition 1 just by replacing \( e^{-w} \) with \( e^{-(1-b+c+e+e+G(z))} \) and \( e^{-w} \) with \( e^{-(1-b+c+e+G(z))} \) from part (i), we can get \((-r_u^* + s)G(z^*_v) \geq 0\), combining with (ii), we can easily draw the conclusion. Following the same way, we can get \( p_v^* \leq p_v^* \).

(iv) From (27) and (54)

\[
\Pi^g_v = \Pi^g_v + \Pi^g_v = 2ae^{b\psi_1} \left\{ \frac{1}{b} [ \frac{1}{b} \psi_1 \left\{ -b[w + (-r_u^* + s)G(z)] + [z - A(z)] \right\} + [z - A(z)] - bsA(z) + bc(1 - z)A(z) \right\} \right\} + \frac{z^*_v - A(z^*_v)}{b}.
\]

Combining with (i) and (ii), we can easily get \( \Pi^g_v \geq \Pi^g_v \). As the channel profit is equally shared between the vendor and the retailer, the other results in (iv) are obvious. \( \square \)

References