Simulation of a multi-wing chaotic system with fractional-order and its circuit realization

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Abstract—In this paper, a fractional-order multi-wing chaotic attractors evolved from Lü system is proposed. Based on the theory of fractional calculus, fractional-order multi-wing chaotic attractors are simulated, and its dynamical characteristics are also given. Numerical simulation reveals that multi-wing chaotic attractors can be generated in Lü system with the system order as low as 2.7. An improved module-based circuit is also designed for realizing fractional-order 10-wing chaotic attractor, and the experimental results are also given, which is in good agreement with the numerical simulation.

Keywords-fractional-order multi-wing chaotic attractor; Lü system; dynamical characteristics; circuit realization

I. INTRODUCTION

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with correspondence between Leibniz and L’Hospital in 1695, where the meaning of derivative of order 1.5 was discussed. Although the fractional operators have a long history, the applications of fractional-order differential systems to physics and engineering are just a recent focus of interest [1-4]. It has been shown that, similarly to the integer-order differential systems, chaos can be found in fractional-order differential systems, with potential applications to chaos control and secure communication [5-7].

To this purpose, in this paper a fractional-order multi-wing chaotic attractors evolved from Lü system are investigated.

II. FRACTIONAL DERIVATIVE AND APPROXIMATION

The well-known Riemann-Liouville definition of fractional derivatives is given by

\[
\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau
\]

(1)

Where \( \Gamma(\cdot) \) is the gamma function and \( n-1 < \alpha < n \), \( n \) is an integer number. Upon considering all the initial conditions to be zero, the Laplace transform of the Riemann-Liouville fractional derivative is

\[
L \left\{ \frac{d^\alpha f(t)}{dt^\alpha} \right\} = s^\alpha L \{ f(t) \}
\]

(2)

Therefore, the fractional integral operator of order \( \alpha \) can be represented by the transfer function \( F(s) = \frac{1}{s^\alpha} \) in the frequency domain.

III. DESIGN OF FRACTIONAL-ORDER MULTI-WING CHAOTIC ATTRACTIONS

In this section, a nonlinear state feedback controller is designed for creating multi-wing chaotic attractors from a fractional-order linear differential system \( D^\alpha X = AX \).

The controlled fractional-order differential system with nonlinear feedback is described as

\[
D^\alpha X = AX + BU(X)
\]

(3)

Where

\[
A = \begin{pmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
    b_{11} & b_{12} & b_{13} \\
    b_{21} & b_{22} & b_{23} \\
    b_{31} & b_{32} & b_{33}
\end{pmatrix},
\]

\[
X = (x, y, z)^T,
\]

\[
D^\alpha X = (D_x^\alpha x, D_y^\alpha y, D_z^\alpha z)^T, \quad \text{and} \quad \alpha_1, \alpha_2, \alpha_3 \in (0,1).
\]

\( U(X) \) is a nonlinear state feedback controller equipped with a even symmetric multi-segment quadratic function, given by:

\[
U(X) = \begin{pmatrix}
    f_1(X) \\
    f_2(X) \\
    f_3(X)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    \frac{1}{2}y^2 + \sum_{i=1}^{N} H_i [\text{sgn}(y + C_i) - \text{sgn}(y - C_i) - 2]
\end{pmatrix}
\]

(4)
Where \( N = 0,1,2,3, \cdots, i = 1,2, \cdots, N \), and
\[
sign(u) = \begin{cases} 
1 & \text{if } u > 0 \\
0 & \text{if } u = 0 \\
-1 & \text{if } u < 0 
\end{cases}
\]

To verify the effectiveness of above controlled fractional-order differential system by numerical simulation, the Grünwald–Letnikov approximation is used. Then, system (1) can be approximated as the following discrete iterative equation:
\[
h^{-a}\sum_{j=0}^{m} \omega_j^a X(m-j) = AX(m) + BU(X(m))
\]
(5)

Where \( h \) is the integration step size, and \( \omega_j^a = \alpha(\alpha - 1) \cdots (\alpha - j + 1)/j! \).

According to system (5), one gets:
\[
\begin{align*}
&\begin{align*}
&x(m) = \begin{cases}
&h^n b_1 (H_1 x^n(m) + \sum_{j=1}^{N} H_j [\text{sgn}(x(m) + C_j) - \text{sgn}(y(m) - C_j) - 2]) \\
&\sum_{j=1}^{N} \omega_j^a x(m-j)
\end{cases}(1-h^a b_1^n) \\
&y(m) = \begin{cases}
&h^n b_2 (H_2 x^n(m) + \sum_{j=1}^{N} H_j [\text{sgn}(x(m) + C_j) - \text{sgn}(y(m) - C_j) - 2]) \\
&\sum_{j=1}^{N} \omega_j^a y(m-j)
\end{cases}(1-h^a b_2^n) \\
&z(m) = \begin{cases}
&h^n b_3 (H_3 x^n(m) + \sum_{j=1}^{N} H_j [\text{sgn}(x(m) + C_j) - \text{sgn}(y(m) - C_j) - 2]) \\
&\sum_{j=1}^{N} \omega_j^a z(m-j)
\end{cases}(1-h^a b_3^n)
\end{align*}
\end{align*}
\]
(6)

In our simulation, let \( \alpha = (\alpha_1, \alpha_2, \alpha_3) = (0.9, 0.9, 0.9) \), integration step size \( h = 5.0 \times 10^{-3} \). And matrices \( A, B \) are given by:
\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} = \begin{bmatrix}
-a & a & 0 \\
0 & c & 0 \\
0 & -b & 0
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & -1/E & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Where \( a = 25, b = 1.5, c = 20, E = \frac{1}{20} \).

According to system (6), the simulation of 2.7-order 10-wing chaotic attractors can be obtained, as shown in Fig. 1.

Fig. 1 shows a 10-wing chaotic attractor with parameters \( N = 4, P = 27, E = \frac{1}{20}, k = 5 \), \( k_1 = 0.33, k_2 = 0.28, k_3 = 0.23, k_4 = 0.18, H_0 = 100 \), \( H_1 = 4.0909, H_2 = 4.8214, H_3 = 5.8696, H_4 = 7.5 \), \( C_1 = 0.27, C_2 = 0.405, C_3 = 0.54, C_4 = 0.675 \). And the maximum Lyapunov exponent \( L_{E_{\text{max}}} \) is 0.2873.

IV. DYNAMICAL CHARACTERISTICS OF FRACTIONAL-ORDER MULTI-WING CHAOTIC SYSTEM

In this section, dynamical characteristics of fractional-order multi-wing chaotic system are analyzed, including equilibrium points and eigenvalues.

Referring to system (3) with (4), according to the definition of equilibrium points of fractional differential system, one gets:
\[
x^{(x\alpha)} = y^{(y\alpha)}
\]
\[
z^{(z\alpha)} = cE
\]

By solving (7), the second type of equilibrium points can be determined:
\[
x^{\pm 0} = \pm \sqrt{bcE}/H_0, x^{\pm n} = \pm \sqrt{bcE + \frac{2\sum_{n=1}^{N} H_j}{H_0}}(n=1,2,\cdots,N)
\]
\[
z^{\pm n} = cE (n=0,1,2,\cdots,N)
\]
(7)

Where \( n = 1,2,3,\cdots,N \). Furthermore, it can be easily deduced the first type of equilibrium points:
\[
x^{\pm n} = \pm 0.5(n+1)PE/k
\]
\[
y^{\pm n} = \pm 0.5(n+1)PE/k
\]
\[
z^{\pm n} = cE
\]
(9)

For example, let \( N = 4, P = 27, E = \frac{1}{20} \), \( k = 5 \), \( k_1 = 0.33, k_2 = 0.28, k_3 = 0.23, k_4 = 0.18, H_0 = 100 \), \( H_1 = 4.0909, H_2 = 4.8214, H_3 = 5.8696, H_4 = 7.5 \), \( C_1 = 0.27, C_2 = 0.405, C_3 = 0.54, C_4 = 0.675 \), from (11), the second type of equilibrium points are given as follows:
\[
Q^{(00)}(x^{(00)}, y^{(00)}, z^{(00)}) = Q^{(00)}(\pm 0.1225, \pm 0.1225, 1.0)
\]
\[
Q^{(11)}(x^{(11)}, y^{(11)}, z^{(11)}) = Q^{(11)}(\pm 0.3112, \pm 0.3112, 1.0)
\]
\[
Q^{(22)}(x^{(22)}, y^{(22)}, z^{(22)}) = Q^{(22)}(\pm 0.4396, \pm 0.4396, 1.0)
\]
\[
Q^{(33)}(x^{(33)}, y^{(33)}, z^{(33)}) = Q^{(33)}(\pm 0.5573, \pm 0.5573, 1.0)
\]
\[
Q^{(44)}(x^{(44)}, y^{(44)}, z^{(44)}) = Q^{(44)}(\pm 0.6787, \pm 0.6787, 1.0)
\]
(10)
Linearizing system (3) with (4), the Jacobian matrix is obtained as:

\[
J_{\mathcal{G}(s)} = \begin{bmatrix}
-a & a & 0 \\
-(1/E)z & c & -(1/E)x \\
0 & 2H_{0}y & -b
\end{bmatrix}
\]

By using (11), the corresponding eigenvalues of the second type of equilibrium points are given as follows:

\[
\begin{align*}
\gamma^{(0)} &= -11.6952, \quad \sigma^{(0)} \approx j\sigma^{(20)} = 2.5976 \pm j11.0232 \\
\gamma^{(1)} &= -16.9385, \quad \sigma^{(1)} \approx j\sigma^{(31)} = 5.2193 \pm j23.3312 \\
\gamma^{(2)} &= -18.9894, \quad \sigma^{(2)} \approx j\sigma^{(32)} = 6.2447 \pm j31.2835 \\
\gamma^{(3)} &= -20.3004, \quad \sigma^{(3)} \approx j\sigma^{(23)} = 6.9002 \pm j38.5045 \\
\gamma^{(4)} &= -21.2806, \quad \sigma^{(4)} \approx j\sigma^{(24)} = 7.3903 \pm j45.9345
\end{align*}
\]

(12)

In the fractional-order differential system, the characteristic of equilibrium points is decided by the corresponding fractional order \(\alpha\) and eigenvalues. Obviously, (12) satisfy the conditions of \(\gamma^{(1)} < 0, \sigma^{(1)} > 0, \sigma^{(i)} \neq 0\). Letting \(\alpha = (\alpha_{1}, \alpha_{2}, \alpha_{3}) = (\alpha_{0}, \alpha_{0}, \alpha_{0}) = (0.9, 0.9, 0.9)\) and \(\alpha_{0} \pi / 2 = 1.413\), one gets:

\[
\begin{align*}
\arctan(\omega^{(0)} / \sigma^{(0)}) &= 1.3934 < \alpha_{0} \pi / 2 \\
\arctan(\omega^{(1)} / \sigma^{(1)}) &= 1.3507 < \alpha_{0} \pi / 2 \\
\arctan(\omega^{(2)} / \sigma^{(2)}) &= 1.3738 < \alpha_{0} \pi / 2 \\
\arctan(\omega^{(3)} / \sigma^{(3)}) &= 1.3935 < \alpha_{0} \pi / 2 \\
\arctan(\omega^{(4)} / \sigma^{(4)}) &= 1.4113 < \alpha_{0} \pi / 2
\end{align*}
\]

According to equation (13), the second type of equilibrium points are saddle-locus points with index 2, each equilibrium point corresponding to a unique wing. So that fractional-order multi-wing chaotic attractors can be obtained with system (3) and equation (4).

V. CIRCUIT DESIGN AND EXPERIMENTAL RESULTS

In this section, the fractional-order 10-wing chaotic attractors are demonstrated by circuit experiment.

According to system (3) with (4), one can design the circuit diagram for realizing 2.7-order 10-wing chaotic attractors as shown in Fig. 2, with the resistance values \(R_{1} = 4k\Omega\), \(R_{2} = 4k\Omega\), \(R_{3} = 0.5k\Omega\), \(R_{4} = 5k\Omega\), \(R_{5} = 100k\Omega\), \(R_{6} = 66.7k\Omega\).

Letting \(\alpha = 0.9\), the approximation of transfer function \(1/s^\alpha\) with an error of approximately 2dB is given by [8]:

\[
\frac{1}{s^\alpha} \approx \frac{2.2675(s+1.292)(s+215.4)}{(s+0.01292)(s+2.154)(s+359.4)}
\]

(14)

A circuit design for realizing equation (14) is shown in Fig. 3, from which one can get the transfer function in the Laplace domain between A and B as follows:

\[
H(s) = \frac{1/C_{0}}{s+1/R_{c}C_{0}} + \frac{1/C_{0}}{s+1/R_{c}C_{0}} + \frac{1/C_{0}}{s+1/R_{c}C_{0}} = \frac{C_{0}}{s(s+1/R_{c}C_{0})(s+1/R_{c}C_{0})(s+1/R_{c}C_{0})}
\]

\[
= \frac{C_{0}}{s+1/R_{c}C_{0}}(s+1/R_{c}C_{0})(s+1/R_{c}C_{0}) = \frac{C_{0}}{s+1/R_{c}C_{0}}(s+1/R_{c}C_{0})(s+1/R_{c}C_{0})
\]

(15)

Where \(C_{0}\) is constant. Letting \(C_{0} = 1\mu F\), one gets \(H(s) = F(s)/C_{0}\). Comparing equation (14) with equation (15), one can calculate the values of the resistances and capacitances in Fig. 3, given by \(R_{c} = 6284\Omega\), \(R_{e} = 25\Omega\), \(R_{e} = 250\Omega\), \(C_{0} = 12.32nF\), \(C_{c} = 18.35nF\), and \(C_{e} = 11nF\).

Based on Figs. 2, the circuit can generate 10-wing chaotic attractors as shown in Figs.4.

VI. CONCLUSIONS

In this paper, we investigate fractional-order multi-wing chaotic attractors. Based on a fractional-order linear differential system, by introducing the nonlinear state feedback controller, fractional-order multi-wing chaotic attractors can be generated. Numerical simulation and circuit experiment have demonstrated the effectiveness of the proposed approach.

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