On the quasilinear elliptic problem with a critical Hardy–Sobolev exponent and a Hardy term

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Abstract

In the present paper, a quasilinear elliptic problem with a critical Sobolev exponent and a Hardy-type term is considered. By means of a variational method, the existence of nontrivial solutions for the problem is obtained. The result depends crucially on the parameters \( p, t, s, \lambda \) and \( \mu \).

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1. Introduction and the main results

In this paper, we consider the elliptic equation

\[
\begin{cases}
-\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \frac{|u|^{p^*(t)-2}u}{|x|^t} - \mu \frac{|u|^{p-2}u}{|x|^p} + \frac{\lambda |u|^{p-2}u}{|x|^s}, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\]

(1.1)

where \( \Omega \subset \mathbb{R}^N \) (\( N \geq 3 \)) is a smooth bounded domain containing the origin 0. \(-\Delta_p u = -\text{div}(\nabla u|^{p-2}\nabla u), 1 < p < N, 0 \leq \mu < \bar{\mu} \triangleq (N - p)p / p^p, \lambda > 0, 0 \leq s, t < p, p^*(t) \triangleq p(N - t)/(N - p) \) is the critical Hardy–Sobolev exponent.

We employ \( D^{1, p}(\Omega) \) to denote the closure of \( C_0^\infty(\Omega) \) with respect to the norm \( (\int_\Omega |\nabla u|^{p} dx)^{1/p} \). The function \( u \in D^{1, p}(\Omega) \) is said to be a solution of the problem (1.1) if \( u \) satisfies

\[
\int_{\Omega} \left( |\nabla u|^{p-2}\nabla u \cdot \nabla v - \mu \frac{|u|^{p-2}uv}{|x|^p} - \frac{|u|^{p^*(t)-2}uv}{|x|^t} - \lambda \frac{|u|^{p-2}uv}{|x|^s} \right) dx = 0
\]

for all \( v \in D^{1, p}(\Omega) \). By the standard elliptic regularity argument we have that any solution \( u \) of (1.1) belongs to \( C^{1, \alpha}(\Omega \setminus \{ 0 \}) \).
Problem (1.1) is related to the following Hardy–Sobolev inequality, which is essentially due to Caffarelli, Kohn and Nirenberg [4]:

\[
\left( \int_\Omega \frac{|u|^p}{|x|^q} \, dx \right)^{\frac{q}{p}} \leq C_{r,t,p} \int_\Omega |\nabla u|^p \, dx, \quad \forall u \in D^{1,p}(\Omega),
\]

where \( p \leq r \leq p^*(t) \). If \( t = r = p \), the above inequality becomes the well-known Hardy inequality [4,7,10]:

\[
\int_\Omega \frac{|u|^p}{|x|^p} \, dx \leq \frac{1}{\mu} \int_\Omega |\nabla u|^p \, dx, \quad \forall u \in D^{1,p}(\Omega).
\]

In the space \( D^{1,p}(\Omega) \) we employ the following norm:

\[
\|u\| = \|u\|_{D^{1,p}(\Omega)} \triangleq \left( \int_\Omega \left( |\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) \, dx \right)^{\frac{1}{p}}, \quad \mu \in [0, \bar{\mu}).
\]

By the Hardy inequality (1.2) this norm is equivalent to the usual norm \( (\int_\Omega |\nabla u|^p \, dx)^{\frac{1}{p}} \). The elliptic operator \( L \triangleq -|\nabla \cdot |^p - \mu \frac{|\nabla u|^p}{|x|^p} \cdot |^{p-2} \cdot \cdot \) is positive in \( D^{1,p}(\Omega) \) if \( 0 \leq \mu < \bar{\mu} \).

By Hardy inequality and Hardy–Sobolev inequality, for \( 0 \leq \mu < \bar{\mu}, 0 \leq t < p \) and \( p \leq r \leq p^*(t) \) we can define the best Hardy–Sobolev constant:

\[
A_{\mu,t,r}(\Omega) \triangleq \inf_{u \in D^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega \left( |\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) \, dx}{\left( \int_\Omega \frac{|u|^r}{|x|^t} \, dx \right)^{\frac{1}{t}}}. \tag{1.3}
\]

In the important case when \( r = p^*(t) \), we simply denote \( A_{\mu,t,p^*(t)} \) as \( A_{\mu,t} \). Note that \( A_{\mu,0} \) is the best constant in the Sobolev inequality, namely,

\[
A_{\mu,0}(\Omega) \triangleq \inf_{u \in D^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega \left( |\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) \, dx}{\left( \int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}}.
\]

Another important parameter is \( A_{\mu,s,p}(\Omega) \), the (general) first eigenvalue of the operator \( L \):

\[
\lambda_1 = A_{\mu,s,p}(\Omega) \triangleq \inf_{u \in D^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega \left( |\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) \, dx}{\int_\Omega \frac{|u|^p}{|x|^p} \, dx}. \tag{1.4}
\]

Furthermore, \( \lambda_1 \) is positive and simple, the corresponding eigenfunction \( \phi_1 \) does not change sign, the operator \( L \) admits a sequence of eigenvalues diverging to \(+\infty \) [14,15]. Without loss of generality, we can assume that \( \phi_1 > 0 \).

Setting

\[
E \triangleq \left\{ u \in D^{1,p}(\Omega) \left| \int_\Omega \phi_1^{p-1} u \frac{|u|^p}{|x|^p} = 0 \right. \right\} \tag{1.5}
\]

and

\[
\lambda^* = \lambda^*_{\mu,s}(\Omega) \triangleq \inf_{u \in E \setminus \{0\}} \frac{\int_\Omega \left( |\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) \, dx}{\int_\Omega \frac{|u|^p}{|x|^p} \, dx}, \tag{1.6}
\]

then we have \( \lambda_1 < \lambda^* \) (see Lemma 2.1 of this paper).

The energy functional corresponding to problem (1.1) is

\[
J(u) \triangleq \frac{1}{p} \int_\Omega \left( |\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) \, dx - \frac{1}{p^*(t)} \int_\Omega \frac{|u|^{p^*(t)}}{|x|^t} \, dx - \frac{\lambda}{p} \int_\Omega \frac{|u|^p}{|x|^t} \, dx.
\]
and then $J(u)$ is well defined in the space $D^{1,p}(\Omega)$ and belongs to $C^1(D^{1,p}(\Omega), \mathbb{R})$. The solution of (1.1) corresponds to the critical point of $J$.

The quasilinear problems related to Hardy inequality and Hardy–Sobolev inequality have been studied by some authors [1,3,7,10–12]. Here we recall the recent work by Abdellaoui, Felli and Peral [1], where the extremal functions for the best constant $A_{\mu,0}$ were studied. The results can be employed to study the problems with critical Sobolev exponents and Hardy terms [1,11]. We also mention that very recently, Kang [12] investigated the extremal functions by which the best constant $A_{\mu,s}$ is achieved (see Lemma 2.2 of this paper). The existence of positive solutions for the problem (1.1) is also obtained in [12].

The main purpose of this paper is to investigate the existence of nontrivial solutions for the problem (1.1). The generalized mountain pass theorem [13] and the orthogonalization technique [9] will be employed. These methods had been applied either to nonlinear elliptic equations or to polyharmonic problems [2,3,5,6,8,9].

We are now ready to state the main results of this paper. The results are new for the case when $0 < \mu < \bar{\mu}$ and $0 < s < p$. We can verify that the intervals for $\lambda$ and $\mu$ in the following theorems are not empty.

**Theorem 1.1.** Assume $N + ps - s - p^2 > 0$, $\lambda \in (\lambda_1, \lambda^*)$ and $0 \leq \mu \leq \mu_1$ with

$$\mu_1 \triangleq \frac{N + ps - s - p^2}{p} \left( \frac{N - s}{p} \right)^{p-1}.$$

Then problem (1.1) has a pair of sign-changing solutions $\pm u \in D^{1,p}(\Omega)$.

**Theorem 1.2.** Assume that $\lambda = \lambda_1$, $0 \leq \mu < \min \{\mu_1, \mu_2\}$, $\bar{\tau}_1 < \bar{\tau}_2$ and

$$N + ps - s - p^2 > 0, \quad p^*(t)(N - 2p + s) + p(p - s) > 0,$$

where $\mu_1$ is defined as in Theorem 1.1,

$$\bar{\tau}_1 \triangleq \frac{pb(\mu) + p - N}{pb(\mu) + s - N}, \quad \bar{\tau}_2 \triangleq \frac{p^*(t)(N - p - pa(\mu))}{(p - s)(p^*(t) - p)},$$

$$\mu_2 \triangleq \left( \frac{N - p}{p} - \frac{(p - s)(p^*(t) - p)}{p^*(t)p} \right)^{p-1} \left( \frac{N - p}{p} + \frac{(p^2 + ps - p)(p^*(t) - p)}{p^*(t)p} \right).$$

Then problem (1.1) has a pair of sign-changing solutions $\pm u \in D^{1,p}(\Omega)$.

**Theorem 1.3.** Assume $0 \leq \mu < \bar{\mu}$, $\lambda > 0$ and $\lambda \in (\bar{\lambda}, \lambda_1)$, where

$$\bar{\lambda} = \lambda_1 - A_{\mu,s} \left( \int_{\Omega} |x|^{\frac{N + ps - N - ps - p^2}{p - s}} \right)^{-\frac{p - s}{N - s}}.$$

Then problem (1.1) has at least one positive solution $u \in D^{1,p}(\Omega)$.

**Remark 1.4.** The existence of one positive solution of (1.1) in the case when $\lambda \in (0, \lambda_1)$ and $\mu \in [0, \mu_1]$ had been obtained in [12].

**Remark 1.5.** The assumptions of Theorem 1.2 are allowable; for example we can take $s \to p$.

This paper is organized as follows. Section 2 deals with some preliminary materials and technical results. Section 3 is devoted to the proofs of Theorems 1.1–1.3. At the end of this section, we explain some notation employed. In the following argument, we denote a positive constant as $C$ and omit $dx$ in the integral for convenience. $L^q(\Omega, |x|^s)$ denotes the weighted $L^q(\Omega)$ space with the weight $|x|^s$, $(D^{1,p}(\Omega))^{-1}$ is the dual space of $D^{1,p}(\Omega)$, $O(\varepsilon^t)$ is the quantity satisfying $|O(\varepsilon^t)|/\varepsilon^t \leq C$, $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$ and $o(1)$ denotes a generic infinitesimal value.

2. Preliminary results

In this section, we will establish several preliminary lemmas. We remark that $D^{1,p}(\Omega) = H \oplus E$, where $H = \langle \phi_1 \rangle \triangleq \text{span}(\phi_1)$, $\phi_1 > 0$ is the eigenfunction corresponding to the first eigenvalue $\lambda_1$ and $E$ is defined as in (1.4).
Lemma 2.1. Let $\lambda_1$ and $\lambda^*$ be defined as in (1.3) and (1.5). Then $\lambda_1 < \lambda^*$.

Proof. The proof follows the same lines as that of Lemma 2 in [2].

To the contrary, we assume that $\lambda_1 = \lambda^*$. Then there exists $\{u_k\} \subset E$ such that $\|u_k\| = 1$ and $\lambda_1 \int_\Omega \frac{|u_k|^p}{|x|^q} \to 1$. By rescaling and setting $v_k = u_k(\int_\Omega \frac{|u_k|^p}{|x|^q})^{-1}$ we have $\int_\Omega \frac{|v_k|^p}{|x|^q} = 1$ and $\|v_k\| \to \lambda_1$. Thus $\{v_k\}$ is a minimizing sequence for (1.3), $v_k \rightharpoonup v$ weakly and $v_k \to v$ strongly in $L^p(\Omega, |x|^{-q})$ for some $v \in D^{1,p}(\Omega)$. Finally, $v_k \to v$ strongly in $D^{1,p}(\Omega)$ since $\|v\| \geq \lambda_1 = \lim_{k \to \infty} \|v_k\|$; which contradicts the fact that $v \notin \langle \phi_1 \rangle$ and $\lambda_1$ is simple. Hence $\lambda_1 < \lambda^*$. □

The following lemma is well known, where we have employed the equivalent norm in $D^{1,p}(\Omega)$, see [10] for the case when $\mu = 0$.

Lemma 2.2. Assume that $0 \leq s \leq p, p \leq q \leq p^*(s)$ and $0 \leq \mu < \bar{\mu}$. Then:

(i) There exists a constant $C > 0$ such that
$$\left(\int_\Omega \frac{|u|^q}{|x|^q} \right)^{\frac{1}{q}} \leq C \|u\|, \quad \forall u \in D^{1,p}(\Omega).$$

(ii) The map $u \to \frac{u}{x^{n/q}}$ from $D^{1,p}(\Omega)$ into $L^q(\Omega)$ is compact if $p \leq q < p^*(s)$.

To continue, we recall a recent result on the extremal functions of $A_{\mu,t}$ [12]. The case when $t = 0$ was studied in [1].

Lemma 2.3 ([12]). Assume that $1 < p < N, 0 \leq t < p$ and $0 \leq \mu < \bar{\mu}$. Then the limiting problem

$$\begin{cases}
-\Delta_p u - \mu \frac{u^{p-1}}{|x|^p} = \frac{u^{p^*(t)-1}}{|x|^p}, & \text{in } R^N \setminus \{0\}, \\
u \in D^{1,p}(\mathbb{R}^N), u > 0 & \text{in } R^N \setminus \{0\},
\end{cases}$$

has positive radial ground states
$$V_\varepsilon(x) \triangleq \varepsilon^{\frac{p-N}{p}} U_{p,\mu} \left(\frac{x}{\varepsilon}\right) = \varepsilon^{\frac{p-N}{p}} U_{p,\mu} \left(\frac{|x|}{\varepsilon}\right), \quad \forall \varepsilon > 0,$$

that satisfy
$$\int_{\mathbb{R}^N} \left|\nabla V_\varepsilon(x)\right|^p - \mu \frac{|V_\varepsilon(x)|^p}{|x|^p} \, dx = \int_{\mathbb{R}^N} \frac{|V_\varepsilon(x)|^{p^*(t)}}{|x|^t} \, dx = (A_{\mu,t})^{\frac{N-t}{p^*-t}},$$

where $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ is the unique radial solution of the limiting problem with
$$U_{p,\mu}(1) = \left(\frac{(N-t)(\bar{\mu} - \mu)}{N-p}\right)^{\frac{p^*-t}{p^*-1}}.$$

Furthermore, $U_{p,\mu}$ have the following properties:

$$\lim_{r \to 0} r^{a(\mu)} U_{p,\mu}(r) = C_1 > 0,$$

$$\lim_{r \to +\infty} r^{b(\mu)} U_{p,\mu}(r) = C_2 > 0,$$

$$\lim_{r \to 0} r^{a(\mu)+1} |U_{p,\mu}'(r)| = C_1 \alpha(\mu) \geq 0,$$

$$\lim_{r \to +\infty} r^{b(\mu)+1} |U_{p,\mu}'(r)| = C_2 \beta(\mu) > 0,$$

where $C_i$ ($i = 1, 2$) are positive constants and $\alpha(\mu)$ and $\beta(\mu)$ are zeros of the function
$$f(\tau) = (p-1)\tau^p - (N-p)\tau^{p-1} + \mu, \quad \tau \geq 0, \quad 0 \leq \mu < \bar{\mu},$$
that satisfy
\[ 0 \leq a(\mu) < \frac{N - p}{p} < b(\mu) \leq \frac{N - p}{p - 1}. \]

In the following, we will give some estimates on the extremal function \( V_\epsilon \) defined in Lemma 2.3. For \( m \in \mathbb{N} \) large, choose \( \varphi(x) \in C_0^\infty(\mathbb{R}^N) \). \( 0 \leq \varphi(x) \leq 1 \), \( \varphi(x) = 1 \) for \( |x| \leq \frac{1}{2m} \), \( \varphi(x) = 0 \) for \( |x| \geq \frac{1}{m} \), \( \| \nabla \varphi(x) \|_{L^p(\Omega)} \leq 4m \), set \( u_\epsilon(x) = \varphi(x)V_\epsilon(x) \). For \( \epsilon \rightarrow 0 \), the behavior of \( u_\epsilon \) has to be the same as that of \( V_\epsilon \), but we need precise estimates of the error terms. For \( 1 < p < N \), \( 0 \leq s, t < p \) and \( 1 < q < p^*(s) \), we have the following estimates [12]:

\[
\| u_\epsilon \|_p = (A_{\mu, t})^{\frac{N-q}{p-t}} + O\left(e^{b(\mu)p^*+p-N}\right),
\]

\[
\int_\Omega \frac{|u_\epsilon|^{p^*(t)}}{|x|^t} = (A_{\mu, t})^{\frac{N-q}{p-t}} + O\left(e^{b(\mu)p^*(t)-N+t}\right),
\]

\[
\int_\Omega \frac{|u_\epsilon|^q}{|x|^q} \geq \begin{cases} \ \ C e^{N-s+(1-\frac{N}{p})q}, & q > \frac{N-s}{b(\mu)}, \\ \ C e^{N-s+(1-\frac{N}{p})q} \ln |\epsilon|, & q = \frac{N-s}{b(\mu)}, \\ \ C e^{q(b(\mu)+1-\frac{N}{p})}, & q < \frac{N-s}{b(\mu)}. \end{cases}
\]

To proceed, another kind of cut-off functions depending on \( m \in \mathbb{N} \) will be employed. Take always \( m \in \mathbb{N} \) large enough such that \( B_{2/m}(0) \subset \Omega \), where \( B_r(0) \) denotes the ball of radius \( r \) with center at 0. Define the cut-off function as

\[ \psi_m(x) = \begin{cases} 0, & x \in B_{1/m}(0), \\ m|x| - 1, & x \in A_m \triangleq B_{2/m}(0) \setminus B_{1/m}(0), \\ 1, & x \in \Omega \setminus B_{2/m}(0), \end{cases} \]

and set \( \phi_m \triangleq \psi_m \varphi_1 \) and \( H_m^- \triangleq (\phi_1^m) \). Then the following approximating lemma holds.

**Lemma 2.4.** Assume that \( a(\mu) \) is the constant defined as in Lemma 2.3 and \( m \) large enough. Then

(i) \( C_1|x|^{-a(\mu)} \leq |\varphi_1(x)| \leq C_2|x|^{-a(\mu)} \), \( |\nabla \varphi_1| \leq C_3|x|^{-(a(\mu)+1)} \), \( \forall x \in B_{2/m}(0) \setminus \{0\} \), where \( C_i \) (\( i = 1, 2, 3 \)) are positive constants.

(ii) \( \phi_1^m \rightarrow \varphi_1 \) in \( D^1, p(\Omega) \) as \( m \rightarrow \infty \).

(iii) For all \( u \in H_m^- \) we have

\[ \| u \|_p \leq \left( \lambda_1 + C_3 m^{-(N-p-a(\mu))} \right) \int_\Omega \frac{|u|^p}{|x|^s}. \]

**Proof.** (i) was already proved in [14]. We need to prove (ii) and (iii).

For \( m \) large enough, from (i) we obtain

\[
\int_\Omega |\nabla (\phi_1^m - \varphi_1)|_p \leq 2^{p-1} \int_{A_m} |\phi_1 \nabla \psi_m|^p + 2^{p-1} \int_{B_{2/m}(0)} |(\psi_m - 1) \nabla \varphi_1|^p \\
\leq 2^{p-1} \int_{A_m} |\phi_1|^p m^p + 2^{p-1} \int_{B_{2/m}(0)} |\nabla \varphi_1|^p \\
\leq C m^p \int_{1/m}^{2/m} r^{-a(\mu)p} r^{N+1} dr + C \int_0^{2/m} r^{-p(a(\mu)+1)+N-1} dr \\
\leq C m^{p+a(\mu)p-N} + C m^{p(a(\mu)+1)-N} \\
\leq C m^{p(a(\mu)+1)-N}. \]

From Lemma 2.3 we have \( p(a(\mu)+1) - N < 0 \), which implies that \( \phi_1^m \rightarrow \varphi_1 \) in \( D^1, p(\Omega) \) as \( m \rightarrow \infty \) and therefore (ii) is verified.
To prove (iii), without loss of generality we can suppose that \( \int_{\Omega} \frac{|\phi_1|^p}{|x|^s} = 1 \). Setting \( \Sigma = \{ u \in D^{1,p}(\Omega) \mid \int_{\Omega} \frac{|u|^p}{|x|^s} = 1 \} \), then for \( v \in H \cap \Sigma \) we have \( v = \pm \phi_1 \) and \( v_m \equiv \pm \psi_m \phi_1 = \pm \phi_1^m \in H_m \). We also define \( \zeta_m(v) = \left( \int_{\Omega} \frac{|v_m|^p}{|x|^s} \right)^{-\frac{1}{p}}, \)

then \( 1 = (\zeta_m(v))^p \left( \int_{\Omega} \frac{|\phi_1|^p}{|x|^s} + \int_{\Omega} \frac{|\psi_m \phi_1|^p - |\phi_1|^p}{|x|^s} \right) \leq (\zeta_m(v))^p, \)

and thus \( \zeta_m(v) \geq 1 \). Furthermore,

\[
\int_{\Omega} \frac{|v_m|^p}{|x|^s} = \int_{A_m} \frac{|\psi_m \phi_1|^p}{|x|^s} - \int_{B_{2/m}(0)} \frac{|\phi_1|^p}{|x|^s} + \int_{\Omega} \frac{|\phi_1|^p}{|x|^s} \\
= 1 + \int_{A_m} \frac{|\psi_m \phi_1|^p}{|x|^s} - \int_{B_{2/m}(0)} \frac{|\phi_1|^p}{|x|^s} \\
\geq 1 - \int_{B_{2/m}(0)} \frac{|\phi_1|^p}{|x|^s} \\
\geq 1 - C \int_0^{2/m} r^{-s-a(\mu)p+N-1} dr \\
\geq 1 - C m^{s+a(\mu)p-N}.
\]

Consequently,

\[
\zeta_m(v) \leq \left( 1 - C m^{s+a(\mu)p-N} \right)^{-\frac{1}{p}} \leq 1 + C m^{s+a(\mu)p-N}.
\]

If \( u \in H_m^\cap \Sigma \), then \( u = \zeta_m(\phi_1) \phi_1^m = \zeta_m(\phi_1) \psi_m \phi_1 \). For \( m \) large enough we obtain

\[
\|u\|^p = (\zeta_m(\phi_1))^p \|\psi_m \phi_1\|^p \\
\leq C (1 + C m^{s+a(\mu)p-N}) \int_{\Omega} \left( |\psi_m \nabla \phi_1 + \phi_1 \nabla \psi_m|^p - \mu \frac{|\psi_m \phi_1|^p}{|x|^s} \right) \\
\leq C \int_{\Omega} \left( |\psi_m \nabla \phi_1|^p + |\phi_1|^p |\nabla \psi_m|^p - \mu \frac{|\psi_m \phi_1|^p}{|x|^s} \right) \\
+ C \int_{\Omega} \left( |\psi_m \phi_1|^{p-1} |\nabla \phi_1| |\nabla \psi_m|^{p-1} + |\psi_m \phi_1|^{p-1} |\nabla \phi_1|^{p-1} \right),
\]

where we have used the fact that \( s + a(\mu)p - N < 0 \) and the elementary inequality

\[
|x + y|^p \leq |x|^p + |y|^p + C_p |x|^p |y|^{p-1} + C_p |x|^p |y|, \quad \forall x, y \in \mathbb{R}^N.
\]

Then the following estimates hold:

\[
\int_{\Omega} \frac{\psi_m}{|x|^s} \left( |\nabla \phi_1|^p - \mu \frac{\phi_1^p}{|x|^s} \right) \leq \int_{\Omega} \left( |\nabla \phi_1|^p - \mu \frac{\phi_1^p}{|x|^s} \right) = \lambda_1.
\]

\[
\int_{\Omega} \frac{|\phi_1|^p |\nabla \psi_m|^s}{|x|^s} \leq C m^p \int_{1/m}^{2/m} r^{N-1-a(\mu)p} dr \leq C m^{p+a(\mu)p-N}.
\]

\[
\int_{\Omega} \frac{|\psi_m \phi_1|^{p-1} |\nabla \phi_1| |\nabla \psi_m|^{p-1}}{|x|^s} \leq C \int_{1/m}^{2/m} m^{p-1} r^{-(a(\mu)+1)+(p-1)a(\mu)+N-1} dr \leq C m^{p+a(\mu)p-N}.
\]

\[
\int_{\Omega} \frac{|\psi_m|^{p-1} |\phi_1| |\nabla \psi_m|^p |\nabla \phi_1|^p}{|x|^s} \leq C \int_{1/m}^{2/m} m^{-a(\mu)+(p-1)a(\mu)+N-1} dr \leq C m^{p+a(\mu)p-N}.
\]
Consequently,
\[ \|u\|^p \leq \lambda_1 + C m^{p+a(\mu) p-N}, \quad \forall u \in H^{-}_{m} \cap \Sigma. \]
Thus we can conclude (iii) by the homogeneity. \( \square \)

Choose \( v \in H^{-}_{m} \oplus \mathbb{R}^+ \{u_{\varepsilon}\} \); then \( v = w + \alpha u_{\varepsilon} \) with \( w \in H^{-}_{m}, \alpha > 0 \) and
\[ \text{supp}(u_{\varepsilon}) \cap \text{supp}(w) = \emptyset. \] (2.4)

We have the following lemma.

**Lemma 2.5.** Assume that \( \lambda \in (0, \lambda^*) \). Then:

(i) There exist constants \( \alpha > 0 \) and \( \rho > 0 \) such that
\[ J(v) \geq \alpha, \quad \forall v \in \partial B_{\rho} \cap H^+, \]
where \( H^+ \triangleq E, B_{\rho} = \{u \in D^{1,p}(\Omega) \|u\| < \rho\} \).

(ii) There exists \( R > \rho \) such that
\[ \max_{v \in \partial Q_m^\varepsilon} J(v) \leq \omega_m \quad \text{with} \quad \omega_m \rightarrow 0 \quad \text{as} \quad m \rightarrow +\infty, \]
where \( Q_m^\varepsilon \triangleq [(\bar{B}_R \cap H^{-}_{m}) \oplus [0, R][u_{\varepsilon}]]. \)

**Proof.** By the Hardy--Sobolev inequality we obtain
\[ J(v) = \frac{1}{p} \|u\|^p - \frac{1}{p^*} \int_{\Omega} |v|^{p^*(t)} \frac{|x|^t}{|x|^s} - \lambda \frac{1}{p} \int_{\Omega} \frac{|v|^p}{|x|^s}, \]
for all \( v \in \partial B_{\rho} \cap H^+ \). Then (i) follows.

To prove (ii), from the facts that
\[ \lim_{m \rightarrow \infty} \max_{w \in H^{-}_{m}} J(w) = 0, \quad \forall \lambda \in (0, \lambda^*) \]
and
\[ J(r u_{\varepsilon}) \leq \frac{r^p}{p} \|u_{\varepsilon}\|^p - \frac{r^q}{q} \int_{\Omega} |u_{\varepsilon}|^{p^*(t)} \frac{|x|^t}{|x|^s}, \]
we deduce that there exists \( R_1 > 0 \) such that \( J(r u_{\varepsilon}) < 0 \) for any \( r \geq R_1 \).

Hence, for all \( v \in H^{-}_{m} \cup H^{-}_{m} \oplus R_1[u_{\varepsilon}] \) we have
\[ J(v) = J(\tau \phi^m_{u_{\varepsilon}}) + J(R_1 u_{\varepsilon}) \leq \omega_m, \quad \omega_m \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty. \]

On the other hand, the fact that
\[ \max_{r \in [0, R]} J(r u_{\varepsilon}) < +\infty \]
implies that there exists \( R \geq R_1 \), such that
\[ J(v) \leq 0, \quad \forall v \in \{ (\partial B_{R} \cap H^{-}_{m}) \oplus [0, R][u_{\varepsilon}]. \}

Therefore, the functional \( J \) satisfies the geometry of the generalized mountain pass theorem [13]. \( \square \)

We recall that a sequence \( \{u_m\} \subset D^{1,p}(\Omega) \) is called a PS sequence for \( J \) at level \( c \) if \( J(u_m) \rightarrow c \) and \( J'(u_m) \rightarrow 0 \) in \( (D^{1,p}(\Omega))^{-1} \). Then we have the following lemma.

**Lemma 2.6.** Suppose \( \{u_m\} \subset D^{1,p}(\Omega) \) is a PS sequence for \( J \). Then there exists \( u \in D^{1,p}(\Omega) \) such that \( u_m \rightharpoonup u \) weakly up to a subsequence and \( J'(u) = 0 \). Moreover, if \( J(u_m) \rightarrow c \) with \( c \in (0, \frac{\rho-1}{\rho(N-1)}(A_{\mu,i})^{\frac{N-i}{p-1}}) \), then \( u \) is a nontrivial solution of the problem (1.1).
The proof is standard and we only sketch it. It is easy to show that \( u_m \) is bounded in \( D^{1,p}(\Omega) \) and there exists \( u \) such that \( u_m \rightharpoonup u \) up to a subsequence. Furthermore, \( J'(u) = 0 \) by the weak continuity of \( J' \).

Assume \( c \in (0, \frac{p-t}{p(N-t)}(\mu_2, \frac{N-t}{p(N-t)}) \) and \( u \equiv 0 \) by contradiction. As the term \( \frac{|u_m|^p}{|x|^t} \) is subcritical, from Lemma 2.2 and by \( (J'(u_m), u_m) = o(1) \) we have

\[
\|u_m\|^p - \int_{\Omega} \frac{|u_m|^p}{|x|^t} = o(1).
\] (2.5)

By the definition of \( A_{\mu,t} \) we obtain

\[
\|u_m\|^p \geq A_{\mu,t} \left( \int_{\Omega} \frac{|u_m|^p}{|x|^t} \right)^{\frac{p}{p-t}}
\]

and

\[
\|u_m\|^p \left( 1 - \frac{p}{p-t} \right) \frac{A_{\mu,t}}{A_{\mu,t}^\frac{N-t}{p(N-t)}} \|u_m\|^p \leq o(1).
\]

If \( \|u_m\| \to 0 \), we contradict the fact that \( c > 0 \). Therefore

\[
\|u_m\|^p \geq (A_{\mu,t})^\frac{N-t}{p(N-t)} + o(1).
\]

From (2.5) we have

\[
J(u_m) = \frac{1}{p} \|u_m\|^p - \frac{1}{p} \left[ \int_{\Omega} \frac{|u_m|^p}{|x|^t} \right] + o(1)
\]

\[
= \frac{p-t}{p(N-t)} \|u_m\|^p + o(1)
\]

\[
\geq \frac{p-t}{p(N-t)} (A_{\mu,t})^\frac{N-t}{p(N-t)} + o(1),
\]

which contradicts the assumption that \( c < \frac{p-t}{p(N-t)} (A_{\mu,t})^\frac{N-t}{p(N-t)} \). Thus \( u \not \equiv 0 \) and therefore \( u \) is a nontrivial solution of problem (1.1). \( \square \)

According to Lemmas 2.5 and 2.6, in order to prove Theorems 1.1–1.3, it suffices to build a PS sequence for \( J \) at a level strictly between 0 and \( \frac{p-t}{p(N-t)} (A_{\mu,t})^\frac{N-t}{p(N-t)} \). To this end, set

\[
\Gamma = \{ h \in C(Q_m^e, D^{1,p}(\Omega)) \mid h(v) = v, \forall v \in \partial Q_m^e \}.
\]

From the generalized mountain pass theorem [13], we can get a PS sequence for the functional \( J \) at the level

\[
c = \inf_{h \in \Gamma} \max_{v \in Q_m^e} J(h(v)) = \max_{v \in Q_m^e} J(h(v)),
\]

where we have used the fact that the identity function is in \( \Gamma \). Thus to prove the theorems, it is sufficient to show that for \( \varepsilon \) small there holds

\[
\max_{v \in Q_m^e} J(v) < \frac{p-t}{p(N-t)} (A_{\mu,t})^\frac{N-t}{p(N-t)} \cdot \mu_1.
\] (2.6)

**Lemma 2.7.** Assume that \( N + ps - s - p^2 > 0, \lambda \in (\lambda_1, \lambda^*) \), \( \varepsilon > 0 \) small enough and \( 0 \leq \mu \leq \mu_1 \), where

\[
\mu_1 \triangleq \frac{N + ps - s - p^2}{p} \left( \frac{N-s}{p} \right)^{p-1}.
\]

Then

\[
\sup_{\tau \geq 0} J(\tau u_\varepsilon) < \frac{p-t}{p(N-t)} (A_{\mu,t})^\frac{N-t}{p(N-t)} \cdot \mu_1.
\]
Proof. By contradiction, assume that for any \( \varepsilon > 0 \), there exists \( \tau_\varepsilon > 0 \) such that

\[
J(\tau_\varepsilon u_\varepsilon) \geq \frac{p-t}{p(N-t)} (A_{\mu,t})^{\frac{N-t}{p-1}},
\]

then we claim that \( \tau_\varepsilon \to \tau_0 > 0 \), up to a subsequence. Otherwise assume that \( \tau_\varepsilon \to +\infty \) as \( \varepsilon \to 0 \), in contradiction with (2.7). Thus \( \{\tau_\varepsilon\} \) is bounded and there exists \( \tau_0 \geq 0 \) such that \( \tau_\varepsilon \to \tau_0 \) up to a subsequence. If \( \tau_0 = 0 \), by (2.1), (2.2) and the fact \( \lim_{\varepsilon \to 0} \int_{\Omega} \frac{|u_\varepsilon|^p}{|x|^t} = 0 \), we have

\[
J(\tau_\varepsilon u_\varepsilon) = \frac{1}{p} \tau_\varepsilon^p \|u_\varepsilon\|^p \| \tau_\varepsilon^p \| \int_{\Omega} \frac{|u_\varepsilon|^{p-t}}{|x|^t} \int_{\Omega} \frac{|u_\varepsilon|^{p^*(t)}}{|x|^t} \leq o(1),
\]

which contradicts (2.7). So \( \tau_\varepsilon \to \tau_0 > 0 \) up to a subsequence if (2.7) holds. Setting

\[
g(\tau) \triangleq \frac{1}{p} \tau^p \|u_\varepsilon\|^p - \frac{1}{p^*(t)} \tau^{p^*(t)} \int_{\Omega} \frac{|u_\varepsilon|^{p^*(t)}}{|x|^t}, \quad \tau \in [0, +\infty),
\]

then \( g(\tau) \) attains its maximum at

\[
\tau' \triangleq \|u_\varepsilon\|^p \| \tau_\varepsilon^p \| \left( \int_{\Omega} \frac{|u_\varepsilon|^{p^*(t)}}{|x|^t} \right)^{-1/p} \quad \| \tau_\varepsilon^p \| \int_{\Omega} \frac{|u_\varepsilon|^{p^*(t)}}{|x|^t}
\]

and

\[
g(\tau') \leq \frac{1}{p} \left( 1 - \frac{1}{p^*(t)} \right) \|u_\varepsilon\|^p \| \tau_\varepsilon^p \| \left( \int_{\Omega} \frac{|u_\varepsilon|^{p^*(t)}}{|x|^t} \right)^{1/p^*(t)} \quad \| \tau_\varepsilon^p \| \int_{\Omega} \frac{|u_\varepsilon|^{p^*(t)}}{|x|^t}
\]

for \( \varepsilon > 0 \) small enough. Consequently,

\[
\frac{1}{p} \int_{\Omega} \frac{|\tau_\varepsilon u_\varepsilon|^p}{|x|^t} \leq \frac{p-t}{p(N-t)} (A_{\mu,t})^{\frac{N-t}{p-1}} + C e^{p^*(t) + p - N}.
\]

If \( p > \frac{N-s}{b(\mu)} \), from (2.3) we have

\[
\int_{\Omega} \frac{|\tau_\varepsilon u_\varepsilon|^p}{|x|^t} \geq C e^{N-s+p-N}.
\]

Furthermore, \( N-s+p-N < pb(\mu) + p - N \).

If \( p = \frac{N-s}{b(\mu)} \), then \( N-s+p-N = pb(\mu) + p - N \). From (2.3) we have

\[
\int_{\Omega} \frac{|\tau_\varepsilon u_\varepsilon|^p}{|x|^t} \geq C e^{N-s+p-N} | \ln \varepsilon |.
\]

Hence, if \( \varepsilon > 0 \) small and \( pb(\mu) - N + s \geq 0 \), then we have

\[
J(\tau_\varepsilon u_\varepsilon) < \frac{p-t}{p(N-t)} (A_{\mu,t})^{\frac{N-t}{p-1}},
\]

which contradicts (2.7).

On the other hand, it is easy to verify that the function

\[
f(\tau) = (p-1) \tau^p - (N-p) \tau^{p-1} + \mu, \quad \tau \geq 0
\]

has the only minimal point \( \tau'' = \frac{N-p}{p} \). Moreover, \( f(\tau) \) is decreasing in \( (0, \tau'') \) and is increasing in \( (\tau'', +\infty) \). Hence,

\[
pb(\mu) - N + s \geq 0 \quad \iff \quad b(\mu) \geq \frac{N-s}{p}
\]
\[
\iffalse = f(b(\mu)) \geq f\left(\frac{N-s}{p}\right)
\]

\[
0 \leq \mu \leq \mu_1
\]

for \( N + ps - s - p^2 > 0 \).

The proof of this lemma is completed. \( \square \)

### 3. Proofs of the theorems

In this section, we give the proofs of Theorems 1.1–1.3.

**Proof of Theorem 1.1.** We prove that (2.6) holds for \( \varepsilon \) small. To the contrary, we assume that

\[
\sup_{v \in Q_m^\varepsilon} J(v) \geq \frac{p-t}{p(N-t)} (A_{\mu,t})^{\frac{N-t}{p-t}}, \quad \forall m \in \mathbb{N}, \forall \varepsilon > 0.
\]

As the set \( \{v \in Q_m^\varepsilon; J(v) \geq 0\} \) is compact, the supremum in (3.1) is attained. Then for all \( \varepsilon > 0 \), there exists \( w \in H_m^{+} \) and \( \tau \geq 0 \) such that

\[
J(v) = \sup_{v \in Q_m^\varepsilon} J(v) \geq \frac{p-t}{p(N-t)} (A_{\mu,t})^{\frac{N-t}{p-t}},
\]

where \( v = w + \tau u \). Thus

\[
\frac{1}{p} \left\| v \right\|^p - \frac{\lambda}{p} \int_\Omega \frac{|v|^p}{|x|^s} - \frac{1}{p^*(t)} \int_\Omega \frac{|v|^{p^*(t)}}{|x|^t} \geq \frac{p-t}{p(N-t)} (A_{\mu,t})^{\frac{N-t}{p-t}}.
\]

According to Lemma 2.5 the sequences \( \{\tau\} \subset \mathbb{R}^+ \) and \( \{w\} \subset H_m^{+} \) are bounded. Up to subsequences we may assume that

\[
\tau \to 0 \geq 0, \quad w \to w_0 \in H_m^{+} \quad \text{as} \quad \varepsilon \to 0.
\]

The convergence of \( \{w\} \) can be viewed as in any norm topology since the space \( H_m^{+} \) is finite dimensional. As \( w \in H_m^{+} \), by Lemma 2.4 and the fact that \( \lambda \in (\lambda_1, \lambda^*) \) we have

\[
J(w) = \frac{1}{p} \left\| w \right\|^p - \frac{\lambda}{p} \int_\Omega \frac{|w|^p}{|x|^s} - \frac{1}{p^*(t)} \int_\Omega \frac{|w|^{p^*(t)}}{|x|^t} \\
\leq \frac{\lambda_1 + C^m (N-p-pa(\mu))}{p} \int_\Omega \frac{|w|^p}{|x|^s} - \frac{\lambda}{p} \int_\Omega \frac{|w|^{p^*(t)}}{|x|^t} \leq 0
\]

for \( m \) large enough (from now on we maintain \( m \) fixed).

On the other hand, (3.3) and Lemma 2.7 imply that

\[
J(v) = J(w) + J(\tau u) \leq J(\tau u) < \frac{p-t}{p(N-t)} (A_{\mu,t})^{\frac{N-t}{p-t}},
\]

which contradicts (3.2). Hence (2.6) holds for \( \varepsilon \) small. By Lemmas 2.5 and 2.6, problem (1.1) has a nontrivial solution \( u \in D^{1,p}(\Omega) \). Since \( \lambda > \lambda_1 \), the solution \( u \) must change sign in \( \Omega \). Therefore \(-u\) is also a sign-changing solution of (1.1). \( \square \)

The proof of Theorem 1.2 follows the same lines as that of Theorem 1.1, but some refinements of the estimates are needed. In the following, we denote \( v, u, w \) and \( \tau \) as \( v^m, u^m, w^m \) and \( \tau_m \), respectively to emphasize the dependence on \( m \).

**Lemma 3.1.** Assume that \( \varepsilon = \varepsilon(m) = o(m^{-1}) \). Then as \( m \to \infty \) we have

\[
\left\| u^m \right\|^p = (A_{\mu,t})^{\frac{N-t}{p-t}} + O\left((\varepsilon m)^{b(\mu)p-p-N}\right)
\]

\[
\int_\Omega \frac{|u_m^m|^{p^*(t)}}{|x|^t} = (A_{\mu,t})^{\frac{N-t}{p-t}} + O\left((\varepsilon m)^{b(\mu)p^*(t)-N+t}\right).
\]
\[
\int_{\Omega} \frac{|u^m|^p}{|x|^s} \geq C \varepsilon^{p-s}. \tag{3.6}
\]

**Proof.** The proof of estimates (3.4) and (3.5) follows the same lines as that of (2.1) and (2.2). On the other hand,
\[
\int_{\Omega} \frac{|u^m|^p}{|x|^s} \geq C \int_{0}^{\bar{\varepsilon}} u^m(r)^p r^{N-1-s} dr

\geq C \int_{0}^{\bar{\varepsilon}} \left( \frac{\bar{\varepsilon} - pN}{p} \right)^p r^{N-1-s} dr

\geq C \varepsilon^{p-s}.
\]

Then the lemma is proved. □

Assuming that
\[ pb(\mu) - N + s > 0 \quad \text{and} \quad p^*(t)(N + s - 2p - pa(\mu)) + p(p - s) > 0, \]
then
\[ \tilde{\tau}_1 \triangleq \frac{pb(\mu) + p - N}{pb(\mu) + s - N} > 1 \quad \text{and} \quad \tilde{\tau}_2 \triangleq \frac{p^*(t)(N - p - pa(\mu))}{(p-s)(p^*(t) - p)} > 1. \]

Furthermore, we assume \( \tilde{\tau}_1 < \tilde{\tau}_2 \). Note that this inequality holds naturally if we take \( s \to p \). Then we can choose \( \varepsilon = m^{-v} \) with \( 1 < \tilde{\tau}_1 < v < \tilde{\tau}_2 \) such that \( \varepsilon = o(m^{-1}) \) and therefore Lemma 3.1 holds. From now on we use the notation \( u^m, v^m \) and \( w^m \) to denote \( u^m_{\varepsilon(m)}, v^m_{\varepsilon(m)} \) and \( w^m_{\varepsilon(m)} \) respectively.

On the other hand, from the proof of Lemma 2.7 we obtain
\[ pb(\mu) - N + s > 0 \iff 0 \leq \mu < \mu_1 \quad \text{and} \quad N + ps - s^2 > 0. \]

Furthermore, if
\[
\frac{N - p}{p} - \frac{(p-s)(p^*(t) - p)}{p^*(t)p} > 0,
\]
then
\[ p^*(t)(N - 2p + s) + p(p - s) > 0, \]
then we have
\[
\tilde{\tau}_2 > 1 \iff p^*(t)(N + s - 2p - pa(\mu)) + p(p - s) > 0

\iff a(\mu) < \frac{N - p}{p} - \frac{(p-s)(p^*(t) - p)}{p^*(t)p}

\iff 0 = f(a(\mu)) > f \left( \frac{N - p}{p} - \frac{(p-s)(p^*(t) - p)}{p^*(t)p} \right)

\iff 0 \leq \mu < \mu_2,
\]

where
\[
\mu_2 \triangleq \left( \frac{N - p}{p} - \frac{(p-s)(p^*(t) - p)}{p^*(t)p} \right)^{p-1} \left( \frac{N - p}{p} + \frac{(p^2 + s - ps - p)(p^*(t) - p)}{p^*(t)p} \right).
\]

**Lemma 3.2.** Assume \( b(\mu)p + s - N > 0, v > \tilde{\tau}_1 \) and \( m \) large enough. Then
\[ J(\tau_m u^m) \leq \frac{p - t}{p(N - t)} \left( A_{\mu,t} \right)^{\frac{N-1}{p-t}} - C m^{-v(p-s)}. \]
Proof. From Lemma 2.3 and the assumption $v > \tilde{\tau}_1$ we have 

$$b(\mu)p + p - N < b(\mu)p^*(t) - N + t$$

and 

$$-(v - 1)(b(\mu)p + p - N) < -v(p - s) < 0.$$ 

Consequently, 

$$J(\tau_m u^m) \leq \frac{p - t}{p(N - t)} \left( A_{\mu,t} \right)^{\frac{N - t}{p - t}} + O(m^{-(v-1)(b(\mu)p+p-N)}) - Cm^{-v(p-s)}$$

$$\leq \frac{p - t}{p(N - t)} \left( A_{\mu,t} \right)^{\frac{N - t}{p - t}} - Cm^{-v(p-s)}. \quad \square$$

Lemma 3.3. Assume $\lambda = \lambda_1$ and $m$ large enough. Then 

$$J(w^m) \leq Cm^{\frac{-(N - p - pa(\mu)p^*(\mu))}{p^*(\mu) - p}}.$$ 

Proof. By Lemma 2.4 and the Holder inequality we have 

$$J(w^m) \leq \frac{1}{p} \parallel w^m \parallel^p - \frac{\lambda_1}{p} \int_{\Omega} \frac{|w^m|^p}{|x|^p} - \frac{1}{p^*(t)} \int_{\Omega} \frac{|w^m|^{p^*(t)}}{|x|^t}$$

$$\leq C_1m^{pa(\mu) + p - N} \int_{\Omega} \frac{|w^m|^p}{|x|^p} - C_2 \int_{\Omega} \frac{|w^m|^{p^*(t)}}{|x|^t}$$

$$\leq C_1m^{pa(\mu) + p - N} \int_{\Omega} \frac{|w^m|^p}{|x|^p} - C_3 \left( \int_{\Omega} \frac{|w^m|^p}{|x|^t} \right)^{\frac{p^*(t)}{p}}.$$ 

By elementary calculus we have 

$$\max_{\tau \geq 0} \left( C_1m^{pa(\mu) + p - N} \tau^p - C_3 \tau^{p^*(t)} \right) = C_4 m^{\frac{-(N - p - pa(\mu)p^*(\mu))}{p^*(\mu) - p}}.$$ 

Therefore the lemma is concluded. \quad \square

Proof of Theorem 1.2. By contradiction, we assume that 

$$J(v^m) = \max_{v \in Q_{m}} J(v) \geq \frac{p - t}{p(N - t)} \left( A_{\mu,t} \right)^{\frac{N - t}{p - t}}, \quad \forall m \in \mathbb{N}, \ v > 0,$$ 

and then $\tau_m \geq C > 0$ and $\parallel w^m \parallel \leq C$.

If $1 < \tilde{\tau}_1 < \tilde{\tau}_2$, by choosing $v \in (\tilde{\tau}_1, \tilde{\tau}_2)$ and from Lemmas 3.2 and 3.3 we have 

$$J(v^m) \leq J(\tau_m u^m) + J(w^m)$$

$$\leq \frac{p - t}{p(N - t)} \left( A_{\mu,t} \right)^{\frac{N - t}{p - t}} - Cm^{-v(p-s)} + Cm^{\frac{-(N - p - pa(\mu)p^*(\mu))}{p^*(\mu) - p}}$$

for $m$ large, which contradicts (3.7). Hence (2.6) holds. By Lemmas 2.5 and 2.6, (1.1) has a nontrivial solution $u \in D^{1,p}(\Omega)$, which changes sign in $\Omega$ and therefore $-u$ is also a sign-changing solution of (1.1).

The proof of Theorem 1.2 is completed. \quad \square

Proof of Theorem 1.3. The proof follows the same lines as that in [2]. Let $u = \tau \phi_1$, $\tau > 0$. Then by the Holder inequality we obtain 

$$J(u) = \frac{\lambda_1 - \lambda}{p} \int_{\Omega} \frac{|u|^p}{|x|^s} - \frac{1}{p^*(t)} \int_{\Omega} \frac{|u|^{p^*(t)}}{|x|^t}.$$
\[ \lambda_1 - \lambda \leq \frac{\lambda_1 - \lambda}{p} \left( \int_{\Omega} \frac{|u|^{p^*(t)}}{|x|^l} \right)^{p^*} \left( \int_{\Omega} |x|^{N+1-Nx-p^*t} \right)^{\frac{p^*-1}{p-1}} - \frac{1}{p^*(t)} \int_{\Omega} |u|^{p^*(t)} \]
\[ \leq (\lambda_1 - \lambda) \frac{N-l}{p-1} \frac{p-t}{p(N-t)} \int_{\Omega} |x|^{N+1-Nx-pt} - t N^{-1} p_{p-t} \]

where we have used the fact that
\[ \max_{\tau \geq 0} (c_1 \tau^p - c_2 \tau^{p^*(t)}) = c_1 (p-t) \left( \frac{c_1 (N-p)}{c_2 (N-t)} \right)^{\frac{N-p}{p-t}}, \quad \forall c_1 > 0, c_2 > 0. \]

If \( \lambda \in (\tilde{\lambda}, \lambda_1) \), then
\[ \max_{\tau \geq 0} J(\tau \phi_1) < \frac{p-t}{p(N-t)} (A_{\mu, t})^{\frac{N-l}{p-1}}. \]

Hence we can obtain a PS sequence in the cone of nonnegative functions, which has a weak limit \( u \) with \( u \geq 0 \) and \( u \neq 0 \). By the maximum principle [16], we obtain that \( u > 0 \) in \( \Omega \) and \( u \) is a positive solution of (1.1). \( \square \)

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