OPTIMAL IMPULSIVE HARVESTING OF A SINGLE-SPECIES WITH GOMPERTZ LAW OF GROWTH

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In this paper we investigate the optimal harvesting problems of a single species with Gompertz law of growth. Based on continuous harvesting models, we propose impulsive harvesting models with constant harvest or proportional harvest. By using the discrete dynamical systems determined by the stroboscopic map, we discuss existence, stability and global attractivity of positive periodic solutions, and obtain the maximum sustainable yield and the corresponding optimal population level. At last, we compare the maximum sustainable yield of impulsive harvest with that of continuous harvest, and point out that proportional harvest is superior to constant harvest.

Keywords: Gompertz Law; Maximum Sustainable Yield; Impulsive Harvest; Global Attractivity.

1. Introduction

The sustainable development of renewable resources is one of the important aspects of human activities. The optimal management of renewable resources, which has a direct relationship to sustainable development, has been extensively studied by a variety of authors such as Chaudhuri,1 Purohit and Chaudhuri,2 Song and Chen,3 Dubey et al.,4 Kar,5 Clark,6 and Alvarez and Shepp.7 Among them Clark6 derived significant results regarding optimal harvesting problem of a single species.

It is well known that the famous Gompertz equation8,9 describes the growth law for a single species. The model reads

$$\frac{dx}{dt} = rx \ln \frac{K}{x},$$

(1.1)
where \( x(t) \) is the density of the population, \( r \), as a positive constant, is called the intrinsic growth rate, the positive constant \( K \) is usually referred to as the environment carrying capacity or saturation level, \( r \ln \frac{K}{x} \) denotes relative growth rate.

In this paper, we want to investigate the optimal harvesting problem of Eq. (1.1). We all know that any over-exploitation can drive the population to extinction. Therefore, a natural problem is how we plan management policy so as to achieve sustainable yields, and avoid the disastrous exploitation. It is an important problem of control in renewable resource management.

The management of renewable resources, where it has been practiced at all, has generally been based on the concept of maximum sustainable yield (commonly abbreviated MSY). This is perhaps the simplest possible management objective that accounts for the fact that a biological resource stock cannot be exploited too heavily without an ultimate loss of productivity.

Suppose that the population described by Eq. (1.1) is subject to harvesting at a rate \( h(t) = h = \text{constant} \) or under the catch-per-unit-effort hypothesis \( h(t) = E \). Then the equations for the harvested population read, respectively

\[
\frac{dx}{dt} = rx \ln \frac{K}{x} - h, \tag{1.2}
\]

\[
\frac{dx}{dt} = rx \ln \frac{K}{x} - Ex, \tag{1.3}
\]

where \( E \) denotes the harvesting effort.

However, in the real world, exploitation by human being is not continuous. It is often the case that harvesting a biological resource is seasonal or occurs in regular pulses. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that the harvesting acts instantaneously, that is, in the form of impulses. The continuous exploitation by human beings is then removed from the model, and replaced by an impulsive perturbation. Impulsive differential equations are studied in almost every domain of applied sciences. Some impulsive differential equations have been recently introduced in population dynamics.\(^{10-16}\) Among them Angelova and Dishliev,\(^{10,11}\) and Zhang et al.\(^{15}\) considered optimization problem for impulsive models in population dynamics.

In the following section, we discuss continuous harvesting problems of the single species which are described by Gompertz equation subjected to constant harvest or proportional harvest. In Sec. 3, we discuss the impulsive harvesting problems of the single species with Gompertz law of growth, which correspond to the above two cases. Existence, stability and attractivity of positive periodic solutions are shown. The maximum sustainable yield and the corresponding optimal population level of the impulsive system are also obtained. We give a brief discussion, and compare the maximum sustainable yield of impulsive system with that of the corresponding continuous system in the last section.
2. Continuous Harvest of the Single Species

We assume that the single species obeys the Gompertz growth law in the absence of harvesting. Then, its equation can be written as

\[ \frac{dx}{dt} = F(x), \]  

where \( F(x) = rx \ln \frac{K}{x} \). One special case of harvest is \( h(t) = h = \text{constant} \), then Eq. (2.1) becomes Eq. (1.2). In case \( h < \max F(x) = \frac{rK}{e} \), Eq. (1.2) possesses two equilibria, \( x_1 \) and \( x_2 \) (\( x_1 < x < x_2 \)). Notice that \( \dot{x} > 0 \) when \( x \) lies between \( x_1 \) and \( x_2 \), while \( \dot{x} < 0 \) elsewhere. It follows that \( x_2 \) is a stable equilibrium and that \( x_1 \) is an unstable equilibrium.

If \( h > \max F(x) \), \( \dot{x} < 0 \), the population approaches to 0 for any initial level \( x(0) \).

In the special case when \( h = \max F(x) \), there is a single equilibrium at \( x_1 = \frac{K}{e} \), which is semi-stable in the sense that \( x(t) \to x_1 \) if \( x(0) > x_1 \), while \( x(t) \to 0 \) if \( x(0) < x_1 \).

There exists a maximum sustainable yield for Eq. (1.2). We denote it as \( h_{MSY} \), here

\[ h_{MSY} = \max F(x) = \frac{rK}{e}, \]

with the property that any larger harvest rate will lead to the depletion of the population (eventually to zero).

If under the catch-per-unit-effort hypothesis, that is, catch-per-unit-effort is proportional to the stock level, then \( h(t) = Ex \), and Eq. (2.1) becomes Eq. (1.3). For any \( E < r \), there exists a unique nonzero equilibrium \( x_1 \) which is always asymptotically stable. The maximum sustainable yield of system (1.3) is

\[ \bar{h}_{MSY} = \max F(x) = \frac{rK}{e}. \]

The optimal harvesting effort is \( E = r \), and the corresponding optimal population level is \( x_1 = \frac{K}{e} \).

3. Impulsive Harvest of the Single Species

If we take the discontinuity of human activity into account, then the continuous harvest model of the single species which obeys Gompertz growth law can be replaced by the following impulsive equations.

\[
\begin{align*}
\frac{dx}{dt} &= rx \ln \frac{K}{x}, \quad t \neq k\tau, \\
\triangle x &= -h(t), \quad t = k\tau, \\
x(0^+) &= x_0,
\end{align*}
\]

where \( \triangle x(t) = x(t^+) - x(t) \), \( h(t) \) denotes the population size harvested each time, \( \tau \) is the period of harvest. The meaning of \( r \) and \( K \) is the same as that of
system (1.1). Assume \( k \in \mathbb{Z}_+ \), \( \mathbb{Z}_+ = \{1, 2, \ldots\} \). Let \( R_+ = [0, +\infty) \). Any solution of the system (3.1) \( x(t) : R_+ \rightarrow (0, +\infty) \) is continuously differential in \( R_+ - \{k\tau\} \), \( x(k\tau^+) = \lim_{t \rightarrow k\tau^+} x(t) \) exists. The global existence and uniqueness of solutions of system (3.1) are guaranteed by the smoothness properties of \( f \), which is the map of the right-hand side of the first equation of system (3.1). For details see Lakshmikantham et al.\(^{17}\)

3.1. Maximum sustainable yield of constant impulsive harvest

Now, we assume that the population size harvested each time is constant. Then the impulsively harvesting model reads

\[
\begin{align*}
\frac{dx}{dt} &= rx \ln \frac{K}{x}, \quad t \neq k\tau, \quad x > 0, \\
x(t^+) &= x(t) - h, \quad t = k\tau, \quad x > h, \\
x(t^+) &= 0, \quad t = k\tau, \quad x \leq h, \\
x(0^+) &= x_0,
\end{align*}
\]

(3.2)

3.1.1. Existence of positive periodic solutions

The function on the right-hand side of the first equation of system (3.2) can be extended continuously to zero when \( x = 0 \). Then the system has the extinction equilibrium, which is made in Angelova and Dishliev.\(^{11}\) In this section, we discuss the existence and stability of the positive periodic solutions.

Integrate the first equation of system (3.2) between \( [0, \tau) \), it is easy to obtain

\[
x(\tau) = K^{(1-e^{-r\tau})}(x_0)^{1-e^{-r\tau}}.
\]

(3.3)

After harvest,

\[
x(\tau^+) = K^{(1-e^{-r\tau})}(x_0)^{1-e^{-r\tau}} - h.
\]

(3.4)

If system (3.2) has periodic solutions, then \( x(\tau^+) = x_0 \), that is

\[
K^{(1-e^{-r\tau})}(x_0)^{1-e^{-r\tau}} - h - x_0 = 0.
\]

(3.5)

Let

\[
f(x_0) = K^{(1-e^{-r\tau})}(x_0)^{1-e^{-r\tau}} - h - x_0.
\]

(3.6)

By differentiating \( f(x_0) \) with respect to \( x_0 \), we have

\[
f'(x_0) = e^{-r\tau}K^{(1-e^{-r\tau})}x_0^{e^{-r\tau} - 1} - 1.
\]

Set \( f'(x_0) = 0 \), then

\[
x^*_0 = Ke^{\left(-\frac{r\tau}{1-e^{-r\tau}}\right)}.
\]

We get \( f'(x_0) > 0 \) when \( x_0 < x^*_0 \), and \( f'(x_0) < 0 \) when \( x_0 > x^*_0 \). This implies \( f(x_0) \) reaches its maximum at \( x_0 = x^*_0 \), namely, \( f(x^*_0) = \max_{x_0 \in (0,K)} f(x_0) \), here

\[
f(x^*_0) = K\left((e^{-\frac{r\tau}{1-e^{-r\tau}}})^{1-e^{-r\tau}} - e^{-\frac{r\tau}{1-e^{-r\tau}}}\right) - h.
\]
Denote
\[ h_{\text{max}} = K \left( e^{-\frac{\tau}{1-e^{-\tau}}} e^{-rt} - e^{-\frac{\tau}{1-e^{-\tau}}} \right). \]
If \( h < h_{\text{max}} \), \( f(x_0^*) = \max_{x_0 \in (0,K)} f(x_0) > 0 \). Since \( f(0) = -h < 0 \) and \( f(K) = -h < 0 \), by the monotonicity of \( f(x_0) \), there exist only two zero solutions for Eq. (3.5) denoted by \( x_{01} \) and \( x_{02} \), where \( x_{01} \in (0,x_0^*), x_{02} \in (x_0^*,K) \). Thus there exist only two positive \( \tau \)-periodic solutions \( x(x_{01},0,t) \) and \( x(x_{02},0,t) \) for system (3.2) with, respectively, \( x_{01} \) and \( x_{02} \) as the initial values.

If \( h > h_{\text{max}} \), \( f(x_0^*) = \max_{x_0 \in (0,K)} f(x_0) < 0 \), and then \( f(x_0) < 0 \) for all \( x_0 \in (0, +\infty) \), there exists no zero solution for Eq. (3.5). Therefore, there exists no periodic solution for system (3.2).

If \( h = h_{\text{max}} \), \( f(x_0^*) = \max_{x_0 \in (0,K)} f(x_0) = 0 \), there exists a unique zero solution for Eq. (3.5), and then there exists a unique positive \( \tau \)-periodic solution \( x(x_0^*,0,t) \) for system (3.2) with \( x_0^* \) the initial value, here
\[ x(x_0^*,0,t) = Ke^{-\frac{\tau}{1-e^{-\tau}}} e^{-rt}. \]

The existence of periodic solutions of system (3.2) is summarized in the following theorem.

**Theorem 3.1.** Assume \( x(x_0,0,t) \) is the solution of system (3.2) with initial value \( x_0 > 0 \).

(i) If \( h < h_{\text{max}} \), then there exist only two positive \( \tau \)-periodic solutions \( x(x_{01},0,t) \) and \( x(x_{02},0,t) \), where \( x_{01} \in (0,x_0^*), x_{02} \in (x_0^*,K) \).

(ii) If \( h = h_{\text{max}} \), then there exists a unique positive \( \tau \)-periodic solution \( x(x_0^*,0,t) \).

(iii) If \( h > h_{\text{max}} \), then there exists no periodic solution.

3.1.2. Stability and global attractivity of positive periodic solutions

Integrate and solve the first equation of system (3.2) between pulses, we have

\[ x(t) = K^{(1-e^{-\tau (t-n\tau)})} (x_0^*) e^{-r(t-n\tau)}, \quad n\tau \leq t < (n+1)\tau, \]

with \( x_0^* = x(n\tau^+) \) the initial value at time \( n\tau \). Setting \( x_n = x(n\tau) \), then we deduce the stroboscopic map of system (3.2)

\[ x_{n+1} = K^{(1-e^{-\tau})} (x_n - h)e^{-rt} \triangleq G(x_n). \]

Equation (3.8) is a difference equation. It describes the population size at a pulse in terms of the value at the previous pulse. We are, in other words, stroboscopically sampling at its pulsing period. The dynamical behavior of system (3.8), coupled with system (3.7), determines the dynamical behaviors of system (3.2).

For the stability and the global attractivity of the periodic solutions for system (3.2), we obtain the following two theorems (their proofs are given in Appendix).

**Theorem 3.2.** If \( h < h_{\text{max}} \), then the periodic solution \( x(x_{02},0,t) \) of system (3.2) is locally stable, and the periodic solution \( x(x_{01},0,t) \) is unstable.
Theorem 3.3. Assume \( x(0,0,t) \) is the solution of system (3.2) with initial value \( x_0 > 0 \).

(i) If \( h < h_{\text{max}} \), then \((x_{01}, +\infty)\) is the attractive region of the positive periodic solution \( x(x_{02}, 0, t) \). When \( x_0 < x_{01} \), there exists an \( n_0 \in \mathbb{Z}_+ \) such that the solution of system (3.2) satisfies \( x(x_0, 0, t) = 0 \) for \( t > n_0 \tau \).

(ii) If \( h = h_{\text{max}} \), then \((x_{0}^*, +\infty)\) is the attractive region of the positive periodic solution \( x(x_{0}^*, 0, t) \). When \( x_0 < x_{0}^* \), there exists an \( n_1 \in \mathbb{Z}_+ \) such that the solution of system (3.2) satisfies \( x(x_0, 0, t) = 0 \) for \( t > n_1 \tau \).

(iii) If \( h > h_{\text{max}} \), there exists an \( n_2 \in \mathbb{Z}_+ \) such that for any \( x_0 > 0 \) the solution of system (3.2) satisfies \( x(x_0, 0, t) = 0 \) for \( t > n_2 \tau \).

When \( h > h_{\text{max}} \), no matter what the initial population level is, the population tends to go extinct in a finite time. The yield of population should be not more than \( h_{\text{max}} \). \( h_{\text{max}} \) is the maximum sustainable yield of system (3.2). Then we obtain the maximum sustainable yield per unit time

\[
Y_{\text{max}} = \frac{K}{\tau} \left( \left( e^{-\frac{r \tau}{1-e^{-r \tau}}} - e^{-\frac{r \tau}{1-e^{-r \tau}}} \right) e^{-r \tau} - e^{-\frac{r \tau}{1-e^{-r \tau}}} \right).
\]

When \( h = h_{\text{max}} \), the unique positive periodic solution is semi-stable. This kind of harvesting policy with constant yield is difficult to control. In the following section, we investigate the harvesting policy with proportional harvest.

3.2. Optimal impulsive harvesting policy for proportional harvest

In this section, under the catch-per-unit-effort hypothesis we assume that the population size harvested each time is proportional to stock level. Then system (3.1) becomes

\[
\begin{align*}
\frac{dx}{dt} &= rx \ln \frac{K}{x}, \quad t \neq k\tau, \\
\triangle x &= -E x, \quad t = k\tau, \\
x(0^+) &= x_0.
\end{align*}
\]

(3.9)

According to the biological meaning, assume \( 0 < E < 1 \).

3.2.1. Existence and global stability of the positive periodic solution

Between pulses, we have already obtained

\[
x(t) = K^{1-e^{-r(t-n\tau)}}(x_n^+) e^{-r(t-n\tau)}, \quad n\tau \leq t < (n+1)\tau,
\]

(3.10)

with \( x_n^+ = x(n\tau^+) \) the initial value at time \( n\tau \). Setting \( x_n = x(n\tau) \). After pulses at time \((n+1)\tau\), more population is reduced, then we deduce the stroboscopic map of system (3.9),

\[
x_{n+1}^+ = (1 - E)K^{1-e^{-r\tau}}(x_n^+) e^{-r\tau} \triangle g(x_n^+).
\]

(3.11)
If Eq. (3.11) has an equilibrium, we denote it as $\tilde{x}$, which satisfies

$$\tilde{x} = (1 - E)K(1 - e^{-\tau r})\tilde{x}.$$  \hspace{1cm} (3.12)

By calculation, we get

$$\tilde{x} = K(1 - E) \frac{1}{1 - e^{-\tau r}}.$$  \hspace{1cm} (3.13)

The equilibrium of Eq. (3.11) corresponds to the periodic solution of system (3.9).

**Theorem 3.4.** There exists a unique positive $\tau$-periodic solution $x(\tilde{x}, 0, t)$ for system (3.9), where

$$x(\tilde{x}, 0, t) = K(1 - E) \frac{e^{-\tau r(n + \tau t)}}{1 - e^{-\tau r}}.$$  \hspace{1cm} (3.14)

With the similar arguments in Sec. 3.1, we have

**Theorem 3.5.** The unique positive periodic solution $x(\tilde{x}, 0, t)$ of system (3.9) is globally asymptotically stable.

### 3.2.2. Optimal harvesting policy

In this section, we show how to plan harvesting policy in order to sustain the resource populations at high levels of productivity. For this purpose, we choose the maximum sustainable yield per unit time as management objective and $E$ as the control variable. Harvest occurs at moment $n\tau$, then the yield per unit time is $Y(E) = \frac{1}{\tau}Ex(n\tau) = \frac{1}{\tau}Ex_n$.

By $x_n^n = x(n\tau^+) = (1 - E)x_n$, we obtain that the fixed point of $\{x_n\}$ is $\tilde{x}$, since the fixed point of $\{x_n^n\}$ is $\tilde{x}$. Then the objective function takes the form

$$Y(E) = \frac{E\tilde{x}}{\tau(1 - E)} = \frac{KE(1 - E)}{\tau(1 - e^{-\tau r})}.$$  \hspace{1cm} (3.15)

We wish to find $0 < E^* < 1$ such that $Y(E)$ reaches its maximum at $E = E^*$, namely, to find the optimal harvesting effort, which is an optimization of a function. It is only necessary to solve

$$Y(E^*) = \max_{E \in (0,1)} Y(E).$$

From Eq. (3.15), we obtain the derivative of $Y(E)$ with respect to $E$,

$$Y'(E) = \frac{K}{\tau} (1 - E)^{\frac{\tilde{x}}{\tau(1 - e^{-\tau r})}} \left( 1 - \frac{e^{-\tau r}}{1 - e^{-\tau r}} \frac{E}{1 - E} \right).$$

Let $Y'(E) = 0$, then

$$1 - \frac{e^{-\tau r}}{1 - e^{-\tau r}} \frac{E}{1 - E} = 0,$$
which yields

\[ E^* = 1 - e^{-r\tau}. \] (3.16)

It is clear that if \( E \) is close to zero \( Y'(E) > 0 \) while it changes sign to negative when \( E \) crosses through \( E^* \). Thus \( Y(E) \) reaches its maximum at \( E = E^* \). Substituting Eq. (3.16) into Eq. (3.15), we get the maximum sustainable yield per unit time,

\[ \bar{Y}_{\text{max}} = Y(E^*) = \frac{K}{\tau} (1 - e^{-r\tau}) (e^{-\frac{r\tau}{1-e^{-r\tau}}} e^{-r\tau}). \]

Substituting Eq. (3.16) into Eq. (3.13), we have

\[ \dot{x} = K e^{-\frac{r\tau}{1-e^{-r\tau}}}. \]

Substituting Eq. (3.16) into Eq. (3.14), we obtain the corresponding optimal population level,

\[ x(\tilde{x}, 0, t) = K(e^{-r\tau}) \frac{e^{-r(t-n\tau)}}{1-e^{-r\tau}}. \] (3.17)

4. Discussion

In this paper, we first discuss continuous harvesting problems of a single species with Gompertz law of growth and obtain the maximum sustainable yield. However, in the real world, exploitation by human beings is not continuous. For example, a fisherman cannot fish the whole day, and only fish for some time everyday, or fish net by net. In other words, harvesting a resource population occurs seasonally or in regular pulses. It is more realistic to consider impulsive harvesting problems. Based on continuous harvesting models, we propose impulsive harvesting models with constant harvest or proportional harvest.

If we take the constant harvesting policy, we obtain the maximum sustainable yield \( h_{\text{MSY}} \). When \( h < h_{\text{MSY}} \), there exist only two positive periodic solutions. When \( h = h_{\text{MSY}} \), there exists a unique positive periodic solution which is semi-stable. This policy cannot keep stationary for perturbations. If we take the proportional harvesting policy, we obtain the optimal harvesting effort, the maximum sustainable yield per unit time \( \bar{Y}_{\text{max}} \) and the corresponding optimal population level, which are equal to those of constant harvest. When \( h = \bar{Y}_{\text{max}} \), there exists a unique globally asymptotically stable positive periodic solution.

It is interesting to compare the maximum sustainable yield of impulsive harvest with that of continuous harvest. It has shown that the maximum sustainable yield for continuous harvest is

\[ h_{\text{MSY}} = \frac{rK}{e}. \]

The maximum sustainable yield per unit time for impulsive harvest is

\[ \bar{Y}_{\text{max}} = \frac{K}{\tau} (1 - e^{-r\tau})(e^{-\frac{r\tau}{1-e^{-r\tau}}} e^{-r\tau}). \]
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It must be noted that the impulsive harvest can be reduced to the continuous harvest if the period of impulse $\tau$ tends to zero. We observe that

$$\lim_{\tau \to 0} \frac{1 - e^{-r\tau}}{\tau} = r,$$

$$\lim_{\tau \to 0} \frac{-r\tau}{1 - e^{-r\tau}} = 1.$$

Thus, we have

$$\lim_{\tau \to 0} \bar{Y}_{\text{max}} = \frac{rK}{e},$$

which is equal to the maximum sustainable yield of continuous harvest. By studying the optimal impulsive harvesting problems, we conclude that proportional harvest is superior to constant harvest. The proportional harvest has a built-in stability mechanism, and is of higher priority to help to reduce over-exploitation.

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References


Appendix

**Proof of Theorem 3.2.** If \( h < h_{\text{max}} \), then there exist only two periodic solutions \( x(x_0, 0, t) \) and \( x(x_2, 0, t) \) for system (3.2). Since the periodic solution \( x(x_2, 0, t) \) of system (3.2) corresponds to the equilibrium \( x^* = x(x_2, 0, n\tau) = x_2 + h \) of Eq. (3.8), so we first discuss the stability of \( x^* \).

From Eq. (3.8), we obtain the derivative of \( G \) with respect to \( x_n \),

\[
G'(x_n) = K(1-e^{-rt})e^{-rt}(x_n - h)(e^{-rt} - 1).
\]

Then

\[
G'(x^*) = K(1-e^{-rt})e^{-rt}(x^* - h)(e^{-rt} - 1) = K(1-e^{-rt})e^{-rt}(x_2)(e^{-rt} - 1).
\]

By

\[
x_0^* = Ke^{-\frac{rt}{1-e^{-rt}}} < x_02,
\]

we know that

\[
K(1-e^{-rt})e^{-rt} < (x_2)(1-e^{-rt}).
\]

Thus \( G'(x^*) < 1 \). It follows that \( x^* \) is locally stable and the corresponding periodic solution \( x(x_2, 0, t) \) is locally stable.

Similarly, we can prove the periodic solution \( x(x_0, 0, t) \) is unstable. This completes the proof.

**Proof of Theorem 3.3.** (i) If \( h < h_{\text{max}} \), according to Eq. (3.4), we can easily obtain

\[
x_1^+ = K(1-e^{-rt})(x_0)e^{-rt} - h.
\]

It follows \( x_2 < x_0 \) if \( x_0 < x_01 \) or \( x_0 > x_02 \).

Similarly, we can prove \( x_n+1 < x_n, \{x_n\} \) is a monotonically decreasing sequence. Therefore, \( \{x_n^+\} \) is also a monotonically decreasing sequence since \( x_n^+ = x_n - h \).
According to the comparison theorem of impulsive differential equations, we have $x_n = x(x_0, 0, n\tau) > x(x_{02}, 0, n\tau) = x^*$ when $x_0 > x_{02}$, i.e. $x^*$ is the lower bound of $\{x_n\}$, and then we know that $x_{02} = x^* - h$ is the lower bound of $\{x^+_n\}$. Therefore, there exists limit for $\{x^+_n\}$ as $n \to \infty$ for $x_0 > x_{02}$. Recall now the stroboscopic map (3.8), we get

$$x^+_{n+1} = K(1-e^{-r\tau})(x^+_n)^{e^{-r\tau}} - h. \quad (A.1)$$

Set $\lim_{n \to \infty} x^+_n = \bar{x}$, then it follows

$$K(1-e^{-r\tau})(\bar{x})^{e^{-r\tau}} - h - \bar{x} = 0. \quad (A.2)$$

With the similar arguments about Eq. (3.5), there exist only two zero solutions for Eq. (A.2) when $h < h_{\text{max}}$, namely $\bar{x} = x_{01}$ or $\bar{x} = x_{02}$. We obtain $\bar{x} = x_{02}$ since $x_{02}$ is the lower bound of $\{x^+_n\}$ when $x_0 > x_{02}$. According to the continuous dependence of solutions with respect to initial values, we further obtain that the entire solution $x(x_0, 0, t)$ of system (3.2) with initial value $x_0 > x_{02}$ converges to the positive periodic solution $x(x_{02}, 0, t)$ (see Fig. 1a).

With the similar arguments, we can prove when $x_{01} < x_0 < x_{02}$, $x^+_1 > x_0$ and then $\{x_n\}$ and $\{x^+_n\}$ are monotonically increasing sequences, and $\{x^+_n\} \to x_{02}$ as $n \to \infty$, at the same time the solution of system (3.2) with initial value $x_0 < x_{02}$ also converges to the positive periodic solution $x(x_{02}, 0, t)$ (see Fig. 1b). Thus, we conclude $(x_{01}, +\infty)$ is the attractive region of positive periodic solution $x(x_{02}, 0, t)$. Therefore, positive periodic solution $x(x_{02}, 0, t)$ is asymptotically stable.

In the following, we prove that when $x_0 < x_{01}$, there exists an $n_0 \in Z_+$ such that $x(t) = 0$ for all $t > n_0 \tau$. We have already obtained that $\{x_n\}$ is a decreasing sequence for $x_0 < x_{01}$, thus we know that there must exist an $n_0 \in Z_+$ such that $x_{n_0} = x(n_0 \tau) \leq h$. According to Eq. (3.2), we have $x(n_0 \tau^+) = 0$, and then $x(t) = 0$.

Fig. 1. Time-series of system (3.2) with $r = 2$, $K = 6$, $\tau = 0.7$, $h = 2$. (a) Initial value $x_0 = 7.6$ and (b) initial value $x_0 = 1$. 

The following diagrams show the time-series of system (3.2) with $r = 2$ and different initial values.
for all \( t > n_0 \tau \). This implies that the population tends to go extinct in a finite time if its initial value is less than \( x_{01} \).

Similarly we can prove (ii). If \( h > h_{\text{max}} \), we have already obtained, \( f(x_0) < 0 \), which implies \( x_1^+ < x_0 \). Thus with the similar arguments we can prove (iii), so we omit them. This completes the proof.