TRAVELING WAVE SOLUTIONS IN DELAYED REACTION DIFFUSION SYSTEMS WITH APPLICATIONS TO MULTI-SPECIES MODELS

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Abstract. This paper is concerned with the existence of traveling wave solutions in delayed reaction diffusion systems which at least contain multi-species competition, cooperation and predator-prey models with diffusion and delays. By introducing the mixed quasimonotone condition and the exponentially mixed quasimonotone condition, we reduce the existence of traveling wave solutions to the existence of a pair of admissible upper-lower solutions. To illustrate our main results, the existence of traveling wave solutions of multi-species competition, cooperation and predator-prey Lotka-Volterra systems with delays is considered. In particular, we show the precisely asymptotic behavior of the traveling wave solutions of these Lotka-Volterra systems.

1. Introduction. It is well known that reaction diffusion systems with delays play an important role in modeling the population evolution, see the survey by Wu [59]. Many dynamical properties of delayed reaction diffusion systems have been widely studied in the past decades, for example, the persistence and extinction [11, 27, 40, 60], stability and convergence [9, 19, 39, 50, 64], periodic solutions [36, 49], traveling wave solutions [4, 5, 10, 12, 13, 37, 38], etc. In particular, the traveling wave solution is one of the most attracted objects in investigating the dynamical properties of delayed evolution systems due to its significant applications in theory and applications, such as in describing the transition between different states of a physical system and domain invasion of species in population dynamics, in understanding the long time behavior of the solutions of the corresponding Cauchy type problems, see [10, 46, 52, 61]. For scalar equations with diffusion and delays, the existence of traveling wave solutions has been widely studied by using the comparison principle [29, 30, 31, 42, 46, 47, 51, 55, 57, 65] and perturbation method [3, 6, 7, 34]. As mentioned in Weinberger et al. [54], the most interesting models in population ecology involve the interaction of multiple species. In population dynamics, three typical and important delayed models with diffusion are listed as follows.

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(i) The multi-species diffusion-competition system [32]

\[ \frac{\partial u_i(x,t)}{\partial t} = d_i \Delta u_i(x,t) + r_i u_i(x,t) \left[ 1 - \sum_{j \in I} a_{ij} u_j(x,t - \tau_{ij}) \right], \quad i \in I, \quad (1) \]

where all the parameters are nonnegative, \( x \in \mathbb{R}, t > 0, u = (u_1, \cdots, u_n) \in \mathbb{R}^n \) and \( I = \{1, 2, \cdots, n\} \).

(ii) The predator-prey Lotka-Volterra system [18, 40]

\[
\begin{cases}
\frac{\partial u_1(x,t)}{\partial t} = d_1 \Delta u_1(x,t) + r_1 u_1(x,t) \left[ 1 - a_{11} u_1(x,t - \tau_{11}) - a_{12} u_2(x,t - \tau_{12}) \right], \\
\frac{\partial u_2(x,t)}{\partial t} = d_2 \Delta u_2(x,t) + r_2 u_2(x,t) \left[ 1 + a_{21} u_1(x,t - \tau_{21}) - a_{22} u_2(x,t - \tau_{22}) \right].
\end{cases} \quad (2)
\]

(iii) The diffusion-cooperation system with delays [16]

\[
\begin{cases}
\frac{\partial u_1(x,t)}{\partial t} = d_1 \Delta u_1(x,t) + r_1 u_1(x,t) \left[ 1 - a_{11} u_1(x,t - \tau_{11}) + a_{12} u_2(x,t - \tau_{12}) \right], \\
\frac{\partial u_2(x,t)}{\partial t} = d_2 \Delta u_2(x,t) + r_2 u_2(x,t) \left[ 1 + a_{21} u_1(x,t - \tau_{21}) - a_{22} u_2(x,t - \tau_{22}) \right].
\end{cases} \quad (3)
\]

Under the Neumann boundary condition, Martin and Smith [32] considered the monotonicity, invariance, comparison and convergence of (1). When the spatial domain is bounded, Ruan and Zhao [40] investigated the persistence and extinction of (1) (with \( n = 2 \)) and (2). If \( n = 2 \) in (1), Li et al. [21] established the existence of traveling wave solutions. For (1) with \( n \geq 3 \) and (2), it remains an open problem on the existence of traveling wave solutions. Moreover, Huang and Zou [16] (also see Li and Wang [23]) proved some existence results for the traveling wave solution of (3). Motivated by the linearly determinate conjecture [54], we believe that the threshold of the wave speed in [16] can be improved and we shall further investigate its traveling wave solutions.

In this paper, our main purpose is to study the existence of traveling wave solutions of the following delayed diffusion system

\[ \frac{\partial u_i(x,t)}{\partial t} = d_i \Delta u_i(x,t) + f_i(u_i(x)), \quad i \in I, \quad (4) \]

where \( x \in \mathbb{R}, t > 0, d_i > 0, i \in I = \{1, 2, \cdots, n\}, u = (u_1, u_2, \cdots, u_n) \in \mathbb{R}^n, \) \( f = (f_1, \cdots, f_n) : C([-\tau,0], \mathbb{R}^n) \to \mathbb{R}^n \) is continuous, and \( u_i(x) \) is an element in \( C([-\tau,0], \mathbb{R}^n) \) parameterized by \( x \in \mathbb{R} \) and given by

\[ u_i(x)(s) = u(x,t + s), s \in [-\tau,0], t \geq 0, x \in \mathbb{R}, \]

hereafter, \( \tau \) denotes the maximal delay involved, and \( \tau = \max_{i,j \in I} \{ \tau_{ij} \} \) in the system (1). We also assume that \( f \) has a trivial equilibrium state \( \mathbf{0} = (0, 0, \cdots, 0) \in \mathbb{R}^n \) and a positive one \( \mathbf{K} = (k_1, k_2, \cdots, k_n) \in \mathbb{R}^n \). It should be noted that \( [\mathbf{0}, \mathbf{K}] \) maybe is not an invariant set of the corresponding ordinary differential equations (for short, ODEs) of (4), for example, the systems (1) and (2) (if they admit positive equilibria, respectively), thus it is difficult to find the traveling wave solution of (4) in the region \( [\mathbf{0}, \mathbf{K}] \), and we should choose a proper region in which we can prove the existence of traveling wave solutions of (4). For the purpose, we assume that for some \( \mathbf{M} \geq \mathbf{K}, [\mathbf{0}, \mathbf{M}] \) is the minimal invariant set of the corresponding ODEs of (4). For convenience, we first give the following definition of traveling wave solutions.
Definition 1.1. A traveling wave solution of (4) is a special solution of the form \( u(x, t) = \Phi(x + ct) \), where \( \Phi = (\phi_1, \phi_2, \cdots, \phi_n) \in C^2(\mathbb{R}, \mathbb{R}^n) \) is the wave profile that propagates through the one-dimensional spatial domain at a constant velocity \( c > 0 \).

Substituting \( u(x, t) = \Phi(x + ct) \) into (4) and denoting \( x + ct \) as \( t \), then (4) has a traveling wave solution \( \Phi \) if and only if \( \Phi \) is the solution of the following functional differential system

\[
d_i \phi_i''(t) - c \phi_i'(t) + f_i^c(\Phi_i) = 0, \quad t \in \mathbb{R}, \quad i \in I,
\]

where \( f_i^c(\Phi_i) := f_i(\Phi_i) \) and \( \Phi_i(s) = \Phi(t + cs), s \in [-\tau, 0] \). Moreover, recalling the background of traveling wave solutions [52, 63] and the importance of coexistence in population dynamics [14, 20, 24, 27, 33, 40], we also require that \( \Phi \) satisfies the following asymptotic boundary conditions

\[
\lim_{t \to -\infty} \Phi(t) = 0, \quad \lim_{t \to +\infty} \Phi(t) = K.
\]

To further consider the traveling wave solution of (4), we should mention some results on the existence of traveling wave solutions of (4), which were essentially based on the comparison principle. In a pioneering work, Schaal [42] systematically studied two scalar reaction diffusion equations with a single discrete delay, both monostable and bistable waves were considered. Since then, many researchers have paid attention to this topic if the reaction term \( f \) satisfies the so-called (exponential) quasimonotonicity [17, 25, 26, 28, 56, 61, 62, 66], (exponentially) partial quasimonotonicity [18], (exponentially) weak quasimonotonicity [21]. These results can be applied to many important models, for example, the delayed Logistic equation [61, 65], the Nicholson’s blowflies model with delay [22, 47], the age-structured model of single species [4, 5], the diffusion-cooperation system [16, 23], the delayed Belousov-Zhabotinskii model [17, 28, 61] and the diffusion-competition model with delays [21]. But these results cannot be applied to the system (1) with \( n \geq 3 \) and the system (2) when their trivial and positive equilibria are concerned, since these systems do not satisfy the monotone conditions mentioned above. Thus new techniques should be developed to consider the problem.

From the viewpoint of ecology, we can assume that any two species of \( n \)-species are competitive or cooperative or predator-prey (see [33]). Thus, let (4) satisfy the following mixed quasimonotone condition (MQM) or the exponentially mixed quasimonotone condition (EMQM) in the sense of general partial ordering in \( \mathbb{R}^n \).

**MQM** There exists a matrix \( \beta = \text{diag}(\beta_1, \beta_2, \cdots, \beta_n), \beta_i > 0, i \in I \) such that

(I): \( f_i^c(\Phi) + \beta_i \Phi_i(0) \) is nondecreasing with respect to \( \phi_i(s) \),

(II): for any \( j \in I_i \), \( f_i^c(\Phi) \) is monotone with respect to \( \phi_j(s), s \in [-\tau, 0] \),

where \( i \in I, \Phi(s) = (\phi_1(s), \cdots, \phi_n(s)) \in C\left([-\tau, 0], \mathbb{R}^n\right) \) and \( 0 \leq \Phi(s) \leq M, s \in [-\tau, 0] \), hereafter \( I_i = \{1, 2, \cdots, i-1, i+1, \cdots, n\} \).

**EMQM** There exists a matrix \( \beta = \text{diag}(\beta_1, \beta_2, \cdots, \beta_n), \beta_i > 0 \) such that

(I): for each \( i \in I \),

\[
f_i^c(\Phi) + \beta_i \Phi_i(0) \geq f_i^c(\tilde{\Phi}_i) + \beta_i \tilde{\Phi}_i(0),
\]

if \( \tilde{\Phi}_i = (\phi_1, \cdots, \phi_{i-1}, \tilde{\phi}_i, \phi_{i+1}, \cdots, \phi_n), \Phi_i = (\phi_1, \cdots, \phi_n) \in C\left([-\tau, 0], \mathbb{R}^n\right) \)

(a): \( 0 \leq \tilde{\Phi}(s) \leq \Phi(s) \leq M, s \in [-\tau, 0] \);

(b): \( e^{\beta_i s} \left[ \phi_i(s) - \tilde{\phi}_i(s) \right] \) is nondecreasing in \( s \in [-\tau, 0] \),
(II): for any \( j \in I \), \( f_j^c(\Phi) \) is monotone with respect to \( \phi_j(s) \), where \( \Phi(s) = (\phi_1(s), \cdots, \phi_n(s)) \in C \([-ct, 0], \mathbb{R}^n) \) and \( 0 \leq \Phi(s) \leq M, s \in [-ct, 0] \).

It is clear that (MQM) and (EMQM) include the monotonicity conditions in \([17, 18, 21, 25, 26, 28, 42, 61, 62, 66]\). Furthermore, similar to these papers, we also assume that \( f \) satisfies the following continuous condition.

(A): For any \( \Phi(s), \Psi(s) \in C([-ct, 0], \mathbb{R}) \) with \( 0 \leq \Phi, \Psi \leq M \), we have

\[
|f^c(\Phi) - f^c(\Psi)| \to 0 \text{ as } \sup_{s \in [-ct, 0]} |\Phi(s) - \Psi(s)| \to 0,
\]

where \( | \cdot | \) denotes the super norm in \( \mathbb{R}^n \).

In order to consider the existence of traveling wave solutions of (4) when (MQM) (or (EMQM)) holds, we will introduce the generalized upper-lower solutions such that the existence of traveling wave solutions of (1), (2) and (3) can be investigated by the same scheme, which is motivated by Pao [35], Ye and Li [63]. By employing the Schauder’s fixed point theorem, we reduce the existence of traveling wave solutions to the existence of a pair of upper-lower solutions. To illustrate our results, we first establish the existence of traveling wave solutions of (1). We note that if \( \tau_{ij} = 0, i, j \in I \), then (1) becomes the following system

\[
\frac{\partial u_i(x, t)}{\partial t} = d_i \Delta u_i(x, t) + r_i u_i(x, t) \left( 1 - \sum_{j \in I} a_{ij} u_j(x, t) \right), i \in I,
\]

which was considered by Ahmad and Lazer [1], Ahmad et al. [2]. Under the condition \( \prod_{i,j \in I} a_{ij} \neq 0 \), they established the existence of traveling wave solutions of (7), while for the case \( \prod_{i,j \in I} a_{ij} = 0 \), as pointed out in [1, 2], it remains open since (7) does not satisfy the condition (A2) in [1, 2]. In addition, van Vuur[53] considered a more general reaction diffusion system than (7). However, the result in [53] cannot be applied to (7). In fact, let \( J(u) \) be the Jacobi matrix given by

\[
J(u) = \left[ \frac{\partial f_i}{\partial u_j} \right]_{i,j \in I},
\]

where \( f_i = r_i u_i \left( 1 - \sum_{j=1}^n a_{ij} u_j \right) \), \( i, j \in I \), then it is clear that \( J(u) \) is not uniformly bounded for \( u = (u_1, \cdots, u_n) \in \mathbb{R}^n_+ \), and so \( J(u) \) does not satisfy the condition 2 in [53, pp.138]. The above facts imply that our results are new even for the undelayed system (7). Moreover, we establish the existence of traveling wave solutions of the system (2) concerning its coexistence equilibrium. For the cooperative system (3), we also give the existence of traveling wave solutions and improve the results in Huang and Zou [16] in a position. In particular, when \( t \to -\infty \), we show the precisely asymptotic behavior of the traveling wave solutions of systems (1)-(3). Notice the study on the stability and the uniqueness of traveling wave solutions (see [31, 41, 52, 58]), we think the asymptotic behavior is very important in investigating more properties of traveling wave solutions.

From the viewpoint of population dynamics, our results imply that there is a transition zone moving from the steady state with no species to the steady state with the coexistence of \( n \) species for Lotka-Volterra system (see (6)) if the coexistence state of the corresponding ODEs is stable, one also refers to [8, 14, 24, 33] for the importance of coexistence in population dynamics. Moreover, our conclusion also indicates that the delays appeared in the interaction terms have no effect on the persistence of traveling wave solutions (see Ou and Wu [34]).
The remainder of this paper is organized as follows. In Section 2, some preliminaries are listed. In Sections 3 and 4, we consider the existence of traveling wave solutions of (4) when (MQM) and (EMQM) hold, respectively. In Section 5, our main results are applied to (1), (2) and (3).

2. Preliminaries. In this paper, \( C(\mathbb{R}, \mathbb{R}^n) \) is defined by

\[
C(\mathbb{R}, \mathbb{R}^n) = \left\{ \Phi \mid \Phi \text{ is a bounded and uniformly continuous vector function from } \mathbb{R} \text{ to } \mathbb{R}^n \right\}.
\]

Then \( C(\mathbb{R}, \mathbb{R}^n) \) is a Banach space with the super norm \( ||\Phi|| \) formulated by

\[
||\Phi|| = \sup_{t \in \mathbb{R}} |\Phi(t)| \text{ for } \Phi \in C(\mathbb{R}, \mathbb{R}^n).
\]

Let

\[
C_{[0,M]}(\mathbb{R}, \mathbb{R}^n) = \{ \Phi \in C(\mathbb{R}, \mathbb{R}^n) : 0 \leq \Phi(t) \leq M, t \in \mathbb{R} \}.
\]

Define the operator

\[
H = (H_1, H_2, \ldots, H_n) : C_{[0,M]}(\mathbb{R}, \mathbb{R}^n) \rightarrow C(\mathbb{R}, \mathbb{R}^n)
\]

by

\[
H_i(\Phi)(t) = f_i'\Phi_i(t) + \beta_i \phi_i(t), \Phi \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^n), i \in I.
\] (8)

Then (5) can be rewritten as follows

\[
d_i \phi''_i(t) - c(\phi'_i(t) - \beta_i \phi_i(t) + H_i(\Phi))(t) = 0, i \in I,
\] (9)

in which \( \beta_1, \ldots, \beta_n \) are given by (MQM) or (EMQM).

Denote

\[
\lambda_{i1} = \frac{c - \sqrt{c^2 + 4\beta_i d_i}}{2d_i}, \lambda_{i2} = \frac{c + \sqrt{c^2 + 4\beta_i d_i}}{2d_i}, i \in I.
\]

Then \( \lambda_{i1} < 0 < \lambda_{i2}, i \in I, \) and

\[
d_i \lambda_{i1}^2 - c \lambda_{i1} - \beta_i = 0, \quad d_i \lambda_{i2}^2 - c \lambda_{i2} - \beta_i = 0, i \in I.
\]

For \( \Phi \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^n) \), define \( F = (F_1, F_2, \ldots, F_n) : C_{[0,M]}(\mathbb{R}, \mathbb{R}^n) \rightarrow C(\mathbb{R}, \mathbb{R}^n) \) by

\[
F_i(\Phi)(t) = \frac{1}{d_i (\lambda_{i2} - \lambda_{i1})} \left[ \int_{-\infty}^{t} e^{\lambda_{i1}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{i2}(t-s)} \right] H_i(\Phi)(s) ds,
\] (10)

which was earlier used in [28, 61]. It is obvious that the operator \( F \) is well defined and satisfies

\[
d_i (F_i(\Phi))''(t) - c(F_i(\Phi))'(t) - \beta_i (F_i(\Phi))(t) + H_i(\Phi)(t) = 0, i \in I, t \in \mathbb{R}.
\] (11)

Thus, a fixed point of the operator \( F \) is a solution of (9). Moreover, if it also satisfies the condition (6), then it is a desired traveling wave solution of (4).

3. The Case (MQM). In this section, we consider the existence of traveling wave solutions of (4) if the delayed reaction term \( f \) satisfies the condition (MQM).

We now introduce the definition of a pair of upper-lower solutions of (5).

**Definition 3.1.** Assume that (MQM) (or (EMQM)) holds. A pair of continuous functions \( \overline{\Phi} = (\overline{\phi}_1, \overline{\phi}_2, \ldots, \overline{\phi}_n) \) and \( \underline{\Phi} = (\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_n) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^n) \) is called an upper solution and a lower solution of (5), respectively, if \( \overline{\Phi}(t), \underline{\Phi}(t) \) with \( \underline{\Phi}(t) \leq \overline{\Phi}(t), t \in \mathbb{R} \) are twice differentiable on \( \mathbb{R} \setminus \mathbb{T} = \{ T_1, T_2, \ldots, T_l \} \), and

(i): \( \overline{\Phi}''(t), \underline{\Phi}''(t) \) and \( \overline{\Phi}(t), \underline{\Phi}(t), t \in \mathbb{R} \setminus \mathbb{T} \) are bounded such that

\[
d_i \overline{\phi}''_i(t) - c \overline{\phi}'_i(t) + f_i''(\overline{\phi}_1, \overline{\phi}_2, \ldots, \overline{\phi}_n) \leq 0, i \in I, t \in \mathbb{R} \setminus \mathbb{T}
\] (12)

for \( \overline{\phi}_i = \overline{\phi}_i \) and all \( \phi_j \in \{ \underline{\phi}_j, \overline{\phi}_j \}, j \in I_i \);
Proof. We first prove that \( \| \cdot \| \) is continuous with respect to the norm \( \| \cdot \| \), those in \([17, 28, 61, 62, 66]\) if the nonlinearity \( f \) satisfies the (exponentially) quasi-monotonicity, \([18]\) if \( f \) satisfies the (exponentially) partial quasimonotonicity, and \([21]\) if the nonlinearity satisfies the (exponentially) weak quasimonotonicity.

In the remainder of this section, we assume that (5) has an upper solution \( \overline{\Phi} = (\overline{\phi}_1, \overline{\phi}_2, \cdots, \overline{\phi}_n) \) and a lower solution \( \underline{\Phi} = (\underline{\phi}_1, \underline{\phi}_2, \cdots, \underline{\phi}_n) \) such that

\[
\begin{align*}
(P1): & \lim_{t \to -\infty} \overline{\Phi}(t) = 0, \lim_{t \to -\infty} \underline{\Phi}(t) = \lim_{t \to -\infty} \overline{\Phi}(t) = K; \\
(P2): & \underline{\Phi}'(t-) \leq \underline{\Phi}'(t+), \overline{\Phi}'(t-) \leq \overline{\Phi}'(t-) \text{ for } t \in \mathbb{T}, \text{ herein } \underline{\Phi}'(t-) = \lim_{s \to t-} \underline{\Phi}'(s), \text{ and similar for } \overline{\Phi}'(t+), \overline{\Phi}'(t-).
\end{align*}
\]

Define the profile set:

\[
\Gamma(\underline{\Phi}, \overline{\Phi}) = \{ \Phi \in C(\mathbb{R}, \mathbb{R}^n) : \underline{\Phi}(t) \leq \Phi(t) \leq \overline{\Phi}(t), \ t \in \mathbb{R} \}.
\]

Then the following result is obvious.

Lemma 3.2. \( \Gamma(\underline{\Phi}, \overline{\Phi}) \) is nonempty and convex. Moreover, it is a bounded and closed subset of \( C(\mathbb{R}, \mathbb{R}^n) \) with respect to the norm \( \| \cdot \| \).

Lemma 3.3. Assume that (A) holds. Then

\[
F = (F_1, F_2, \cdots, F_n) : C_{[0,M]}(\mathbb{R}, \mathbb{R}^n) \to C(\mathbb{R}, \mathbb{R}^n)
\]

is continuous with respect to the norm \( \| \cdot \| \).

Proof. We first prove that \( H_1 : C_{[0,M]}(\mathbb{R}, \mathbb{R}^n) \to C(\mathbb{R}, \mathbb{R}) \) is continuous with respect to the norm \( \| \cdot \| \). For any \( \Phi = (\phi_1, \phi_2, \cdots, \phi_n), \Psi = (\psi_1, \psi_2, \cdots, \psi_n) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^n) \), it is clear that

\[
|H_1(\Phi)(t) - H_1(\Psi)(t)| = |f_1^*(\Phi_t) - f_1^*(\Psi_t) + \beta_1(\phi_1(t) - \psi_1(t))| \\
\leq |f_1^*(\Phi_t) - f_1^*(\Psi_t)| + \beta_1 |\phi_1(t) - \psi_1(t)|,
\]

which implies that

\[
\sup_{t \in \mathbb{R}} |H_1(\Phi)(t) - H_1(\Psi)(t)| \to 0 \text{ as } \|\Phi - \Psi\| \to 0
\]

by the assumption (A). Note that

\[
\begin{align*}
&|F_1(\Phi)(t) - F_1(\Psi)(t)| \\
&= \frac{1}{d_1(\lambda_{12} - \lambda_{11})} \left[ \left| \int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right| [H_1(\Phi)(s) - H_1(\Psi)(s)] ds \right] \\
&\leq \frac{1}{d_1(\lambda_{12} - \lambda_{11})} \left[ \frac{1}{\lambda_{12}} - \frac{1}{\lambda_{11}} \right] \sup_{s \in \mathbb{R}} |H_1(\Phi)(s) - H_1(\Psi)(s)| \\
&= \frac{1}{\beta_1} \sup_{s \in \mathbb{R}} |H_1(\Phi)(s) - H_1(\Psi)(s)|.
\end{align*}
\]

Then the continuity of \( H_1 \) indicates that

\[
\sup_{t \in \mathbb{R}} |F_1(\Phi)(t) - F_1(\Psi)(t)| \to 0 \text{ if } \|\Phi - \Psi\| \to 0.
\]
Similarly, we can prove that
\[ \sup_{t \in \mathbb{R}} |F_i(\Phi(t)) - F_i(\Psi(t))| \to 0 \text{ if } \|\Phi - \Psi\| \to 0, \quad i \in I, \]
namely, we obtain
\[ \|F(\Phi) - F(\Psi)\| \to 0 \text{ if } \|\Phi - \Psi\| \to 0. \]

The proof is complete. \(\square\)

**Lemma 3.4.** Assume that (MQM) holds. Then
\[ F : \Gamma(\Phi, \Psi) \to \Gamma(\Phi, \Psi). \]

**Proof.** For \(i = 1\), by (MQM), it is sufficient to prove that
\[ \phi_1(t) \leq F_1(\phi_1, \phi_2, \ldots, \phi_n)(t), \quad F_1(\phi_1, \phi_2, \ldots, \phi_n)(t) \leq \phi_1(t) \quad (14) \]
for all \(\phi_i(t) \in \{\phi_i(t), \bar{\phi}_i(t)\}, t \in \mathbb{R}\) and \(j \in I_1\), for which one refers to [21, 28, 61].

Without loss of generality, we assume that \(T_1 < T_2 < \cdots < T_l\) and denote \(T_0 = -\infty, T_{l+1} = \infty\). By the definition of upper-lower solutions, if \(t \notin T\), then
\[
F_1(\phi_1, \phi_2, \cdots, \phi_n)(t) \\
= \frac{1}{d_1(\lambda_{11} - \lambda_{11})} \left[ \int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{21}(t-s)} \right] H_1(\phi_1, \phi_2, \cdots, \phi_n)(s) ds \\
= \frac{1}{d_1(\lambda_{21} - \lambda_{11})} \sum_{j=0}^{l} \int_{T_j}^{T_{j+1}} \min \left\{ e^{\lambda_{11}(t-s)}, e^{\lambda_{21}(t-s)} \right\} H_1(\phi_1, \phi_2, \cdots, \phi_n)(s) ds \\
\geq \frac{1}{d_1(\lambda_{21} - \lambda_{11})} \sum_{j=0}^{l} \int_{T_j}^{T_{j+1}} \min \left\{ e^{\lambda_{11}(t-s)}, e^{\lambda_{21}(t-s)} \right\} \left[ \phi_1'(s) + \beta_1 \phi_1(s) - d_1 \phi_1''(s) \right] ds \\
= \phi_1(t) + \frac{1}{(\lambda_{11} - \lambda_{11})} \sum_{j=1}^{l} \min \left\{ e^{\lambda_{11}(T_j+T_j)}, e^{\lambda_{21}(T_j+T_j)} \right\} \left[ \phi_1'(T_j) - \phi_1'(T_j) \right] \left[ \phi_1'(T_j) - \phi_1'(T_j) \right] \\
\geq \phi_1(t) \text{ (by the assumption (P2))}. 
\]

Then the continuity of \(F_1(\phi_1, \phi_2, \cdots, \phi_n)(t), \phi_1(t)\) for \(t \in \mathbb{R}\) implies that
\[ \phi_1(t) \leq F_1(\phi_1, \phi_2, \cdots, \phi_n)(t), \quad t \in \mathbb{R}. \]

Similarly, we can prove that for any given \(i \in I\),
\[ \phi_j(t) \leq F_i(\phi_1, \phi_2, \cdots, \phi_i, \cdots, \phi_n)(t), \quad F_i(\phi_1, \phi_2, \cdots, \phi_i, \cdots, \phi_n)(t) \leq \phi_i(t) \]
for all \(\phi_j \in [\bar{\phi}_1, \bar{\phi}_j], \quad j \in I_i\). This completes the proof. \(\square\)

**Lemma 3.5.** Assume that (A) and (MQM) hold. Then
\[ F : \Gamma(\Phi, \Psi) \to \Gamma(\Phi, \Psi) \]
is compact with respect to the norm \(\|\cdot\|\).
Proof. For any \( \Phi \in \Gamma(\Phi, \Phi) \), it is evident that \( F(\Phi)(t) \) is totally bounded and equicontinuous with respect to the norm \( \| \cdot \| \). Let \( n \in \mathbb{N} \), define \( F^{n}(\Phi) \) by

\[
F^{n}(\Phi)(t) = \begin{cases} 
F(\Phi)(t), & t \in [-n,n]; \\
F(\Phi)(n), & t \in (n,\infty); \\
F(\Phi)(-n), & t \in (-\infty,-n). 
\end{cases}
\]

Then the Ascoli-Arzela Lemma implies that \( F^{n}(\Phi) \) maps \( \Gamma(\Phi, \Phi) \) into a precompact subset of \( C(\mathbb{R}, \mathbb{R}^{n}) \). Moreover, Lemma 3.4 and (P1) indicate that

\[
\|F^{n}(\Phi) - F(\Phi)\| \to 0 \quad \text{as} \quad n \to \infty
\]

for any \( \Phi \in \Gamma(\Phi, \Phi) \). Thus, \( F : \Gamma(\Phi, \Phi) \to \Gamma(\Phi, \Phi) \) is compact. The proof is complete.

Due to Lemmas 3.2-3.5, we may apply the Schauder’s fixed point theorem to the operator \( F \) and get the following result because of (11).

**Theorem 3.6.** Assume that (A) and (MQM) hold. If (5) has a pair of upper-lower solutions such that (P1)-(P2) are satisfied, then (5) with (6) has a positive solution which is a desired traveling wave solution of (4).

4. The Case (EMQM). In this section, we shall consider the existence of traveling wave solutions of (5) when \( f \) satisfies the condition (EMQM).

In the rest of the current section, we suppose that (5) has an upper solution \( \Phi(t) \) and a lower solution \( \Phi(t) \) satisfying (P1), (P2) and

(P3): \( e^{\beta_{i}s}[\phi_{i}(s) - \phi_{i}(s)] \) are nondecreasing for \( s \in \mathbb{R} \) and \( i \in I \).

Define the following profile set

\[
\Gamma^{*}(\Phi, \Phi) = \left\{ \Phi \in C(\mathbb{R}, \mathbb{R}^{n}) : \begin{array}{ll} 
(i) & \Phi \leq \Phi \leq \Phi, \\
(ii) & e^{\beta_{i}s}[\phi_{i}(s) - \phi_{i}(s)], \quad e^{\beta_{i}s}[\bar{\phi}_{i}(s) - \Phi_{i}(s)]
\end{array} \right\}.
\]

By (P3), it is obvious that \( \Gamma^{*}(\Phi, \Phi) \) is nonempty and the following lemma is true.

**Lemma 4.1.** \( \Gamma^{*}(\Phi, \Phi) \) is convex. Moreover, it is bounded and closed with respect to the super norm \( \| \cdot \| \).

**Lemma 4.2.** Assume that (EMQM) holds. If \( c > 1 - \min_{1 \leq i \leq n} \{ \beta_{i}d_{i} \} \), then

\( F : \Gamma^{*}(\Phi, \Phi) \to \Gamma^{*}(\Phi, \Phi) \).

**Proof.** For any \( \Phi = (\phi_{1}, \phi_{2}, \cdots, \phi_{n}) \in \Gamma^{*}(\Phi, \Phi) \), similar to the proof of Lemma 3.4, we can show

\[
\Phi(t) \leq F(\Phi)(t) \leq \Phi(t) \quad \text{for} \quad t \in \mathbb{R},
\]

...
which implies that $F(\Phi)$ satisfies (i) of $\Gamma^* (\Phi, \overline{\Phi})$. We now prove (ii) of $\Gamma^* (\Phi, \overline{\Phi})$.

Let $F_1(\Phi) = \psi_1$ for $\Phi \in \Gamma^*$, then $\psi_1 \in C^2_{[0, M]}(\mathbb{R}, \mathbb{R})$. Moreover, if $t \not\in T$, then

$$
e^{-\beta t} \left[ \phi_1(t) - \psi_1(t) \right] = \frac{e^{\beta t}}{d_1(\lambda_{12} - \lambda_{11})} \sum_{j=0}^{T_{j+1}} \int_{T_j} \min \left\{ e^{\lambda_{11}(t-s)}, e^{\lambda_{12}(t-s)} \right\} \left\{ \beta_1 \phi_1(s) + c \phi_1'(s) - d_1 \phi_1''(s) - \psi_1(s) \right\} ds \right.$$ 

Similar to Lemmas 3.3 and 3.5, we can verify the following result.

**Lemma 4.3.** Assume that (A) and (EMQM) hold. Then $F : \Gamma^* (\Phi, \overline{\Phi}) \to \Gamma^* (\Phi, \overline{\Phi})$ is completely continuous with respect to the norm $\| \cdot \|$.

Since Lemmas 4.1-4.3 and (11) are true, then the Schauder’s fixed point theorem indicates the following conclusion.

**Theorem 4.4.** Assume that (A) and (EMQM) hold. Suppose that (5) has a pair of upper-lower solutions satisfying (P1)-(P3). Then, for any $c > 1 - \min_{1 \leq i \leq n} \{ \beta_i d_i \}$, (5) with (6) has a positive solution which is a traveling wave solution of (4).

**Remark 2.** If (EMQM) is satisfied, then we can always choose $\beta_i > 0$ sufficiently large such that $c > 1 - \min_{1 \leq i \leq n} \{ \beta_i d_i \}$.

**Remark 3.** Comparing Theorems 3.6 and 4.4 with our results in [21], we did not use the decay norm in this paper, which weakens the continuous condition in [21].
Remark 4. Comparing Theorems 3.6 and 4.4 with the theory in [17, 28], we need more requirements on the lower solution. However, if \( K \) is a local stable equilibrium of the corresponding ODEs, it is easy to construct lower solution \( \Phi \) such that \( \lim_{t \to -\infty} \Phi(t) = K \), see the examples given in Section 5.

Remark 5. To apply the Schauder’s fixed point theorem in the proof of Theorems 3.6 and 4.4, what we needed was the upper-lower solutions of the operator \( F \). However, it is very difficult to check the definition of upper-lower solutions for \( F \) (see Wang [55] for the result of a scalar equation). In Lemmas 3.4 and 4.3, we achieved the purpose by the definition of upper-lower solutions of differential equations, which is easy to verify in practice.

5. Multi-species diffusion model. In this section, we shall establish the existence of traveling wave solutions of the following delayed Lotka-Volterra systems

\[
\frac{\partial u_i(x, t)}{\partial t} = d_i \Delta u_i(x, t) + r_i u_i(x, t) \left[ 1 - c_{ii} u_i(x, t) + \sum_{j \in I_i} c_{ij} u_j(x, t - \tau_{ij}) \right], \tag{15}
\]

and

\[
\frac{\partial u_i(x, t)}{\partial t} = d_i \Delta u_i(x, t) + r_i u_i(x, t) \left[ 1 - c_{ii} u_i(x, t - \tau_{ii}) + \sum_{j \in I_i} c_{ij} u_j(x, t - \tau_{ij}) \right], \tag{16}
\]

where \( d_i > 0, r_i > 0, \tau_{ij} \geq 0, c_{ij} \in \mathbb{R} \) with \( c_{ii} > 0, i, j \in I \). Clearly, (15) and (16) at least contain the systems (1), (2) and (3).

Let \( \Phi = (\phi_1, \cdots, \phi_n) \) denote the wave profile and \( c > 0 \) be the wave speed, then the corresponding wave systems of (15) and (16) are

\[
d_i \phi_i''(t) - c \phi_i'(t) + r_i \phi_i(t) \left[ 1 - c_{ii} \phi_i(t) + \sum_{j \in I_i} c_{ij} \phi_j(t - c\tau_{ij}) \right] = 0, \quad i \in I, \tag{17}
\]

and

\[
d_i \phi_i''(t) - c \phi_i'(t) + r_i \phi_i(t) \left[ 1 - c_{ii} \phi_i(t - c\tau_{ii}) + \sum_{j \in I_i} c_{ij} \phi_j(t - c\tau_{ij}) \right] = 0, \quad i \in I, \tag{18}
\]

respectively.

Assume that \([0, \mathbf{M}]\) is invariant in the sense of the corresponding ODEs of (15) or (16). For \( 0 \leq \Phi(s) \leq \mathbf{M}, s \in [-c\tau, 0] \), define \( h = (h_1, \cdots, h_n) \) and \( g = (g_1, \cdots, g_n) : C([-c\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n \) by

\[
h_i = r_i \phi_i(0) \left[ 1 - c_{ii} \phi_i(0) + \sum_{j \in I_i} c_{ij} \phi_j(-c\tau_{ij}) \right], \quad i \in I,
\]

\[
g_i = r_i \phi_i(0) \left[ 1 - c_{ii} \phi_i(-c\tau_{ii}) + \sum_{j \in I_i} c_{ij} \phi_j(-c\tau_{ij}) \right], \quad i \in I,
\]

where \( \tau = \max_{i,j \in I} \{\tau_{ij}\} \). Then the following result is true.

Lemma 5.1. The functional \( h \) in (17) satisfies (MQM), and the functional \( g \) in (18) satisfies (EMQM) provided that \( \tau_{ii}, i \in I, \) are small enough.
The proof is complete. Thus, if \( \tau \) in which \( \beta \) holds and \( e^{\beta t}(\phi_1(s) - \psi_1(s)) \) is nondecreasing for \( s \in [-c\tau, 0] \) with some \( \beta_1 > 0 \) which will be clarified later. It is evident that
\[
g_1(\Psi) = -r_1\sum_{j \in I} c_{1j} \phi_j(-c\tau_{1j}) + 1 - c_{11} \psi_1(-c\tau_{11}) \left[ \phi_1(0) - \psi_1(0) \right]
\]
\[
-r_1 \sum_{j \in I} c_{1j} \phi_j(-c\tau_{1j}) + 1 - c_{11} \psi_1(-c\tau_{11}) - c_{11} \phi_1(0) e^{\beta_{12} c\tau_{11}} \left[ \phi_1(0) - \psi_1(0) \right]
\]
\[
+ r_1 \sum_{j \in I} c_{1j} \phi_j(-c\tau_{1j}) + 1 - c_{11} \psi_1(-c\tau_{11}) \left[ \phi_1(0) - \psi_1(0) \right] + c_{11} M_1 e^{\beta_{12} c\tau_{11}} + \sum_{j \in I} |c_{1j}| M_j - 1 .
\]

If \( \tau_{11} \) is small enough, then we can choose \( \beta_1 > 0 \) such that
\[
r_1 \left[ c_{11} M_1 e^{\beta_{12} c\tau_{11}} + \sum_{j \in I} |c_{1j}| M_j - 1 \right] \leq \beta_1 .
\]
Thus, \( f_1 \) satisfies (EMQM). Similarly, we can prove that \( f_2, \cdots, f_n \) satisfy (EMQM). The proof is complete.

In order to apply our abstract results in Sections 3 and 4, we need to construct proper upper-lower solutions for systems (17) and (18). If \( c > \max_{1 \leq i \leq n} \{ 2 \sqrt{d_i r_i} \} \), then there exist \( \gamma_1, \gamma_2 \) such that \( 0 < \gamma_1 < \gamma_2 \) and
\[
d_i \gamma_{1i}^2 - c\gamma_{1i} + r_i = d_i \gamma_{2i}^2 - c\gamma_{2i} + r_i = 0 \text{ for } i \in I .
\]
(19)
Assume that \( q > 1 \) holds and \( \eta \) satisfies
\[
\eta \in \left( 1, \min_{i,j \in I} \left\{ \frac{\gamma_{2i}}{\gamma_{1i}}, \frac{\gamma_{1i} + \gamma_{2i}}{\gamma_{1i}} \right\} \right) .
\]
(20)
Define
\[
l_i(t) = e^{\gamma_{1i} t} - q e^{\eta \gamma_{1i} t} .
\]
Then each \( l_i \) has a global maximum \( m_i > 0 \). Denote constants \( t_{i3}, i \in I \) such that
\[
t_{i3} = \max \left\{ t : l_i(t) = \frac{m_i}{2} \right\} .
\]
Let \( \gamma \in (0, 1) \) be small enough such that
\[
k_i - (k_i - \frac{m_i}{2}) e^{-\gamma t_{i2}} \geq \frac{m_i}{4}, i \in I .
\]
For every \( i \in I \), define the continuous function \( \Phi = (\phi_i, \cdots, \phi_n) \) by
\[
\phi_i(t) = \begin{cases} e^{\gamma_{1i} t} - q e^{\eta \gamma_{1i} t}, & t \leq t_{i2}, \\ k_i - (k_i - \frac{m_i}{2}) e^{-\gamma t}, & t \geq t_{i2} , \end{cases}
\]
(21)
5.1. Competition model with delays.

**Example 5.2.** Let us consider the following delayed system

\[
\frac{\partial u_i(x,t)}{\partial t} = d_i \Delta u_i(x,t) + r_i u_i(x,t) \left[ 1 - a_{ii} u_i(x,t) - \sum_{j \in I_i} a_{ij} u_j(x,t - \tau_{ij}) \right],
\]

(22)

where \( a_{ii} > 0, a_{ij} \geq 0 \) for all \( j \in I_i \) and \( i \in I \). Moreover, we assume that there exists \( K = (k_1, k_2, \ldots, k_n), k_i > 0, i \in I \) such that \( \sum_{j \in I} a_{ij} k_j = 1, i \in I \) and

\[
a_{ii} k_i > \sum_{j \in I_i} a_{ij} k_j, i \in I.
\]

(23)

By (17), the corresponding wave system of (22) is

\[
d_i \phi_i''(t) - c \phi_i'(t) + r_i \phi_i(t) \left[ 1 - a_{ii} \phi_i(t) - \sum_{j \in I_i} a_{ij} \phi_j(t - c r_{ij}) \right] = 0
\]

(24)

with the following asymptotic boundary conditions

\[
\lim_{t \to -\infty} \phi_i(t) = 0, \lim_{t \to -\infty} \phi_i(t) = k_i, i \in I.
\]

(25)

Let \( M = (M_1, \ldots, M_n) \) with \( M_i a_{ii} = 1, i \in I \) in this example. Then \([0, M]\) is an invariant region of the corresponding ODEs. Define

\[
\overline{\phi}_i(t) = \min \left\{ M_i, e^{r_i t}, k_i + k_i e^{-c r_i} \right\}, i \in I,
\]

and we further assume that there exist constants \( t_{i1}, i \in I \) such that

\[
\overline{\phi}_i(t) = k_i + k_i e^{-c r_i}, t \geq t_{i1} \text{ and } \overline{\phi}_i(t) = \min \left\{ M_i, e^{r_i t} \right\}, t \leq t_{i1}, i \in I.
\]

**Remark 6.** We now show the choice of the constants \( q \) and \( \gamma \) is admissible in the above continuous functions. We claim these by the following three steps.

**Step 1** In view of (23), there exist \( \tau_i \in (0, k_i), i \in I \) such that

\[
a_{ii} k_i > \sum_{j \in I_i} a_{ij} \tau_j, \quad a_{ii} \tau_i > \sum_{j \in I_i} a_{ij} k_j
\]

hold for any \( \epsilon_i \in [\tau_i, k_i), i \in I \).

**Step 2** Let \( q > 1 \) such that (a) \( q > \max_{i \in I} \left\{ \frac{\sum_{j \in I} a_{ij}}{(d_i (r_{ij})^2 - c r_i), + 1} \right\} \); (b) \( k_i - \frac{m}{q} > 0 \); (c) \( \frac{\ln k_i}{r_i} \geq \frac{\ln q}{1 + (q^2 - 1)/2} + c r_i \) for all \( i \in I \).

**Step 3** Let \( \gamma > 0 \) be small enough such that \( k_i - (k_i - \frac{m}{q}) e^{-c r_i > \frac{m}{q}, i \in I}. \)

**Lemma 5.3.** Assume that (20) and (23) hold. Then \( (\overline{\phi}_i(t), \overline{\phi}_2(t), \ldots, \overline{\phi}_n(t)) \) and \( (\overline{\phi}_i(t), \overline{\phi}_2(t), \ldots, \overline{\phi}_n(t)) \) are a pair of upper-lower solutions of (24).

**Proof.** For \( \overline{\phi}_i(t) \), it suffices to prove that

\[
\mathcal{L}_i(\Phi)(t) := d_i \overline{\phi}_i''(t) - c \overline{\phi}_i'(t) + r_i \overline{\phi}_i(t) \left[ 1 - a_{ii} \overline{\phi}_i(t) - \sum_{j \in I_i} a_{ij} \overline{\phi}_j(t - c r_{ij}) \right] \leq 0
\]

(26)

except several finite points. When \( t \leq t_{i1} \), the result is clear if \( \overline{\phi}_i(t) = M_i \).

If \( \overline{\phi}_i(t) = e^{r_i t} \), then

\[
\mathcal{L}_i(\Phi)(t) \leq d_i \overline{\phi}_i''(t) - c \overline{\phi}_i'(t) + r_i \overline{\phi}_i(t) = [d_i \gamma_{i1}^2 - c r_{i1} + r_i] e^{r_i t} = 0.
\]
If \( t > t_{11} \), then Remark 6 (step 2, item (c)) implies that
\[
\mathcal{L}_i(\Phi)(t) = k_ie^{-\gamma t} \left\{ d_i\gamma^2 + c\gamma + r_i (1 + e^{-\gamma t}) \left( \sum_{j \in I_i} a_{ij}(k_i - \frac{m_j}{2})e^{\gamma \tau_{ij}} - a_{ii}k_i \right) \right\}.
\]
Let
\[
R_{i1}(\gamma) = d_i\gamma^2 + c\gamma + r_i (1 + e^{-\gamma t}) \left( \sum_{j \in I_i} a_{ij}(k_i - \frac{m_j}{2})e^{\gamma \tau_{ij}} - a_{ii}k_i \right).
\]
Since \( t \geq t_{11} \), then \( R_{i1}(\gamma) \) is uniformly continuous for \( \gamma \in [0, 1] \). Moreover,
\[
R_{i1}(0) = 2r_i \left( \sum_{j \in I_i} a_{ij}(k_i - \frac{m_j}{2}) - a_{ii}k_i \right) < 0,
\]
then there exists a constant \( \gamma_{11}^* \in (0, 1) \) such that \( R_{i1}(\gamma) < 0 \) for any \( \gamma \in (0, \gamma_{11}^*) \).
Thus, \( \mathcal{L}_i(\Phi)(t) \leq 0 \) holds for \( t > t_{11} \) if \( \gamma \in (0, \gamma_{11}^*) \).
For \( \phi(t) \), it is sufficient to verify that
\[
\mathcal{L}_i(\Phi)(t) := d_i\phi''(t) - c\phi'(t) + r_i\phi(t) \left[ 1 - a_{ii}\phi(t) - \sum_{j \in I_i} a_{ij}\phi_j(t - \tau_{ij}) \right] \geq 0 \quad (27)
\]
for \( t \in \mathbb{R} \setminus t_{12} \).
If \( t < t_{12} \), then \( \phi_j(t - \tau_{ij}) \leq e^{\gamma_i(t - \tau_{ij})} \) and
\[
\mathcal{L}_i(\Phi)(t) \geq - \left[ d_i(\eta_{i1})^2 - c\gamma_i + r_i \right] qe^{\eta_{i1}t} - a_{ii} \left( e^{\gamma_i t} - qe^{\eta_{i1}t} \right) e^{\gamma_i(t - \tau_{ij})} - \sum_{j \in I_i} a_{ij}e^{\eta_{i1}t}e^{\gamma_i(t - \tau_{ij})},
\]
Let \( q > 1 \) satisfy Remark 6 (step 2, item (a)), then (20) shows that
\[
- \left[ d_i(\eta_{i1})^2 - c\gamma_i + r_i \right] qe^{\eta_{i1}t} - a_{ii}e^{2\eta_{i1}t} - \sum_{j \in I_i} a_{ij}e^{\gamma_i t}e^{\gamma_j(t - \tau_{ij})} > 0.
\]
If \( t > t_{12} \), then \( e^{\gamma_i(t - \tau_{ij})} \leq k_j + kj e^{-\gamma(t - \tau_{ij})} \) and
\[
\mathcal{L}_i(\Phi)(t) \geq e^{-\gamma t} \left\{ d_i\gamma^2 \left( k_i - \frac{m_j}{2} \right) - c\gamma \left( k_i - \frac{m_j}{2} \right) \right\} + r_i \left( k_i - \left( k_i - \frac{m_j}{2} \right) e^{-\gamma t} \right) \]
\[
	imes \left\{ 1 - a_{ii} \left( k_i - \left( k_i - \frac{m_j}{2} \right) e^{-\gamma t} \right) - \sum_{j \in I_i} a_{ij} \left( k_j + kj e^{-\gamma(t - \tau_{ij})} \right) \right\}
\]
\[
= e^{-\gamma t} \left\{ d_i \left( k_i - \frac{m_j}{2} \right) \right\} \gamma^2 - c \left( k_i - \frac{m_j}{2} \right) \gamma
\]
\[
+ r_i \left( k_i - \left( k_i - \frac{m_j}{2} \right) e^{-\gamma t} \right) \left\{ a_{ii} \left( k_i - \frac{m_j}{2} \right) - \sum_{j \in I_i} a_{ij}kj e^{\gamma \tau_{ij}} \right\}.
\]
Let \( \gamma \in (0, \gamma^*_{12}) \) such that
\[
a_{ii} \left( k_i - \frac{m_i}{2} \right) - \sum_{j \in I_i} a_{ij} k_j e^{\gamma c \tau_{ij}} > 0, \quad k_i - \left( k_i - \frac{m_i}{2} \right) e^{-\gamma t} \geq \frac{m_i}{4}
\]
Then, \( \gamma \in (0, \gamma^*_{12}) \) indicates that
\[
L(\Phi)(t) \geq e^{-\gamma t} \left\{ d_i e_i \gamma^2 + \frac{r_{ii} m_i}{4} \left[ a_{ii} \left( k_i - \frac{m_i}{2} \right) - \sum_{j \in I_i} a_{ij} k_j e^{\gamma c \tau_{ij}} \right] \right\}.
\]
It is evident that there exists a constant \( \gamma^*_{03} \in (0, \gamma^*_{12}) \) such that \( L(\Phi)(t) \geq 0 \) for any \( \gamma \in (0, \gamma^*_{03}) \) and \( t > t_{i2} \).

By what we have done, the following result is true.

**Theorem 5.4.** Assume that (23) holds. Then for any given \( c > \max_{1 \leq i \leq n} \{ 2\sqrt{d_i r_i} \} \), (24) with (25) has a positive solution \( \Phi(t) = (\phi_1, \phi_2, \cdots, \phi_n) \) such that \( 0 < \phi_i(t) < \frac{1}{a_{ii}}, \lim_{t \to -\infty} \phi_i(t)e^{-\gamma_i t} = 1 \) for \( i \in I, t \in \mathbb{R} \).

**Example 5.5.** Let us consider the following delayed system
\[
\frac{\partial u_i(x,t)}{\partial t} = d_i u_i(x,t) + r_{ii} u_i(x,t) \left[ 1 - \sum_{j \in I} a_{ij} u_j(x,t - \tau_{ij}) \right], \quad (28)
\]
in which all the parameters are positive such that (23) holds.

By (18), the corresponding wave system of (28) is
\[
d_i \phi_i'''(t) - c \phi_i''(t) + r_{ii} \phi_i(t) \left[ 1 - \sum_{j \in I} a_{ij} \phi_j(t - c \tau_{ij}) \right] = 0 \quad (29)
\]
with the following asymptotic boundary conditions
\[
\lim_{t \to -\infty} \phi_i(t) = 0, \quad \lim_{t \to -\infty} \phi_i(t) = k_i, i \in I. \quad (30)
\]

Similar to that of Example 5.2 and by the uniform boundedness of \( \Phi, \Phi', \Phi'', \Phi \) and \( \Phi'' \), we can prove the following result.

**Lemma 5.6.** Assume that (23) holds and \( \max_{1 \leq i \leq n} \{ \tau_{ii} \} \) is small enough. Then the continuous functions \( (\phi_1(t), \cdots, \phi_n(t)) \) and \( (\phi_1'(t), \cdots, \phi_n'(t)) \) in the Example 5.2 are a pair of upper-lower solutions of (29) if \( q > 1, \frac{1}{\gamma} > 1 \) are large enough.

Thus Theorem 4.4 implies the following conclusion.

**Theorem 5.7.** Assume that \( \max_{1 \leq i \leq n} \{ \tau_{ii} \} \) is small enough and (23) holds. Then for any given \( c > \max_{1 \leq i \leq n} \{ 2\sqrt{d_i r_i} \} \), (29) with (30) has a positive solution \( \Phi(t) = (\phi_1, \cdots, \phi_n) \) such that
\[
0 < \phi_i(t) \leq \frac{1}{a_{ii}}, \quad \lim_{t \to -\infty} \phi_i(t)e^{-\gamma_i t} = 1 \quad \text{for } i \in I, t \in \mathbb{R}.
\]

For the system (7), the following result is true by Theorem 5.4.

**Theorem 5.8.** Assume that (23) holds. Then for any given \( c > \max_{1 \leq i \leq n} \{ 2\sqrt{d_i r_i} \} \), (7) has a traveling wave solution \( \Phi(x + ct) = (\phi_1, \phi_2, \cdots, \phi_n) \) connecting 0 with \( K \) and satisfying \( 0 < \phi_i(\xi) < \frac{1}{a_{ii}}, \lim_{\xi \to -\infty} \phi_i(\xi) e^{-\gamma_i \xi} = 1 \) for \( \xi = x + ct, i \in I \).
Remark 7. Theorem 5.8 cannot ensure the monotonicity of the traveling wave solutions (see [1, 2, 48]), but it provides the precisely asymptotic behavior of the traveling wave solution of (7) when $\xi \to -\infty$.

Remark 8. We conjecture the existence of nonmonotone traveling wave solutions if the delay is not small enough, which is motivated by the existence of periodic solutions of the corresponding functional differential equations [15] and will be further studied in our recent research.

5.2. Cooperative model with delays.

Example 5.9. Let us consider the following diffusion system

\[
\begin{align*}
\frac{\partial u_1}{\partial t}(x,t) &= d_1 \Delta u_1(x,t) + r_1 u_1(x,t) [1 - a_{11} u_1(x,t) + a_{12} u_2(x,t - \tau_1)], \\
\frac{\partial u_2}{\partial t}(x,t) &= d_2 \Delta u_2(x,t) + r_2 u_2(x,t) [1 - a_{21} u_1(x,t - \tau_2) - a_{22} u_2(x,t)],
\end{align*}
\]

in which all the parameters are positive such that $a_{11} a_{22} > a_{12} a_{21}$.

By (17), the corresponding wave system of (31) is

\[
\begin{align*}
d_1 \phi''_1(t) - c \phi'_1(t) + r_1 \phi_1(t) [1 - a_{11} \phi_1(t) + a_{12} \phi_2(t - c \tau_1)] &= 0, \\
d_2 \phi''_2(t) - c \phi'_2(t) + r_2 \phi_2(t) [1 - a_{22} \phi_2(t) + a_{21} \phi_1(t - c \tau_2)] &= 0
\end{align*}
\]

with the following asymptotic boundary conditions

\[
\lim_{t \to -\infty} \phi_i(t) = 0, \quad \lim_{t \to -\infty} \phi_i(t) = k_i, \quad i = 1, 2,
\]

where

\[
k_1 = \frac{a_{11} + a_{21}}{a_{11} a_{22} - a_{12} a_{21}} \quad \text{and} \quad k_2 = \frac{a_{22} + a_{12}}{a_{11} a_{22} - a_{12} a_{21}}.
\]

Clearly, $(k_1, k_2)$ is a stable equilibrium of the corresponding ODEs of (3), and we can define $M_i = k_i, i = 1, 2$, in this example. Assume that $(\gamma_{11}, \gamma_{12}) \cap (\gamma_{21}, \gamma_{22})$ is nonempty. Then there exists $\kappa \in (\gamma_{11}, \gamma_{12}) \cap (\gamma_{21}, \gamma_{22})$ such that

\[
\kappa \in (\gamma_{11}, \min(\gamma_{11} + \gamma_{21}, \gamma_{22})) \cap (\gamma_{21}, \min(\gamma_{11} + \gamma_{21}, \gamma_{22})).
\]

Define continuous functions

\[
\overline{\phi}_1(t) = \min \{e^{\gamma_{11} t} + k_1 p e^{\kappa t}, k_1\}, \quad \overline{\phi}_2(t) = \min \{e^{\gamma_{11} t} + k_2 p e^{\kappa t}, k_2\}.
\]

Lemma 5.10. Assume that $(\gamma_{11}, \gamma_{12}) \cap (\gamma_{21}, \gamma_{22})$ is nonempty. Then $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\overline{\phi}_1, \overline{\phi}_2)$ are a pair of upper-lower solutions of (32) if $p > 1, q > 1$ are large enough and $\gamma > 0$ is small enough.

Proof. For $\overline{\phi}_1(t)$, it suffices to prove that

\[
\overline{L}_1(\Phi)(t) := \frac{d_1 \overline{\phi}'_1(t) - c \overline{\phi}_1(t) + r_1 \overline{\phi}_1(t) [1 - a_{11} \overline{\phi}_1(t) + a_{12} \overline{\phi}_2(t - c \tau_1)]}{a_{11} \overline{\phi}_1(t) + a_{12} \overline{\phi}_2(t - c \tau_1)} \leq 0 \tag{35}
\]

if $e^{\gamma_{11} t} + k_1 p e^{\kappa t} \neq k_1$, and (35) is clear if $\overline{\phi}_1(t) = k_1$. If $\overline{\phi}_1(t) = e^{\gamma_{11} t} + k_1 p e^{\kappa t}$, then $\overline{\phi}_1(t) \leq e^{\gamma_{11} t} + k_2 p e^{\kappa t}$ and

\[
\overline{L}_1(\Phi)(t) \leq \left[ d_1 \kappa^2 - c \kappa + r_1 \right] p e^{\kappa t} - r_1 \left( e^{\gamma_{11} t} + k_1 p e^{\kappa t} \right)
\]

\[
\times \left[ a_{11} \left( e^{\gamma_{11} t} + k_1 p e^{\kappa t} \right) - a_{21} \left( e^{\gamma_{11} (t - c \tau_1)} + k_2 p e^{\kappa (t - c \tau_1)} \right) \right].
\]

Note that $a_{11} k_1 - a_{21} k_2 = 1$, then (35) holds provided that

\[
(d_1 \kappa^2 - c \kappa + r_1) k_1 p e^{\kappa t} - r_1 \left( e^{\gamma_{11} t} + k_1 p e^{\kappa t} \right) \left[ a_{11} e^{\gamma_{11} t} - a_{21} e^{\gamma_{11} (t - c \tau_1)} \right] \leq 0,
\]
which is clear if \( p > 1 \) is large enough because of (34) and \( d_1 \kappa^2 - c \kappa + r_1 < 0 \).

Similarly, we can prove that
\[
d_{2}^{2}\phi_{2}(t) - c\phi_{2}(t) + r_{2}\phi_{2}(t)\left[1 - a_{21}\phi_{2}(t) + a_{22}(t - ct_{2})\right] \leq 0
\]
with \( e^{\gamma_{1}t} + k_{2}e^{\kappa t} \neq k_{2} \) if \( p > 1 \) is large enough.

When \( \phi_{1}(t) \) is involved, we only need to verify that
\[
d_{1}^{2}\phi_{1}'(t) - c\phi_{1}'(t) + r_{1}\phi_{1}(t)\left[1 - a_{11}\phi_{1}(t) + a_{12}(t - ct_{1})\right] \geq 0
\]
for \( t \neq t_{12} \). If \( \phi_{1}(t) = e^{\gamma_{1}t} - qe^{\gamma_{1}t} \), then \( \phi_{1}(t - ct) > 0 \) indicates that (36) holds once
\[
d_{1}^{2}\phi_{1}'(t) - c\phi_{1}'(t) + r_{1}\phi_{1}(t)\left[1 - a_{11}\phi_{1}(t)\right] \geq 0,
\]
which is obvious when \( q > \frac{r_{1}a_{11}}{d_{1}(r_{1} - m_{1} + m_{2})} + 1 \) (one also refers to Zou [65]).

Let \( q > 1 \) be large enough such that
\[
1 - a_{11}\phi_{1}(t) + a_{12}(t - ct_{1}) > 1 - \frac{a_{11}m_{1}}{2}
\]
and then we only need to prove that
\[
d_{1}\left(k_{1} - m_{1}/2\right)\gamma^{2}e^{-\gamma t} + c\left(k_{1} - m_{1}/2\right)\gamma e^{-\gamma t} + \frac{m_{1}}{8} \geq 0.
\]
(38)

Let \( \gamma \rightarrow 0 \), then (38) is clear since \( t \) is bounded.

If \( \phi_{1}(t) = k_{1} - \left(k_{1} - m_{1}/2\right) e^{-\gamma t} \) with \( t > 0 \), then
\[
1 - a_{11}\phi_{1}(t) + a_{12}(t - ct_{1}) = a_{11}\left(k_{1} - m_{1}/2\right) e^{-\gamma t} - a_{12}\left(k_{2} - m_{2}/2\right) e^{-\gamma t},
\]
and (36) is clear if \( \gamma \rightarrow 0 \) since (37) holds.

Similarly, we can prove that \( \phi_{2}(t) \) is the lower solution if \( q \) and \( \frac{1}{\gamma} \) are large enough. The proof is complete. \( \Box \)

Note that \( \overline{\phi}_{1}, \overline{\phi}_{2} \) and \( \left(\overline{\phi}_{1}, \overline{\phi}_{2}\right) \) satisfy the conditions (P1) and (P2) in Theorem 3.6, then the following result holds by Theorem 3.6.

**Theorem 5.11.** Assume that \( c > \max_{1 \leq i \leq 2}\left\{ 2\sqrt{d_{i}}r_{i}\right\} \) such that \( (\gamma_{11}, \gamma_{12}) \subset (\gamma_{21}, \gamma_{22}) \) is nonempty. Then (31) has a traveling wave solution \((u_{1}(x, t), u_{2}(x, t)) = (\phi_{1}(x + ct), \phi_{2}(x + ct)) \) connecting 0 with \( (k_{1}, k_{2}) \) and satisfying
\[
0 \leq \phi_{i}(\xi) \leq k_{1}, \quad \lim_{\xi \rightarrow -\infty} \phi_{i}(\xi)e^{-\gamma_{i}t\xi} = 1, \quad i = 1, 2, \quad x + ct = \xi \in \mathbb{R}.
\]

**Remark 9.** From the monotonicity [43] and the invariance [32] of the system (31) in \([0, k_{1}] \times [0, k_{2}], \) Theorem 5.11 also implies that (32)-(33) has a monotone solution \((\psi_{1}(t), \psi_{2}(t)) \) such that
\[
\lim_{t \rightarrow -\infty} \psi_{i}(t)e^{-\gamma_{i}t} = 1, \quad i = 1, 2.
\]

In fact, define the set \( \Omega \) as follows
\[
\Omega = \left\{ (\phi_{1}, \phi_{2}) \in C(\mathbb{R}, \mathbb{R}^{2}) \mid (i) (\phi_{1}, \phi_{2}) \leq (\overline{\phi}_{1}, \overline{\phi}_{2}) \leq (\phi_{1}, \phi_{2}), \quad (ii) (\phi_{1}, \phi_{2}) \text{ is nondecreasing} \right\}.
\]
It is clear that $\Omega$ is closed and bounded with respect to the super norm. Moreover, it is also nonempty and convex. By the discussion similar to that in Section 3, then the result is evident.

**Example 5.12.** Let us consider the following diffusion system

$$
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= d_1 \Delta u_1(x,t) + r_1 u_1(x,t) \left[1 - a_{11} u_1(x,t) - a_{12} u_2(x,t) - a_{13} u_3(x,t)\right], \\
\frac{\partial u_2(x,t)}{\partial t} &= d_2 \Delta u_2(x,t) + r_2 u_2(x,t) \left[1 + a_{21} u_1(x,t) - a_{22} u_2(x,t) - a_{23} u_3(x,t)\right],
\end{align*}
$$

where all the parameters are positive and $a_{11}a_{22} > a_{12}a_{21}$.

By (18), its wave system is

$$
\begin{align*}
d_1 \phi_1''(t) - c \phi_1'(t) + r_1 \phi_1(t) \left[1 - a_{11} \phi_1(t) - c \tau_1\right] &= 0, \\
d_2 \phi_2''(t) - c \phi_2'(t) + r_2 \phi_2(t) \left[1 - a_{22} \phi_2(t) - c \tau_4\right] &= 0,
\end{align*}
$$

with the asymptotic boundary conditions

$$
\lim_{t \to -\infty} \phi_i(t) = 0, \quad \lim_{t \to \infty} \phi_i(t) = k_i, \quad i = 1, 2,
$$

where

$$
k_1 = \frac{a_{11} + a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \quad \text{and} \quad k_2 = \frac{a_{22} + a_{12}}{a_{11}a_{22} - a_{12}a_{21}}.
$$

Similar to the proof of Lemma 5.10, we can get the following result.

**Lemma 5.13.** Assume that $(\gamma_{11}, \gamma_{12}) \cap (\gamma_{21}, \gamma_{22})$ is nonempty, $\tau_1$ and $\tau_4$ are small enough. Then $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ in the Example 5.9 are a pair of upper-lower solutions of (41) if $p > 1, q > 1$ are large enough and $\gamma > 0$ is small enough.

The proof is similar to those of Lemmas 5.3 and 5.10. Moreover, it is clear that $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ satisfy (P1)-(P3) in Theorem 4.4 when $\tau_1$ and $\tau_2$ are small enough. Thus, we have the following result.

**Theorem 5.14.** Assume that $c > \max_{i=1,2} \{2 \sqrt{d_i r_i}\}$ such that $(\gamma_{11}, \gamma_{12}) \cap (\gamma_{21}, \gamma_{22})$ is nonempty, $\tau_1$ and $\tau_4$ are small enough. Then (41) with (42) has a positive solution $(\phi_1(t), \phi_2(t))$ satisfying

$$
0 \leq \phi_i(t) \leq k_i, \quad \lim_{t \to -\infty} \phi_i(t) e^{-\gamma_i t} = 1, \quad i = 1, 2,
$$

Remark 10. Let $\beta_1, \beta_2$ be the constants given in Lemma 5.1, then $\beta_1, \beta_2$ can be large enough if $\tau_1$ and $\tau_4$ are small enough. Define the set $\Omega^*$ as follows

$$
\Omega^* = \left\{ (\phi_1, \phi_2) \in C(\mathbb{R}, \mathbb{R}^2) \mid \begin{array}{l}
(i) \quad (\phi_1, \phi_2) \leq (\overline{\phi}_1, \overline{\phi}_2) \leq (\underline{\phi}_1, \underline{\phi}_2) \\
(ii) \quad (\phi_1, \phi_2) \text{ is nondecreasing; } \\
(iii) \quad e^{\beta_1 t} \left(\phi_1(t) - \phi_1(t)\right), e^{\beta_2 t} \left(\phi_2(t) - \phi_2(t)\right), e^{\beta_1 t} \left(\phi_1(t) - \underline{\phi}_1(t)\right) \text{ and } e^{\beta_2 t} \left(\phi_2(t) - \underline{\phi}_2(t)\right) \text{ are nondecreasing in } t \in \mathbb{R}; \\
(iv) \quad \text{for every } s > 0, e^{\beta_1 t} \left(\phi_1(t + s) - \phi_1(t)\right), e^{\beta_2 t} \left(\phi_2(t + s) - \phi_2(t)\right) \text{ are nondecreasing in } t \in \mathbb{R}
\end{array} \right\}.
$$

It is clear that $\Omega^*$ is closed and bounded with respect to the super norm. It is also nonempty and convex. In fact, let $\beta_1, \beta_2 > 0$ be large enough, then

$$
(\frac{k_1}{1 + e^{-\gamma_1 t}}, \frac{k_2}{1 + e^{-\gamma_2 t}}) \in \Omega^*
$$
if \( q > 1 \) is large enough and \( \gamma > 0 \) is small enough. By the nonstandard order or the exponential order (see [43, 44, 45, 60, 61, 62]), (40) has a monotone traveling wave solution \((\psi_1(x + ct), \psi_2(x + ct))\) such that

\[
0 < \psi_i(\xi) < k_i, \quad \lim_{\xi \to -\infty} \psi_i(\xi)e^{-\gamma_1 \xi} = 1, i = 1, 2, \xi \in \mathbb{R}.
\]

**Remark 11.** From the asymptotic behavior (43), our result is different from that in Huang and Zou [16]. Moreover, if \( d_1 = d_2 \), then \((\gamma_{11}, \gamma_{12}) \cap (\gamma_{21}, \gamma_{22})\) is nonempty for any \( c > \max_{i=1,2} \{2\sqrt{d_i r_i}\} \).

### 5.3. Predator-Prey system with delays.

**Example 5.15.** Consider the following predator-prey model with delays [40]

\[
\begin{align*}
\frac{\partial u_1(x, t)}{\partial t} &= d_1 \Delta u_1(x, t) + r_1 u_1 [1 - a_{11} u_1(x, t - \tau_1) - a_{12} u_2(x, t - \tau_2)], \\
\frac{\partial u_2(x, t)}{\partial t} &= d_2 \Delta u_2(x, t) + r_2 u_2 [1 + a_{21} u_1(x, t - \tau_3) - a_{22} u_2(x, t - \tau_4)],
\end{align*}
\]

(44)

where all the parameters are positive and \( a_{22} > a_{12} \) holds.

By (17), the corresponding wave system of (44) is

\[
\begin{align*}
d_1 \phi_1''(t) - c \phi_1(t) + r_1 \phi_1(t) [1 - a_{11} \phi_1(t - c \tau_1) - a_{12} \phi_2(t - c \tau_2)] &= 0, \\
d_2 \phi_2''(t) - c \phi_2(t) + r_2 \phi_2(t) [1 + a_{21} \phi_1(t - c \tau_3) - a_{22} \phi_2(t - c \tau_4)] &= 0,
\end{align*}
\]

(45)

with the following asymptotic boundary conditions

\[
\lim_{t \to -\infty} (\phi_1(t), \phi_2(t)) = (0, 0), \quad \lim_{t \to -\infty} (\phi_1(t), \phi_2(t)) = (k_1, k_2),
\]

(46)

where \( k_1 \) and \( k_2 \) are defined by

\[
k_1 = \frac{a_{22} - a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, \quad k_2 = \frac{a_{21} + a_{11}}{a_{11} a_{22} + a_{12} a_{21}}.
\]

Furthermore, we also assume that \( a_{11} k_1 > a_{12} k_2 \) holds such that \((k_1, k_2)\) is a stable equilibrium of the corresponding ODEs.

Due to the invariance of the corresponding ODEs of (46), we choose constants \( M_1, M_2 \) in this example as follows

\[
M_1 = \frac{1}{a_{11}} \quad \text{and} \quad M_2 = \frac{a_{11} + a_{21}}{a_{11} a_{22}}.
\]

In order to apply Theorem 4.4, we further define the continuous functions

\[
\begin{align*}
\overline{\phi}_1(t) &= \min \{e^{\gamma_1 t}, M_1, k_1 e^{-\gamma_1 t}\}, \\
\overline{\phi}_2(t) &= \min \{e^{\gamma_2 t} + p e^{p \gamma_1 t}, M_2, k_2 e^{-\gamma_2 t}\},
\end{align*}
\]

where the notations are the same as those in (19) and (20).

**Lemma 5.16.** Assume that \( \tau_1, \tau_4 \) are small enough. Then \((\overline{\phi}_1, \overline{\phi}_2)\) and \((\underline{\phi}_1, \underline{\phi}_2)\) are a pair of upper-lower solutions of (45) if \( p, q \) and \( 1/\gamma \) are large enough.

**Proof.** For \( \overline{\phi}_1(t) \), it suffices to prove that

\[
d_1 \overline{\phi}_1''(t) - c \overline{\phi}_1(t) + r_1 \overline{\phi}_1(t) [1 - a_{11} \overline{\phi}_1(t - c \tau_1) - a_{12} \overline{\phi}_2(t - c \tau_2)] \leq 0
\]

(47)

except several finite points on \( \mathbb{R} \). The proof of (47) is similar to that of Lemma 5.3, and we omit it here.

For \( \overline{\phi}_2(t) \), we need to verify that
\[ d_2 \phi''_2(t) - c \phi'_2(t) + r_2 \phi_2(t) \left[ 1 + a_{21} \phi_1(t - c \tau_3) - a_{22} \phi_2(t - c \tau_4) \right] \leq 0 \]

except several finite points on \( \mathbb{R} \). If \( \phi_2(t) = e^{\gamma t} + pe^{\eta_1 t} \), then

\[ d_2 \phi''_2(t) - c \phi'_2(t) + r_2 \phi_2(t) \left[ 1 + a_{21} \phi_1(t - c \tau_3) - a_{22} \phi_2(t - c \tau_4) \right] \leq p(d_2 \eta^2 \gamma^2 + c \eta \gamma + r_2)e^{\eta_1 t} + a_{21}e^{\gamma t} \left( e^{\gamma t} + pe^{\eta_1 t} \right). \]

Let

\[ p = \max \left\{ \frac{-2a_{11}}{d_2 \eta^2 \gamma^2 + c \eta \gamma + r_2} + 1, \frac{-2a_{21} M \gamma^2}{d_2 \eta^2 \gamma^2 + c \eta \gamma + r_2} + 1 \right\}. \]

Then

\[ p(d_2 \eta^2 \gamma^2 + c \eta \gamma + r_2)e^{\eta_1 t} + a_{21}e^{\gamma t} \left( e^{\gamma t} + pe^{\eta_1 t} \right) < 0 \]

is clear. If \( \phi_2(t) = M_2 \), then the conclusion is obvious. If \( \phi_2(t) = k_2 + k_2e^{-\gamma t} \), the proof is similar to that of Lemma 5.3 and is omitted here.

We now consider the result on \( \phi_1(t) \) and it is sufficient to prove that

\[ d_1 \phi''_1(t) - c \phi'_1(t) + r_1 \phi_1(t) \left[ 1 - a_{11} \phi_1(t - c \tau_1) - a_{12} \phi_2(t - c \tau_2) \right] \geq 0, t \neq t_{12}. \tag{48} \]

Let \( q > 1 \) be large enough such that

\[ \sqrt{\phi_2(t)} \leq e^{\gamma t} + pe^{\eta_1 t} \leq 2e^{\gamma t}, t < t_{12}. \]

Then the result is clear if \( q > 1 \) is large enough and \( \gamma > 0 \) is small enough.

For \( \phi_2(t) \), we need to prove that

\[ d_2 \phi''_2(t) - c \phi'_2(t) + r_2 \phi_2(t) \left[ 1 + a_{21} \phi_1(t - c \tau_3) - a_{22} \phi_2(t - c \tau_4) \right] \geq 0, t \neq t_{22}. \tag{49} \]

By the discussion similar to those in Lemmas 5.3 and 5.10, we can prove (49) if \( q > 1 \) is large enough and \( \gamma > 0 \) is small enough. The proof is complete. \( \square \)

**Remark 12.** From the above proof, we can define \( p \), and then choose \( q, \gamma \). Since \( p \) is independent of \( q, \gamma \), then the choice is also admissible.

By Theorem 4.4, we now get the following result.

**Theorem 5.17.** Assume that \( \tau_1, \tau_4 \) are small enough and \( c > \max \{2 \sqrt{d_1 \tau_1}, 2 \sqrt{d_2 \tau_2} \} \) holds. Then (45) with (46) has a positive solution \((\phi_1, \phi_2)\) such that

\[ 0 < \phi_i(t) < M_i, \quad \lim_{t \to \infty} \phi_i(t)e^{-\gamma_1 t} = 1, i = 1, 2. \]

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REFERENCES


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