Atanassov's Interval-valued Intuitionistic Linguistic Multi-criteria Group Decision-making Method Based on Trapezium Cloud Model

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Abstract—The cloud model, which can implement uncertain transformation between a qualitative concept and its quantitative instantiations, has attracted great attentions in multi-criteria decision-making problems with linguistic information. This paper proposes some operations and a possibility degree of trapezium clouds as well as several new aggregation operators: the trapezium cloud weighted arithmetic averaging (TCWAA) operator, the trapezium cloud ordered weighted arithmetic averaging (TCOWA) operator and the trapezium cloud hybrid arithmetic (TCHA) operator. Moreover, a method with Atanassov’s interval-valued intuitionistic linguistic numbers (AIVILNs) based on trapezium clouds is presented, which can provide solutions for multi-criteria group decision-making problems with Atanassov’s interval-valued intuitionistic linguistic information. In this method, AIVILNs are firstly converted into trapezium clouds and aggregated by trapezium cloud aggregation operators. Then, the ranking of alternatives is determined by the possibility degree matrix of trapezium clouds. Finally, an illustrative example confirming the validity and feasibility of the proposed method is conducted.

Index Terms—group decision-making; Atanassov’s interval-valued intuitionistic linguistic number; trapezium cloud; aggregation operator; possibility degree

I. INTRODUCTION

In 1986, Atanassov [1] introduced a concept of intuitionistic fuzzy sets (AIFSs), which is a generalization of fuzzy sets (FSs) introduced by Zadeh [2]. AIFSs have been widely studied [3]–[6]. In 1993, Gau and Buehren [7] proposed the concept of vague sets (VSs), but Bustince and Burillo [8] proved that VSs were the same as AIFSs. Both AIFSs and VSs are well-known generalizations of FSs, which consider the membership degree and non-membership degree of each element to the set [9]. Atanassov and Gargov [10] further generalized AIFSs to Atanassov’s interval-valued intuitionistic fuzzy sets (AIVIFSs), where the membership degree and non-membership degree are interval values. AIVIFSs have the virtue of complementing AIFSs, which are more flexible and practical than AIFSs in dealing with fuzziness and uncertainty.

In this sense, AIVIFSs provide a more reasonable mathematical framework to handle imprecise facts or imperfect information. AIVIFSs have attracted wide attention and many researchers have focused on their operations [11], [12], distance measures [13]–[15], relation measures [16]–[19], score functions [20]–[22] and operators [23]–[26]. Moreover, the multi-criteria decision-making (MCDM) methods with known or incompletely known criteria weight information under Atanassov’s interval-valued intuitionistic fuzzy environment have been studied [27]–[30]. Mathematically, no matter whether FSs, AIFSs or AIVIFSs are restricted to describe quantitative information, in case of referring to quantitative expression, the descriptions made by them can be ill-defined, which are inconvenient in application. Therefore, it is more suitable to provide assessments by means of linguistic values rather than numerical ones.

In the past several decades, many researchers have investigated linguistic MCDM problems and presented a number of linguistic MCDM methods which can be mainly classified into three types.

1) The linguistic computational model based on membership functions, which can convert linguistic information into triangular fuzzy numbers or other fuzzy numbers by means of membership functions [31]–[33]. Although fuzzy numbers such as triangular fuzzy numbers and trapezium fuzzy numbers have advantages in operations and can be easily understood, this method could lead to information distortion and loss to a certain extent in the transformation, which results from the difficulties in providing pre-determined fuzzy numbers corresponding to the language phrases.

2) The method based on symbols, which makes computations on the indices of the linguistic terms, is of great convenience and simplicity [34]–[36]. However, the essence of this method is converting the linguistic variables into integral numbers, which totally abandons the fuzziness of qualitative information.

3) The method based on the 2-tuple linguistic representation model, which can overcome the difficulties in information distortion and loss, occurs formerly in the linguistic information processing. However, just similar to the linguistic symbolic models, the 2-tuple linguistic model is actually to transform linguistic information into real numbers [37]–[42], i.e., it directs the uncertain decision-making into the precise domain, which violates the original intention of fuzzy methods.
However, there are such cases that a single linguistic term can’t accurately express the evaluation information for an object when the evaluation information hovers between two terms, and then the researchers propose uncertain linguistic variables. Afterwards, Wang and Li [43] proposed Atanassov’s intuitionistic linguistic number (AILN), which is a linguistic term closest to the evaluation information with the membership degree and non-membership degree to this linguistic term. AILNs have some unique advantages: (1) It is more accurate to depict fuzziness than uncertain linguistic variable. For example, given a linguistic set \( S = \{s_1, s_2, s_3, s_4, s_5, s_6\} \) = {extremely poor, very poor, poor, fair, good, very good, extremely good}, perhaps, the performance evaluation result is then thought to be higher than “good” and lower than “very good”. If it is expressed with uncertain linguistic variables, it will be \([s_4, s_5]\), while the preference degree of \( s_4 \) or \( s_5 \) is not clear. When AILNs are used, the performance evaluation will be expressed as \(< s_4, (0.8, 0.0) >\), which is absolutely more precise than uncertain linguistic variables. (2) Atanassov’s intuitionistic linguistic variables are more flexible to express fuzzy information than Atanassov’s intuitionistic fuzzy numbers. With regard to an Atanassov’s intuitionistic fuzzy number, it represents the membership degree and non-membership degree of elements belonging to a specific qualitative concept, while the specific qualitative concept may not be suitable for describing all performance evaluations, but AILNs will not be limited to a specific qualitative concept. It is applied to a linguistic term set, which makes the expressions with fuzzy information much more flexible. Inspired by Atanassov’s interval-valued intuitionistic fuzzy numbers, there exist such situations that the membership degree and non-membership degree are interval values corresponding to AILNs, which are Atanassov’s interval-valued intuitionistic linguistic numbers (AIVILNs).

Regarding MCDM problems based on AILNs or AIVILNs, we may take advantage of traditional linguistic MCDM methods aforementioned, but the deficiencies of them will pass down as well. For instance, Liu presented some generalized dependent aggregation operators with AILNs [44] and Wang introduced a group decision-making approach based on Atanassov’s interval-valued intuitionistic linguistic geometric aggregation operators [45]. Both of them dealt with linguistic terms according to the linguistic symbolic model, and thus the rough descriptions of the uncertainties of qualitative information are inevitable. What’s worse, the linguistic terms and Atanassov’s intuitionistic fuzzy numbers are computed separately, which will lead to information distortion.

Fortunately, there is still a way for linguistic computation without the shortcomings of aforementioned methods – the cloud model. It’s a description of qualitative concept, which was developed on the foundation of the probability theory and the fuzzy set theory and allows a stochastic disturbance of the membership degree encircling a determined central value rather than a fixed number. Moreover, the cloud model perfectly depicts the uncertainty of qualitative concept with three numerical characteristics, in which the objective and interchangeable transformation between qualitative concepts and quantitative values becomes possible [46]. Therefore, the cloud model is good at depicting qualitative concepts and would put forward new solutions for Atanassov's intuitionistic linguistic MCDM problems.

With the development of the cloud model theory, cloud models have received more and more attention and have been applied in many fields. Over the past decades, cloud models have been successfully applied in data mining [47]–[49], knowledge discovery [48], [50], intelligent control [51], [52], network security [53]–[55] and algorithm improvement [56]–[58]. As a transformation between the qualitative and quantitative matters, usually existing in linguistic multi-criteria group decision-making (MCGDM) problems, Wang proposed a method of generating 5 clouds from the linguistic term set of 5 labels on the basis of the Golden Ratio [59]. However, the limitation of generating 5 clouds makes it a narrow range of applications.

Despite that the cloud model is a description of qualitative concept, it is depicted by three numerical characteristics, which are the expected value \( Ex \), the entropy \( En \) and the hyper entropy \( He \). \( Ex \) is regarded as the most representative and typical sample of qualitative concepts, where the normal cloud equipped with the crisp expected value is suitable for certain linguistic variables. But an AIVILN is in essence an uncertain linguistic variable, for which a trapezium cloud [60]–[62] with the interval expected value would be appropriate.

In this paper, the trapezium cloud is applied in the foundation of previous work. A knowledge representation and a processing method based on trapezium clouds, and an MCDM method with AIVILNs are proposed. To do that, the rest of this paper is organized as follows. In Section 2, the trapezium cloud and the related operations, as well as a possibility degree for the comparison between two trapezium clouds, are proposed. In Section 3, some new operators for aggregating trapezium clouds are presented, such as the trapezium cloud ordered weighted arithmetic averaging (TCOWA) operator, the trapezium cloud ordered weighted arithmetic averaging (TCWAA) operator and the trapezium cloud hybrid arithmetic (TCHA) operator. In Section 4, the notion of AIVILNs and a conversion method between AIVILNs and trapezium clouds are presented. In Section 5, an Atanassov’s interval-valued intuitionistic linguistic multi-criteria group decision-making method based on trapezium clouds is proposed. In Section 6, an illustrative example is provided, followed by a comparison analysis with other methods. Finally, some summary remarks are given in Section 7.

II. TRAPEZIUM CLOUDS AND RELATED CONCEPTS

The trapezium cloud is more general than the normal cloud. The normal cloud has a crisp value that certainly belongs to the linguistic term, but the trapezium cloud allows an interval value belonging to the linguistic term. For instance, because of various subjective cognitions, most people may have different opinions on whether a person of 25 years old is considered as “Youth” or not, but they would approve the range of age from 18 to 25.

Definition 1 [61]: Cloud is a conversion model using natural language to describe the uncertainty between the qualitative concepts and their values. Let \( U = \{x\} \) be a concerned domain, and \( T \) be the linguistic value related to \( U \),
and then the qualitative concept’s membership $G_{x} (x)$ related to the element $x$ in the set $U$ is a random number with stable tendency. The element $x$ on the domain $U$ is called a membership cloud, or simply, a cloud drop. $G_{x} (x)$ is valued in $[0, 1]$, and a cloud is a mapping from the domain $U$ to $[0, 1]$ . The numerical characteristics of a trapezium cloud are described by the expected interval-value $[\bar{E}_{x}, \overline{E_{x}}]$, the entropy $E_{n}$ and the hyper entropy $H_{e}$ . The trapezium cloud can be written as $\gamma (\bar{E}_{x}, \overline{E_{x}}, E_{n}, H_{e})$ .

1) The expected interval-value $[\bar{E}_{x}, \overline{E_{x}}]$ is the interval-value that could represent granularity of a certain quality concept. Let $x \in [\bar{E}_{x}, \overline{E_{x}}]$, and then the membership degree of the element $x$ of the qualitative concept equals 1, i.e., the membership degree is a hundred percent. Especially when $\bar{E}_{x} = \overline{E_{x}}$, there is only one element whose membership degree equals 1.

2) The entropy $E_{n}$ is the uncertainty measurement of the qualitative concept. From the perspective of probability theory, it is similar to standard variance of random variables. In the fuzzy set theory, it represents the value scope, of which the drop is acceptable by the concept, and it defines the support set of the concept with the membership degrees larger than 0. Consequently, the correlation of randomness and fuzziness is reflected by the same numerical character.

Many researchers introduced entropy measures from different points of view. The entropy of Burillo and Bustince [63] was proposed with an axiomatic structure. The entropy of Szmidt and Kacprzyk [64] was nonprobabilistic-type. In [65], a new axiomatic characterization of the entropy measures was proposed using a 2-tuple for representation of uncertainty of an AIFS: the fuzziness and the non-specificity. In this paper, the entropy of cloud $E_{n}$ is a measure of the fuzziness of the qualitative concept within the universe of discourse defined on the basis of [66]. Nevertheless, in future research, we intend to extend the entropy of cloud with the entropy measures in [63]–[65].

3) The hyper entropy $H_{e}$ is the uncertainty measurement of the entropy $E_{n}$, i.e., the second-order entropy of the entropy $E_{n}$.

Here, how to determine the three numerical parameters of the trapezium cloud? A subjective method and an objective method are presented. The subjective method is based on human’s experience, i.e., the experts gives the three parameters of clouds manually. By objective method, the parameters can be obtained by cloud transforming method called backward cloud generator [66]. The backward cloud generator decomposes a data distribution to the sum of several clouds. The backward cloud generator is described as follows.

**Input:** The quantitative positions of $g$ cloud drops, $x_{i} (i = 1, 2, \ldots, g)$ and the certainty degree that each cloud drop can represent a linguistic notion $y_{i} (i = 1, 2, \ldots, g)$.

**Output:** The three numerical parameters $[\bar{E}_{x}, \overline{E_{x}}]$, $E_{n}$ and $H_{e}$, which can represent the corresponding linguistic notions.

The algorithm is implemented in 4 steps:

1) Calculate the value of $[\bar{E}_{x}, \overline{E_{x}}]$. If $y_{i} (i = 1, 2, \ldots, g) = 1$, let $x_{j} = x_{j} (j = 1, 2, \ldots, t) = t = \text{the count of } x_{j}$, and then

$$E_{x} = \min \{x_{j}' \}, \quad \overline{E_{x}} = \max \{x_{j}' \}.$$ 

2) For each pair of $(x_{j}, y_{i})$,

$$\frac{(E_{x} - x_{j})}{\sqrt{-2 \ln y_{i}}}, \quad x_{j} \leq E_{x} \quad E_{n_{i}} = 0,$$

$$\frac{(E_{x} - \overline{E_{x}})}{\sqrt{-2 \ln y_{i}}}, \quad x_{j} \geq \overline{E_{x}}.$$ 

3) Calculate the mean value of $E_{n_{i}}$: $\overline{E_{n}} = \frac{1}{g} \sum_{i=1}^{g} E_{n_{i}}$

4) Calculate the standard deviation of $\overline{E_{n}}$:

$$H_{e} = \sqrt{\frac{1}{8} \sum_{i=1}^{g} (\overline{E_{n}} - E_{n_{i}})^{2}}$$

Here, a real-world example is given to illustrate the trapezium cloud. Some researchers have carefully analyzed the membership degree of the fuzzy notion “young” through statistical methods [67]. This is a typical large-scale examination on the membership degrees. 129 students have been chosen to give the most suitable age range of the notion “young” respectively. Then, the result calculated for each group is shown in Table I .

1) Calculate the value of $[\bar{E}_{x}, \overline{E_{x}}]$: $[\bar{E}_{x}, \overline{E_{x}}] = [20, 25]$. 2) Calculate the value of $E_{n_{i}}$, and the results are listed in Table II .

3) Calculate the mean value of $E_{n_{i}}$: $\overline{E_{n}} = 3.803$.

4) Calculate the standard deviation of $\overline{E_{n}}$: $H_{e} = 1.36$.

<table>
<thead>
<tr>
<th>Age</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_{i}$</td>
<td>0.186</td>
<td>0.194</td>
<td>0.209</td>
<td>0.225</td>
<td>0.581</td>
<td>0.620</td>
<td>0.760</td>
</tr>
<tr>
<td>$x_{j}$</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>23</td>
</tr>
<tr>
<td>$y_{i}$</td>
<td>0.791</td>
<td>0.798</td>
<td>0.985</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_{j}$</td>
<td>24</td>
<td>25</td>
<td>26</td>
<td>27</td>
<td>28</td>
<td>29</td>
<td>30</td>
</tr>
<tr>
<td>$y_{i}$</td>
<td>1</td>
<td>1</td>
<td>0.992</td>
<td>0.798</td>
<td>0.783</td>
<td>0.767</td>
<td>0.620</td>
</tr>
<tr>
<td>$x_{j}$</td>
<td>31</td>
<td>32</td>
<td>33</td>
<td>34</td>
<td>35</td>
<td>36</td>
<td>36</td>
</tr>
<tr>
<td>$y_{i}$</td>
<td>0.597</td>
<td>0.209</td>
<td>0.209</td>
<td>0.202</td>
<td>0.202</td>
<td>0.008</td>
<td>0.008</td>
</tr>
</tbody>
</table>

| TABLE II. The value of $E_{n_{i}}$ |
|-----|----|----|----|----|----|----|----|
| Age | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| $E_{n_{i}}$ | 5.452 | 4.968 | 4.523 | 4.051 | 5.761 | 5.115 | 5.395 |
| Age | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| $E_{n_{i}}$ | 4.377 | 2.978 | 5.658 | 0 | 0 | 0 | 0 |
| Age | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| $E_{n_{i}}$ | 0 | 0 | 7.991 | 2.978 | 4.288 | 5.497 | 5.115 |
| Age | 31 | 32 | 33 | 34 | 35 | 36 | 36 |
| $E_{n_{i}}$ | 5.906 | 3.958 | 4.523 | 5.029 | 5.588 | 5.388 | 5.535 |

Here, the linguistic notion can be explicitly presented by the graph as shown in Figure 1. [61].
Definition 2: Given any two trapezium clouds \(Y_1(\overline{E_{x1}}, \overline{E_{x1}}, E_n, H_e)\) and \(Y_2(\overline{E_{x2}}, \overline{E_{x2}}, E_n, H_e)\), the arithmetic operations of \(Y_1\) and \(Y_2\) can be summarized as follows:

1) \(Y_1 + Y_2 = (\overline{E_{x1}} + \overline{E_{x2}}, \overline{E_{x1}} + \overline{E_{x2}}, \overline{E_{x1}} + E_n, \overline{E_{x2}} + H_e)\);

2) \(Y_1 - Y_2 = (\overline{E_{x1}} - \overline{E_{x2}}, \overline{E_{x1}} - \overline{E_{x2}}, \overline{E_{x1}} - E_n, \overline{E_{x2}} - H_e)\);

3) \(\lambda Y_1 = (\lambda \overline{E_{x1}}, \lambda \overline{E_{x2}}, \lambda E_n, \lambda H_e), \lambda \geq 0\).

As an extension of the arithmetic operations of normal clouds [46], the arithmetic operations of trapezium clouds are defined. Especially, if the entropy and hyper entropy of one of trapezium clouds are 0, the arithmetic operations of trapezium clouds will be reduced to the arithmetic operations of a trapezium cloud and an interval-value, which will be introduced in detail in Definition 3.

With the operations of trapezium clouds above, the following properties can be proved.

1) The addition operation of trapezium clouds satisfies the commutative property and the associative property, i.e., \(Y_1 + Y_2 = Y_2 + Y_1\), and \((Y_1 + Y_2) + Y_3 = Y_1 + (Y_2 + Y_3)\).

2) The arithmetic operations of trapezium clouds result in the increasing uncertainty; therefore, if \(Y_1 + Y_2 = Y_3\), \(Y_2 - Y_1 = Y_3\), cannot be obtained.

Example 1: Assume that the decision-maker constructs two trapezium clouds \(Y_1\) and \(Y_2\), and if \(Y_1(8, 20, 3, 2)\) and \(Y_2(6, 10, 4, 1)\), then

1) \(Y_1 + Y_2 = (14, 30, 5, \sqrt{5})\);

2) \(Y_1 - Y_2 = (2, 10, 5, \sqrt{5})\);

3) \(2Y_1 = (16, 40, 3, 2\sqrt{2})\).

Definition 3: Given one trapezium cloud \(Y_1(\overline{E_{x1}}, \overline{E_{x1}}, E_n, H_e)\) and an interval-value \(A = [a, b]\),

1) \(Y_1 + A = (\overline{E_{x1}} + a, \overline{E_{x1}} + b, E_n, H_e)\);

2) \(Y_1 - A = (\overline{E_{x1}} - b, \overline{E_{x1}} - a, E_n, H_e)\).

The operations on a trapezium cloud and an interval-value do not increase the uncertainty, and especially when \(a = b\), the operations of a trapezium cloud and an interval-value are reduced to the operations of a trapezium cloud and real numbers.

Example 2: Given a trapezium cloud \(Y_1(4, 5, 2, 0.5)\) and an interval-value \(A = [2, 4]\),

1) \(Y_1 + A = (6, 9, 2, 0.5)\);

2) \(Y_1 - A = (0, 3, 2, 0.5)\).

In recent years, many studies focus on the ranking method of interval values. In [68], two approaches comparing two interval values were defined based on the area measurement with the integral value about the membership function of intervals. In [69], [70], the degree of possibility to compare interval values was defined, which has been widely used. In [71], a method of an interval-valued Choquet integral with respect to an admissible order was presented to compare intervals. A method for building linear orders between intervals by means of two continuous aggregation functions was proposed [72].

It is worth noting that there exist difficulties in utilizing the method in [68], [71], [72] to compare two trapezium clouds, since a trapezium cloud is described by not only expected values, but also entropy and hyper entropy. For convenience, a possibility degree for the comparison between two trapezium clouds is proposed in this paper.

Definition 4: Given two trapezium clouds \(Y_1(\overline{E_{x1}}, \overline{E_{x1}}, E_n, H_e)\) and \(Y_2(\overline{E_{x2}}, \overline{E_{x2}}, E_n, H_e)\), the possibility degree for the comparison between two trapezium clouds \(Y_1\) and \(Y_2\) can be represented as follows:

\[
p(Y_1 \geq Y_2) = \min \left( \frac{\min \{l_i, l_j\}, \max \{s_i, s_j\}}{l_i + l_j} \right)
\]

where \(l_i = \frac{1 - \sqrt{E_n^2 + H_e^2}}{\sqrt{E_n^2 + H_e^2}} (\overline{E_{x1}} - \overline{E_{x}}),\)

\(l_j = \frac{1 - \sqrt{E_n^2 + H_e^2}}{\sqrt{E_n^2 + H_e^2}} (\overline{E_{x2}} - \overline{E_{x}}),\)

And \(s_i = 1 - \frac{\sqrt{E_n^2 + H_e^2}}{\sqrt{E_n^2 + H_e^2}},\)

\(s_j = 1 - \frac{\sqrt{E_n^2 + H_e^2}}{\sqrt{E_n^2 + H_e^2}}.\)

If \(E_n = H_e = E_n = H_e = 0\),

then \(\sqrt{E_n^2 + H_e^2} = 0,\)

\(\sqrt{E_n^2 + H_e^2} = 0,\)

\(\sqrt{E_n^2 + H_e^2} = 0.\)

Now, the possibility degree between two trapezium clouds is reduced to the possibility degree between two interval-values [69].

Theorem 1: Given two trapezium clouds \(Y_1(\overline{E_{x1}}, \overline{E_{x1}}, E_n, H_e)\) and \(Y_2(\overline{E_{x2}}, \overline{E_{x2}}, E_n, H_e)\), then the followings are true.

1) \(0 \leq p(Y_1 \geq Y_2) \leq 1;\)

2) \(p(Y_1 \geq Y_2) = 1 \Leftrightarrow s_1 \overline{E_{x1}} = s_1 \overline{E_{x2}};\)

3) \(p(Y_1 \geq Y_2) = 0 \Leftrightarrow s_1 \overline{E_{x1}} = s_1 \overline{E_{x2}};\)

4) if \(s_1 \overline{E_{x1}} = s_1 \overline{E_{x2}} = 1\), especially \(p(Y_1 \geq Y_2) = 1/2;\)

5) if \(s_1 \overline{E_{x1}} = s_1 \overline{E_{x2}} = 1\), then \(p(Y_1 \geq Y_2) \geq 1/2,\)

and especially if \(s_1 \overline{E_{x1}} + s_1 \overline{E_{x2}} = s_1 \overline{E_{x1}} + s_1 \overline{E_{x2}},\) then
\[ p(Y_1 \geq Y_2) = 1/2. \]

**Proof.** For computational convenience, suppose

\[ A_1 = \sqrt{E_1^2 + \mu_1^2} \quad \text{and} \quad A_2 = \sqrt{E_2^2 + \mu_2^2}, \]

and then

\[ s_1 \bar{E}_1 - s_2 \bar{E}_2 = \frac{A_1 \bar{E}_1 - A_2 \bar{E}_2}{A_1 + A_2}, \]

\[ l_i + l_i = \frac{A_i (\bar{E}_1 - \bar{E}_2)}{A_1 + A_2}. \]

The theorem is proved as follows:

1. Since \( l_i + l_i \geq 0 \), then

\[ \min \{l_i + l_i, \max \{s_1 \bar{E}_1 - s_2 \bar{E}_2, 0\}\} \geq 0, \]

and

\[ \min \{l_i + l_i, \max \{s_1 \bar{E}_1 - s_2 \bar{E}_2, 0\}\} \leq l_i + l_i, \]

therefore, \( 0 \leq p(Y_1 \geq Y_2) \leq 1 \).

2. Assume \( s_2 \bar{E}_2 \leq s_1 \bar{E}_1 \), and then

\[ \min \{l_i + l_i, \max \{s_1 \bar{E}_1 - s_2 \bar{E}_2, 0\}\} \leq l_i + l_i, \]

therefore, \( 0 \leq p(Y_1 \geq Y_2) \leq 1 \).

3. Assume \( s_2 \bar{E}_2 \geq s_1 \bar{E}_1 \), and then

\[ \min \{l_i + l_i, \max \{s_1 \bar{E}_1 - s_2 \bar{E}_2, 0\}\} \geq 0, \]

and \( p(Y_1 \geq Y_2) = 0 \).

4. Assume \( s_2 \bar{E}_2 \geq s_1 \bar{E}_1 \), and then

\[ \min \{l_i + l_i, \max \{s_1 \bar{E}_1 - s_2 \bar{E}_2, 0\}\} \geq 0, \]

and \( p(Y_1 \geq Y_2) = 0 \).

Therefore, \( s_1 \bar{E}_1 - s_2 \bar{E}_2 \leq 0 \) and \( s_1 \bar{E}_1 - s_2 \bar{E}_2 = 0 \).

Since \( \max \{s_1 \bar{E}_1 - s_2 \bar{E}_2, 0\} = (\bar{E}_1 - \bar{E}_2)/2 \), and

\[ l_i + l_i = \bar{E}_1 - \bar{E}_2, \]

then

\[ \min \{l_i + l_i, \max \{s_1 \bar{E}_1 - s_2 \bar{E}_2, 0\}\} = \frac{\bar{E}_1 - \bar{E}_2}{2}. \]

Therefore, \( p(Y_1 \geq Y_2) = 1/2 \).

5. Assume \( s_2 \bar{E}_2 \leq s_1 \bar{E}_1 \), then \( s_1 \bar{E}_1 \geq s_2 \bar{E}_2 \) (the situation of \( s_1 \bar{E}_1 \leq s_2 \bar{E}_2 \) does’t exist) and \( \max \{s_1 \bar{E}_1 - s_2 \bar{E}_2, 0\} = s_1 \bar{E}_1 - s_2 \bar{E}_2 \).

Given \( s_2 \bar{E}_2 \leq s_1 \bar{E}_1 \),

\[ l_i + l_i = \frac{s_1 \bar{E}_1 - s_2 \bar{E}_2}{A_1 + A_2} = \frac{A_1 \bar{E}_1 - A_2 \bar{E}_2}{A_1 + A_2} = \frac{A_1 \bar{E}_1 - A_2 \bar{E}_2}{A_1 + A_2} \]

\[ \frac{s_1 \bar{E}_1 - s_2 \bar{E}_2}{A_1 + A_2} \geq \frac{s_1 \bar{E}_1 - s_2 \bar{E}_2}{A_1 + A_2} = s_1 \bar{E}_1 - s_2 \bar{E}_2. \]

Therefore, \( p(Y_1 \geq Y_2) = 1/2 \).

Assume that there are \( g \) trapezium clouds \( Y_i (i = 1, 2, \cdots, g) \),
and each trapezium cloud \( Y_j \) is compared to all trapezium clouds \( Y_j(j = 1,2,\ldots,g) \) by using Eq. (1), i.e.,

\[
p_g = p(Y_j \geq Y_j) = \frac{\min \{l_y + l_y', \max \{s, E_x - s, F_x, 0\}\}}{l_y + l_y'}.
\]

Then a complementary matrix can be set up as follows:

\[
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1g} \\
p_{21} & p_{22} & \cdots & p_{2g} \\
\vdots & \vdots & \ddots & \vdots \\
p_{g1} & p_{g2} & \cdots & p_{gg}
\end{bmatrix},
\]

where \( p_g \geq 0, p_g + p_{g} = 1 \), and \( p_g = 1/2 \).

According to [73], Theorem 2 can be obtained as follows.

**Theorem 2:** Let \((p_g)_g \) be a fuzzy complementary judgment matrices, and then its priority vector \( \nu \) can be obtained as follows:

\[
\nu = (v_1, v_2, \cdots, v_g), \quad \text{where} \quad v = \frac{1}{g(g - 1)} \left( \sum_{i=1}^{g} p_i + \frac{g}{2} - 1 \right).
\]

The advantages of utilizing the priority vector are mainly as follows: the decision-making information is made best use of; the amount of calculation is very small.

**Definition 5:** Given two trapezium clouds \( Y_1(\overline{E}_{x_1}, \overline{E}_{x_1}, E_{x_1}, H_{e_1}) \) and \( Y_2(\overline{E}_{y_2}, \overline{E}_{y_2}, E_{y_2}, H_{e_2}) \), whose priority indices are \( v_1 \) and \( v_2 \), then

1) if \( v_1 > v_2 \), then \( Y_1 > Y_2 \);
2) if \( v_1 = v_2 \), then \( Y_1 = Y_2 \);
3) if \( v_1 < v_2 \), then \( Y_1 < Y_2 \).

\( Y_1 \geq Y_2 \iff p(Y_1 \geq Y_2) \geq p(Y_2 \geq Y_1) \) and \( Y_1 = Y_2 \iff p(Y_1 \geq Y_2) = p(Y_2 \geq Y_1) = 1/2 \) can be obtained from Definition 5.

**Proposition 1:** Let \( F \) be the set of all trapezium clouds, and the relation on \( F \) is defined as follows:

\( F, \leq \) satisfies:

1) reflexivity: \( \forall Y, Y \in F, Y \leq Y \);
2) transitivity: \( \forall Y_1, Y_2, Y_3 \in F, \text{ if } Y_1 \leq Y_2 \text{ and } Y_2 \leq Y_3, \text{ then } Y_1 \leq Y_3 \);
3) linear order: \( \forall Y_1, Y_2 \in F, Y_1 \leq Y_2 \text{ or } Y_2 \leq Y_1 \text{ or } Y_1 = Y_2 \).

**Definition 6:** Given \( g \) trapezium clouds \( (Y_1, Y_2, \cdots, Y_g) \), whose priority vectors are \( (v_1, v_2, \cdots, v_g) \), and then the larger the element of priority vector is, the larger the trapezium cloud will be.

**Example 3:** Let \( Y_1(20.3,36.75,14.5,0.321) \), \( Y_1(15.16,31.37,12.52,0.24) \), \( Y_1(13.34,41.23,13.72,0.12) \), \( Y_2(28.67,33.24,17.61,0.099) \), \( Y_1(22.88,24.56,10.99,0.45) \) be five trapezium clouds, and the possibility degree matrix can be obtained by applying Eq. (1) as follows:

\[
P = \begin{bmatrix}
0.500 & 0.545 & 0.504 & 0.650 & 0.353 \\
0.455 & 0.500 & 0.472 & 0.565 & 0.294 \\
0.496 & 0.528 & 0.500 & 0.783 & 0.513 \\
0.350 & 0.435 & 0.217 & 0.500 & 0.101 \\
0.647 & 0.706 & 0.487 & 0.899 & 0.500
\end{bmatrix}.
\]

By applying Eq. (2), \( \nu = (0.203,0.189,0.216,0.155,0.237) \). Therefore, the ranking of the trapezium clouds is \( x_5 > x_4 > x_2 > x_3 > x_1 \).

**III. SOME AGGREGATION OPERATORS FOR TRAPEZIUM CLOUDS**

This paper provides a method for Atanassov's interval-valued intuitionistic linguistic decision-making problems based on trapezium clouds. In the decision-making process, Atanassov's interval-valued intuitionistic linguistic information represented by trapezium clouds should be aggregated to obtain the overall evaluation. However, the related aggregation operators of trapezium clouds have not been reported yet, and therefore, in the following part, some aggregation operators for trapezium clouds will be introduced.

**Definition 7:** Let \( Y_j(\overline{E}_{x_j}, \overline{E}_{x_j}, E_{x_j}, H_{e_j})(j = 1,2,\cdots,g) \) be a set of trapezium clouds. The weighted arithmetic averaging (TCWAA) operator is defined as

\[
TCWAA(Y_1,Y_2,\cdots,Y_g) = \left( \sum_{j=1}^{g} w_j \overline{E}_{x_j}, \sum_{j=1}^{g} w_j \overline{E}_{x_j}, \sum_{j=1}^{g} w_j E_{x_j}, \sum_{j=1}^{g} w_j H_{e_j} \right),
\]

where \( W = (w_1, w_2, \cdots, w_g) \) is the weight vector of \( Y_j(j = 1,2,\cdots,g) \), \( w_j \in [0,1] \) and \( \sum_{j=1}^{g} w_j = 1 \). Especially, if \( W = \left( \frac{1}{g}, \frac{1}{g}, \cdots, \frac{1}{g} \right) \), then the TCWAA operator is reduced to an arithmetic averaging (TCAA) operator for trapezium clouds.

**Theorem 3:** Let \( Y_j(\overline{E}_{x_j}, \overline{E}_{x_j}, E_{x_j}, H_{e_j})(j = 1,2,\cdots,g) \) be a set of trapezium clouds. Then their aggregated result by applying the TCWAA operator is also a trapezium cloud, and

\[
TCWAA(Y_1,Y_2,\cdots,Y_g) = \left( \sum_{j=1}^{g} w_j \overline{E}_{x_j}, \sum_{j=1}^{g} w_j \overline{E}_{x_j}, \sum_{j=1}^{g} w_j E_{x_j}, \sum_{j=1}^{g} w_j H_{e_j} \right),
\]

The theorem is proved by using mathematical induction as follows.

**Proof.**
1) For \( g = 2 \) : since \( w_1 = (w_1 \overline{E}_{x_1}, w_1 \overline{E}_{x_1}, E_{x_1}, H_{e_1}) \), and \( w_2 = (w_2 \overline{E}_{x_2}, w_2 \overline{E}_{x_2}, E_{x_2}, H_{e_2}) \), then

\[
TCWAA(Y_1,Y_2) = w_1 Y_1 + w_2 Y_2 = \left( w_1 \overline{E}_{x_1} + w_2 \overline{E}_{x_2}, w_1 \overline{E}_{x_1} + w_2 \overline{E}_{x_2}, \sqrt{w_1 E_{x_1}^2 + w_2 E_{x_2}^2}, \sqrt{w_1 H_{e_1}^2 + w_2 H_{e_2}^2} \right)
\]

2) Assume Eq. (4) holds for \( g = k \), namely,

\[
TCWAA(Y_1,Y_2,\cdots,Y_k) = \left( \sum_{j=1}^{k} w_j \overline{E}_{x_j}, \sum_{j=1}^{k} w_j \overline{E}_{x_j}, \sum_{j=1}^{k} w_j E_{x_j}, \sum_{j=1}^{k} w_j H_{e_j} \right).
\]

When \( g = k + 1 \),

\[
TCWAA(Y_1,Y_2,\cdots,Y_k,Y_{k+1}) = \left( w_{k+1} \overline{E}_{x_{k+1}} + \sum_{j=1}^{k} w_j \overline{E}_{x_j}, w_{k+1} \overline{E}_{x_{k+1}} + \sum_{j=1}^{k} w_j \overline{E}_{x_j}, \sqrt{w_{k+1} E_{x_{k+1}}^2 + \sum_{j=1}^{k} w_j E_{x_j}^2}, \sqrt{w_{k+1} H_{e_{k+1}}^2 + \sum_{j=1}^{k} w_j H_{e_j}^2} \right).
\]
be the weight vector of \( g \in \Omega \), the following is obtained:

\[
\omega = \left( \sum_{i=1}^{k+1} w_i g_i^1, \sum_{i=1}^{k+1} w_i g_i^2 \right)
\]

So, Eq. (4) holds for \( g = k + 1 \). Thus, Eq. (4) holds for all \( n \).

**Example 4:** Let \( Y_1(5,10,2,0.2) \), \( Y_2(3,7,1,0.1) \), \( Y_3(7,10,2.5,0.3) \), \( Y_4(8,10,3,0.2) \) be four trapezium clouds, and \( W = (0.25,0,3,0.4,0.05) \) be the weight vector of \( Y_j(j = 1,2,3,4) \). Then by applying Eq. (4), TCOWA(Y_1, Y_2, Y_3, Y_4) = (5.35,9.1,2.06,0.23) is obtained.

**Definition 8:** Let \( Y_j(\overline{E}_x,E_x,\overline{H}_e,E_h) \)(\( j = 1,2,\ldots, g \)) be a set of trapezium clouds. The ordered weighted averaging (TCOWA) operator is defined as:

\[
\text{TCOWA}(Y_1, Y_2, \ldots, Y_g) = \left( \sum_{j=1}^{g} w_j Y_j^1, \sum_{j=1}^{g} w_j Y_j^2 \right)
\]

where \( W = (w_1, w_2, \ldots, w_g) \) is the weight vector of \( \{Y_1, Y_2, \ldots, Y_g\} \), satisfying \( 0 \leq w_j \leq 1 \) and \( \sum_{j=1}^{g} w_j = 1 \), and \( Y_j^* \) is the \( j \)-th largest one of \( \{Y_1, Y_2, \ldots, Y_g\} \), satisfying \( 0 \leq w_j \leq 1 \) and \( \sum_{j=1}^{g} w_j = 1 \), and \( g \) is a balance factor.

**Theorem 5:** Let \( Y_j(\overline{E}_x,E_x,\overline{H}_e,E_h) \)(\( j = 1,2,\ldots, g \)) be a set of trapezium clouds. Then their aggregated result by applying the TCOWA operator is also a trapezium cloud, and

\[
\text{TCOWA}(Y_1, Y_2, \ldots, Y_g) = \left( \sum_{j=1}^{g} w_j \overline{E}_x^1, \sum_{j=1}^{g} w_j \overline{E}_x^2 \right)
\]

**Example 6:** Let \( Y_1(5,10,2,0.2) \), \( Y_2(3,7,1,0.1) \), \( Y_3(7,10,2.5,0.3) \), \( Y_4(8,10,3,0.2) \) be four trapezium clouds, and \( W = (0.25,0,3,0.4,0.05) \) be the weight vector of \( Y_j(j = 1,2,3,4) \). Then by applying Eq. (8) with the weight vector \( \omega = (0,0.4,0.5,0.1) \), the following is obtained:

\[
\text{TCOWA}(Y_1, Y_2, \ldots, Y_4) = (5.82,10.46,2.86,0.30)
\]

**IV. THE CONVERSION BETWEEN AIVILNS AND TRAPEZIUM CLOUDS**

Regarding uncertain decision-making problems, decision-makers incline to express the evaluation information with linguistic descriptors. However, it seldom occurs that a single linguistic term can completely accurately describing the performance evaluation, i.e., the membership degree of objects to a specific linguistic term may not be 1. AIVILN, a linguistic term with the membership degree and non-membership degree taking the form of interval values, could depict the uncertainty of qualitative information at great length. For MCDM problems with Atanassov's interval-valued intuitionistic linguistic information, the extension of traditional linguistic decision-making methods to Atanassov's interval-valued intuitionistic linguistic decision-making methods will pass the traditional linguistic decision-making methods' deficiencies down, but Atanassov's interval-valued intuitionistic linguistic decision-making based on trapezium clouds can handle it without the shortcomings of the aforementioned methods, as long as the conversion between AIVILNs and trapezium clouds is implemented.

Let \( H = \{ h_i \mid i = 0,1,\ldots,2t, t \in N^+ \} \) be a finite and linguistic term set, where \( h_i \) represents a possible value for a linguistic variable, and it should satisfy the following characteristics [34].

1) The set is ordered: \( h_i > h_j \), if \( i > j \) ;
2) There is the negation operator: \( h_i = \neg(h_i) \), where \( i + j = 2t \);
3) Max operator: \( \max\{h_i, h_j\} = h_i \), if \( i \geq j \);
4) Min operator: \( \min\{h_i, h_j\} = h_j \), if \( i \geq j \).

**Definition 10 [35]:** Let a set \( X = \{x_1, x_2, \ldots, x_n\} \) be fixed and \( h_i(x) \in H \). An AILFS \( A \) in \( X \) is an object:

\[
A = \{x, h_i(x), \mu_i(x), \nu_i(x) \mid x \in X\}
\]
where $\mu_i(x) \to [0,1]$ and $v_i(x) \to [0,1]$, under the condition $0 \leq \mu_i(x) + v_i(x) \leq 1$. The numbers $\mu_i(x)$ and $v_i(x)$ determine respectively the membership degree and non-membership degree of the element $x$ to the linguistic value $h_{0i}$. For each AILFS $A$ in $X$, if $\pi_i(x) = 1 - \mu_i(x) - v_i(x)$, $x \in X$, then $\pi_i(x)$ is called the Atanassov’s intuitionistic index of the element $x$ in the set $A$. It is a hesitancy degree of $x$ to $A$. It is obvious that $0 \leq \pi_i(x) \leq 1, x \in X$. For computational convenience, $\alpha = h_{0i}, \mu_i(x), v_i(x)$ is called an AILN.

Especially if $\mu_i(x) = 1$ and $v_i(x) = 0$, then the AILN $A$ is reduced to a linguistic term set; if the linguistic term set includes a single value, then the AILN $A$ is reduced to an AIFS.

The AILN $\alpha = h_{0i}, \mu_i(x), v_i(x)$ has a physical interpretation, for example, if $< h_{0i}, \mu_i(x), v_i(x) >= < h_i, 0.5, 0.3 >$, then it can be interpreted as “the vote for resolution which is good ($h_i$) is 5 in favor, 3 against, and 2 abstentions”.

Actually, it may not be easy to identify the exact value of the membership degree and non-membership degree of an element to a linguistic term set. In this case, a range of values may be a more appropriate measure to accommodate the vagueness. Thus, this paper introduces the notion of Atanassov’s interval-valued intuitionistic linguistic fuzzy set (AIVILFS).

**Definition 11:** Let a set $X = \{x_1, x_2, \ldots, x_n\}$ be fixed and $h_{0j} \in H$. An AIVILFS $\tilde{A}$ in $X$ is an object:

$$\tilde{A} = \{< x, h_{0j}, \tilde{\mu}_j(x), \tilde{v}_j(x) > | x \in X \}$$

where $\tilde{\mu}_j(x) \subseteq [0,1]$ and $\tilde{v}_j(x) \subseteq [0,1]$ are intervals, and for every $x \in X, 0 \leq \sup \tilde{\mu}_j(x) + \sup \tilde{v}_j(x) \leq 1$.

Similarly, the intervals $\tilde{\mu}_j(x)$ and $\tilde{v}_j(x)$ determine the membership degree and non-membership degree of the element $x$ to the linguistic value $h_{0j}$, respectively, but for each $x \in X, \tilde{\mu}_j(x)$ and $\tilde{v}_j(x)$ are closed intervals rather than real numbers and their lower and upper boundaries are denoted by $\tilde{\mu}_j^-(x), \tilde{\mu}_j^+(x), \tilde{v}_j^-(x), \tilde{v}_j^+(x)$, respectively.

Therefore, another equivalent expression of an AIVILFS $\tilde{A}$ is:

$$\tilde{A} = \{< x, h_{0j}, [\tilde{\mu}_j^-(x), \tilde{\mu}_j^+(x)]; [\tilde{v}_j^-(x), \tilde{v}_j^+(x)] > | x \in X \}$$

where $\tilde{\mu}_j^-(x) + \tilde{v}_j^+(x) \leq 1$, $\tilde{\mu}_j^-(x) \geq 0$, and $\tilde{v}_j^-(x) \geq 0$.

Similar to AILSs, for each element $x \in X$, the hesititation interval relative to $\tilde{A}$ can be calculated:

$$\tilde{\pi}_j(x) = [\tilde{\pi}_j^-(x), \tilde{\pi}_j^+(x)] = [1 - \tilde{\mu}_j^+(x) - \tilde{v}_j^-(x), 1 - \tilde{\mu}_j^-(x) - \tilde{v}_j^+(x)]$$

For any given $x$, $< h_{0j}, \tilde{\mu}_j(x), \tilde{v}_j(x) >$ is referred to as an AIVILN. For convenience, $< h_{0j}, \tilde{\mu}_j(x), \tilde{v}_j(x) >$ is often denoted by $< h_{0j}, [a, b], [c, d] >$, where $[a, b] \subseteq [0,1]$, $[c, d] \subseteq [0,1]$ and $b + d \leq 1$.

**Example 7:** Assume that a decision-maker constructs an AIVILN $\alpha = < h_i, [0.5, 0.6], [0.2, 0.3] >$, in which $[0.5, 0.6]$ and $[0.2, 0.3]$ are the membership degree and the non-membership degree of the assessment object belonging to the linguistic term $h_i$ ($h_i$ is good).

**Definition 12:** For the linguistic term $h_j$, in a linguistic term set $H$, where $H = \{h_j | j = 0, 1, \ldots, 2t, t \in N^+\}$. If $\theta \in [0,1]$ is a numerical value, then the linguistic scale function $f$, which conducts the mapping from $h_j$ to $\theta_j (j = 0, 1, \ldots, 2t)$, is defined as follows:

$$f : h_j \rightarrow \theta_j \quad \text{for} \quad (j = 0, 1, \ldots, 2t)$$

The value of $a$ can be obtained through experiments or subjective methods. [74] states that $a$ is most likely to be obtained in the interval of $[1.36, 1.4]$ according to a mess of experimental researches. In addition, $a$ can also be determined by a subjective method, i.e., assuming the indicator $A$ is far more important than the indicator $B$, and the importance ratio is $m$, then $a^* = k \times m$. At present, most scholars believe that $m = 9$ is the upper limit of the importance ratio; therefore, with respect to the scale level of 7, $a = \sqrt{5} \approx 2.236$ can be calculated.

Eq. (9) is a remarkable function with the advantages of the index scale and $0 \sim 2t$ scale, where the index scale is a multi-granular assessment scale on the basis of Weber-Fehner law and $0 \sim 2t$ scale is under the psychological discipline of decision-makers assessing objects with two poles: good pole and bad pole [74].

Based on Definition 12, a method for converting AIVILN $\alpha = < h_j, [a, b], [c, d] >$ into the corresponding trapezium clouds $Y_j(\mathbf{Ex}_j, \mathbf{Ex}_j, x_j, H_j)$ is introduced as follows.

1) Calculate $\theta_j$ and $x_j$.

Calculate $\theta_j$ by applying Eq. (9), and supposing the effective domain is $U = [X_{min}, X_{max}]$, then $x_j = X_{max} + \theta_j(X_{max} - X_{min})$.

2) Calculate $\mathbf{Ex}_j$ and $\mathbf{Ex}_j$.

For $x = < h_j, [a, b], [c, d] >$, its average hesitant degree $\gamma$ which reflects the uncertainty of the expected value in the trapezium cloud can be denoted by $1 - \frac{1}{2} (a + b + c + d)$.

$$\mathbf{Ex}_j = \mathbf{Ex}_j - \frac{a - a^*}{2a - 2}$$

For $x_j = \frac{1}{2} (a + b + c + d)$,

$$\mathbf{Ex}_j = \mathbf{Ex}_j - \frac{a - a^*}{2a - 2}$$

$$\mathbf{Ex}_j = \mathbf{Ex}_j - \frac{1}{2} (a + b + c + d)$$
Let \( m_1 = \left[ \frac{1}{2}, \frac{1}{4}(a + b + c + d) \right] \) and \( m_2 = \left[ \frac{3}{2}, \frac{1}{4}(a + b + c + d) \right] \), then \( \overline{E_j} = m_1 E_j \) and \( \underline{E_j} = m_2 E_j \).

3) Calculate \( E_{n_j} \).

A droplet of the trapezium cloud is denoted by \((x, y)\), the distribution of \( \{x|x \leq \underline{E_j}\} \) is a left normal cloud and the distribution of \( \{x|x \geq \overline{E_j}\} \) is a right normal cloud, so there are \( x \sim N(\overline{E_j}, E_{n_j}^\prime) \), \( E_{n_j}^\prime - N(E_{n_j}, He_{n_j}) \) for \( x \leq \underline{E_j} \) and \( x \sim N(\overline{E_j}, E_{n_j}^\prime) \), \( E_{n_j}^\prime - N(E_{n_j}, He_{n_j}) \) for \( x \geq \overline{E_j} \).

According to the 3\( \sigma \) principle of the normal distribution, in case of estimating \( E_{n_j}^\prime \), we should act up to \( 3E_{n_j}^\prime = \max\{X_{\max} - \underline{E_j}, \overline{E_j} - X_{\max}\} \).

For \( \gamma \in [0,1] \), \( \max\{E_{n_j}\} = \frac{\max\{X_{\max} - \underline{E_j}, \overline{E_j} - X_{\max}\}}{3} \) and \( \min\{E_{n_j}\} = \frac{\max\{X_{\max} - \underline{E_j}, \overline{E_j} - X_{\max}\}}{3} \).

Furthermore, since \( E_{n_j}^\prime - N(E_{n_j}, He_{n_j}) \), the statistic of \( E_{n_j}^\prime \), can be regarded as \( E_{n_j} \), i.e., \( E_{n_j} = \frac{\max\{E_{n_j}\} + \min\{E_{n_j}\}}{2} \).

4) Calculate \( He_{n_j} \).

Due to \( E_{n_j}^\prime - N(E_{n_j}, He_{n_j}) \), the hyper entropy should satisfy the 3\( \sigma \) principle of the normal distribution. Therefore, \( He_{n_j} = \frac{\max\{E_{n_j}\} - E_{n_j}}{3} \).

**Example 8:** Given the universe \( \{X_{\min}, X_{\max}\} = [5,95] \) and the linguistic term set \( H = \{h_0, h_1, h_2, h_3, h_4, h_5\} \), then the AIVILN \( x = < h_4, [0,6,0.7], [0,1,0.3]> \) can be transformed into trapezium clouds, and the transformation procedures are shown as below.

1) Applying Eq. (9) with \( a = 1.4 \), the value of \( \theta_j \) can be obtained as follows:
\[ \theta_h = 0.615, \quad E_{x^h} = 60.35. \]
2) \( E_{x^h}(x) = 55.82, \quad E_{\overline{x^h}}(x) = 64.88. \)
3) \( \max\{E_{n_j}\} = \frac{\max\{X_{\max} - \underline{E_j}, \overline{E_j} - X_{\max}\}}{3} = 18.45 \), and \( \min\{E_{n_j}\} = \frac{\max\{X_{\max} - \underline{E_j}, \overline{E_j} - X_{\max}\}}{3} = 8.39 \), then
\[ E_{n_j} = \frac{18.45 + 8.39}{2} = 13.42. \]
4) \( He_{n_j} = \frac{18.45 - 13.42}{3} = 1.68. \)

Therefore, the AIVILN \( x = < h_4, [0,6,0.7], [0,1,0.3]> \) can be transformed into a trapezium cloud: \((55.82, 64.88, 13.42, 1.68)\).

V. ATANASSOV’S INTERVAL-VALUED INTUITIONISTIC LINGUISTIC MULTI-CRITERIA GROUP DECISION-MAKING METHOD BASED ON TRAPEZIUM CLOUDS

Consider an Atanassov’s interval-valued intuitionistic linguistic multi-criteria group decision-making (AVISL-MCGDM) problem. Assume that there are \( q \) alternatives \( X = \{x_1, x_2, \ldots, x_q\} \) and \( m \) criteria \( C = \{c_1, c_2, \ldots, c_m\} \) with the weight vector \( W = (w_1, w_2, \ldots, w_m) \) associated with \( C \), where \( w_j \in [0,1] \) and \( \sum_{j=1}^{m} w_j = 1 \). Assume that there are \( n \) decision-makers \( D = \{d_1, d_2, \ldots, d_n\} \) whose corresponding weight vectors are \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), where \( \lambda_i \in [0,1] \) and \( \sum_{i=1}^{n} \lambda_i = 1 \). The evaluation matrix of the \( k \)-th decision-maker \( d_k \) is denoted by \( R_k = (x_{kj})_{m \times n} \), where \( x_{kj} \) is in the form of AIVILNs with the linguistic term set \( H \). Assume that all the decision-makers are risk neutral. Now a method is given to determine the ranking of alternatives as follows.

Step 1: Convert the decision-making information into trapezium clouds.

Use Definition 12 and the conversion method of decision-making information in Section 4, which is in the form of AIVILNs and given by various decision-makers, into the trapezium cloud \( Y(E_{x^h}, E_{\overline{x^h}}, En, He) \).

Step 2: Aggregate the trapezium clouds of each decision-maker.

Utilizing the TCWA operator to aggregate the trapezium clouds of each decision-maker, the individual value of the alternative \( x \), can be derived.
\[ z_i^h = TCWA(x_{i1}, x_{i2}, \ldots, x_{in}) = \sum_{j=1}^{n} w_j x_{ij}^h. \]

Step 3: Aggregate the trapezium clouds of all decision-makers.

Utilize the TCHA operator to derive the overall values \( z_i^h \) of the alternative \( x_i \),
\[ z_i = TCHA(z_i^1, z_i^2, \ldots, z_i^n) = \sum_{i=1}^{n} \alpha_i z_i^i, \]
where the associated weight vector \( \omega = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) of the TCHA operator is determined as follows:
\[ \alpha_k = Q(k/l) - Q((k-1)/l), k = 1, 2, \ldots, n, \]
where \( \alpha_k \in [0,1] \) and \( \sum_{i=1}^{n} \alpha_i = 1 \), and the regular unimodal quantifier is shown as follows:
\[ Q = \begin{cases} 0, r < a \\ r - a, a \leq r \leq b \\ 1, r > b \end{cases}. \]

Here, we utilize the principle of antonym pairs many and \((a,b)\) equals \((0.3, 0.8)\).

Step 4: Rank the alternatives according to the possibility degree matrix of trapezium clouds, and then obtain the best option.
VI. ILLUSTRATIVE EXAMPLE

The AIVIL-MCGDM method introduced in Section V should be put into practice, so the entire process of the AIVIL-MCGDM method based on trapezium clouds is demonstrated via an illustrative example. Assume that an investment company wants to invest a sum of money with the best option. There are five possible alternatives \{x_1, x_2, x_3, x_4, x_5\} evaluated under the following three criteria: the degree of risk \(c_1\), the growth rate \(c_2\) and the environmental impact \(c_3\) with the weight vector \(W = (0.2, 0.3, 0.5)\). And there are three decision-makers \(\{d_1, d_2, d_3\}\), of which the weight vector is \(\lambda = (0.5, 0.3, 0.2)\), given the evaluation information being in the form of AIVILNs with the linguistic term set \(H = \{h_1 = \text{extremely poor}, h_2 = \text{very poor}, h_3 = \text{poor}, h_4 = \text{fair}, h_5 = \text{good}, h_6 = \text{very good}, h_7 = \text{extremely good}\}\). Then, how to get the best option?

The evaluation information given by the three decision-makers is listed in Tables III–V:

**TABLE III.** The evaluation information from the decision-maker \(d_1\)

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(c_1)</th>
<th>(c_2)</th>
<th>(c_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(h_1)</td>
<td>(h_1)</td>
<td>(h_0)</td>
<td>(h_0)</td>
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<td>(h_0)</td>
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<td>(h_0)</td>
<td>(h_0)</td>
<td>(h_0)</td>
<td>(h_1)</td>
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</table>

**TABLE IV.** The evaluation information from the decision-maker \(d_2\)

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(c_1)</th>
<th>(c_2)</th>
<th>(c_3)</th>
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</thead>
<tbody>
<tr>
<td>(h_0)</td>
<td>(h_1)</td>
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<td>(h_0)</td>
<td>(h_0)</td>
<td>(h_0)</td>
<td>(h_1)</td>
</tr>
</tbody>
</table>

**TABLE V.** The evaluation information from the decision-maker \(d_3\)

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(c_1)</th>
<th>(c_2)</th>
<th>(c_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(h_0)</td>
<td>(h_1)</td>
<td>(h_0)</td>
<td>(h_0)</td>
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<tr>
<td>(h_0)</td>
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<td>(h_0)</td>
<td>(h_1)</td>
<td>(h_0)</td>
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A. Procedures of the AIVIL-MCGDM method based on trapezium clouds

**Step 1:** Convert the decision-making information into trapezium clouds.

Firstly, set the effective domain \([X_{min}, X_{max}] = [5, 95]\), by applying Eq. (9) (where \(a = 1.4\)), and the followings can be obtained:

\[
\theta_b = 0, \ \theta_1 = 0.775, \ \theta_2 = 0.385, \ \theta_3 = 0.500,
\]

So, \(Ex_0 = 5\), \(Ex_1 = 25.25\), \(Ex_2 = 39.65\), \(Ex_3 = 50\), \(Ex_4 = 60.35\), \(Ex_5 = 74.75\) and \(Ex_6 = 95\).

Then, by using the conversion method in Section IV, the evaluated information can be converted into trapezium clouds, as shown in Tables VI–VIII.

**Step 2:** Aggregate the trapezium clouds of each decision-maker. By utilizing the TCWAA operator to aggregate the trapezium clouds of each decision-maker, the individual value of the alternative \(x_i\) can be derived.

The integrated trapezium clouds for the alternatives \(x_1, x_2, x_3, x_4, x_5\) of the decision-maker \(d_i\) are shown as follows:

**TABLE VI.** The trapezium clouds of evaluation information given by \(d_1\)

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(c_1)</th>
<th>(c_2)</th>
<th>(c_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(41.45, 52.79, 17.30, 1.37)</td>
<td>(43.58, 50.21, 12.29, 1.31)</td>
<td>(42.27, 52.34, 17.06, 1.41)</td>
<td>(36.59, 44.78, 17.35, 1.22)</td>
</tr>
</tbody>
</table>

The integrated trapezium clouds for the alternatives \(x_1, x_2, x_3, x_4, x_5\) of the decision-maker \(d_2\) are listed in the following:

\(Y_1^1 = (39.51, 45.64, 14.71, 1.22), \ Y_2^1 = (33.53, 39.65, 18.13, 1.08), \ Y_3^1 = (44.79, 51.07, 13.87, 1.36), \ Y_4^1 = (38.30, 45.59, 14.73, 1.19), \) and \(Y_5^1 = (36.41, 45.06, 17.35, 1.22)\).

The integrated trapezium clouds for the alternatives \(x_1, x_2, x_3, x_4, x_5\) of the decision-maker \(d_3\) are outlined below: \(Y_1^2 = (41.71, 51.72, 15.54, 1.35), \ Y_2^2 = (31.94, 39.17, 17.46, 1.03), \ Y_3^2 = (37.11, 48.40, 14.00, 1.19)\), and \(Y_4^2 = (34.03, 43.30, 17.99, 1.15)\).
Therefore, the ranking of the alternatives is $Y_1 > Y_2 > Y_3 > Y_4 > Y_5$. The procedures of the method based on Atanassov’s interval-valued intuitionistic uncertain linguistic (AIVIUL) aggregation operators in [75] are shown as follows.

The AIVIUL weighted arithmetic averaging (AIVIULWA) operator is used to aggregate the criteria values of each alternative. The individual overall evaluation values $Z_i'$ of the alternatives $x_i$ given by $d_i$ are shown in Table IX, which take the form of Atanassov’s interval-valued intuitionistic linguistic variables.

The AIVIUL hybrid averaging (AIVIULHA) operator has not been introduced yet, but according to the operations defined in [75], the hybrid averaging operator can be extended to AIVIUL information. By applying the AIVIULHA operator with the associated weight vector $\omega = (0.0666, 0.6667, 0.2667)$, the aggregation of overall evaluation values $Z_i$ of the alternative $x_i$ is shown as follows:

$$Z_i = h_{243\cdot\{0.58, 0.72\}, \{0, 0.254\}},$$

Table IX. The individual overall evaluation values $Z_i$

<table>
<thead>
<tr>
<th>$d_i$</th>
<th>$Z_1$</th>
<th>$Z_2$</th>
<th>$Z_3$</th>
<th>$Z_4$</th>
<th>$Z_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td></td>
<td>$h_{243\cdot{0.58, 0.72}, {0, 0.254}}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_3$</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

$Z_2 = h_{224\cdot\{0.59, 1.00\}, \{0, 0\}}$, $Z_3 = h_{224\cdot\{0.55, 0.69\}, \{0, 0.28\}}$, $Z_4 = h_{224\cdot\{0.44, 0.58\}, \{0, 0.40\}}$, $Z_5 = h_{224\cdot\{0.52, 0.72\}, \{0, 0.27\}}$.

In [75], the comparison of AIVIULNs is based on the expected value and the accuracy function, where for $a = [s_{(x_i)}, s_{(r_{(x_i)})}]$, $[\mu^i(a), \mu^i(a)], [v^i(a), v^i(a)]$, the expected value $E(a)$ of $a$ is presented as

$$E(a) = T \cdot \left( \frac{\mu^i(a) + \mu^i(a)}{2} + 1 - v^i(a) + v^i(a) \right) \times S_{(\theta_{(x_i)}, r_{(a)})}/2,$$

and the accuracy function

$$H(a) = \left( \frac{\mu^i(a) + \mu^i(a)}{2} + v^i(a) + v^i(a) \right) \times S_{(\theta_{(x_i)}, r_{(a)})}/2,$$

By applying Eq. (2), $\nu' = 0.2146, 0.1, 0.2816, 0.1696, 0.2342$ is obtained; therefore, the ranking of the alternatives is $x_5 > x_1 > x_3 > x_4 > x_2$ and $x_5$ is the optimal one.

### B. Comparison analysis and discussion

Considering the results of the illustrative example above, a comparison analysis is going to be conducted using other two main MCDM methods in [75] and [76]. The comparison analysis is based on the same illustrative example.
Thus, the ranking of the alternatives is
\[ x_3 > x_1 \succ x_5 > x_2 > x_4 \].

A comparison among the three models is shown in Table X.

<table>
<thead>
<tr>
<th>Models</th>
<th>Ranking results</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIVIUL aggregation operators in Ref. [75]</td>
<td>( x_1 \succ x_2 \succ x_3 \succ x_4 )</td>
</tr>
<tr>
<td>Distance operator in Ref. [76]</td>
<td>( x_1 \succ x_2 \succ x_3 \succ x_4 )</td>
</tr>
<tr>
<td>Trapezium cloud model</td>
<td>( x_1 \succ x_2 \succ x_3 \succ x_4 )</td>
</tr>
</tbody>
</table>

It can be learned that the rankings of the alternatives with the proposed method and the method in [75] are not exactly the same. The difference lies in the positions of \( x_2 \) and \( x_4 \). Regarding the alternatives \( x_2 \) and \( x_4 \), the expected value of \( x_3 \) is greater than that of \( x_2 \) when they are tackled with the trapezium-cloud-based method proposed in this paper, but the situation is reversed using the method in [75]. This is because a uniform granular linguistic assessment scale is used in the method of [75], but a multi-granular linguistic assessment scale is applied in the trapezium-cloud-based method. Contrasting to the uniform granular linguistic assessment scale, the multi-granular linguistic assessment scale is a better psychological sense. The AIVIULWAA operator and the AIVIUULH operator defined in [75] handle the linguistic term on the basis of linguistic symbolic model. As was mentioned above, it totally abandons the uncertainty of qualitative concepts. Moreover, the operations of AIVIULNs regard the linguistic term and the Atanassov's intuitionistic fuzzy number being independent of each other, which may lead to information loss. In addition, the method in [75] considers nothing but the overall average level of each alternative. However, the trapezium-cloud-based method completes an objective and interchangeable transformation between qualitative concepts and quantitative information, and takes both the average level and the fluctuation and stability degree of qualitative concepts into consideration. As a result, the deficiencies inherited in the method of [75] will be eliminated by the trapezium-cloud-based method.

The difference of the results between the proposed method and the method in [76] lies in the sequence of \( x_2 \) and \( x_4 \). Defuzzification of the method in [76] causes that the results cannot reflect the actual situation. However, the proposed method can avoid the problem of defuzzification.

Considering the factors above, the proposed method is much more precise and reliable than the AIVIUL aggregation operators.

VII. THE CONVERSION BETWEEN AIVILNs AND TRAPEZIUM CLOUDS

In this paper, the operations, the possibility degree and some new aggregation operators of trapezium clouds are defined. Then, a model converting AIVILNs into trapezium clouds is developed. An Atanassov's interval-value intuitionistic linguistic group decision-making method based on trapezium clouds is introduced. Under the background of linguistic decision-making problems, this paper makes four contributions with respect to the existing studies. Firstly, the introduction of AIVILNs makes great sense. AIVILNs can better reflect the uncertainty of human thinking for the advantages in expressing fuzzy linguistic variables. Secondly, the conversion between AIVILNs and trapezium clouds realized the transformation between qualitative concepts and quantitative information. Thirdly, the possibility degree for comparing trapezium clouds is proposed. Lastly, an MCGDM method with AIVIL information based on trapezium cloud aggregation operators is developed, thus it can provide solutions for uncertain linguistic decision-making. The validity and feasibility of this method have been demonstrated by a numerical example together with the corresponding comparison analysis with other methods. In future researches, the problems focusing in entropy measures of trapezium cloud and methods to rank trapezium clouds will be studied.

REFERENCES


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