A mixed convection flow and heat transfer of pseudo-plastic power law nanofluids past a stretching vertical plate

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ABSTRACT
A mixed convection flow and heat transfer of pseudo-plastic power law nanofluid past a stretching vertical plate is investigated. Three types of nano-particles, such as copper (Cu), aluminum oxide (Al2O3) and titanium oxide (TiO2), are considered. The generalized Fourier law proposed by Zheng for varying thermal conductivity of nanofluids, which is dependent on the power law of velocity gradient as well as the nanoparticles property, is taken into account. Dual solutions are obtained by Bvp4c with Matlab. The stability of the solution also is discussed by introducing the unsteady governing equations. Furthermore, a new interesting phenomenon is found: the local Nusselt number do not maintain the similar characteristics of Newtonian fluid near the point where the velocity ratio is equal to 0.5.

1. Introduction
In recent years, many investigations have been made to explore the nanofluid technology in the field of enhanced heat transfer and new generation cooling technology, which overcomes the limitation of many conventional heat transfer fluids with low thermal conductivity, such as water, oil, and ethylene glycol mixture. Furthermore, due to the tiny size of nanoelements and the low volume fraction of nanoelements required for conductivity enhancement, nanofluids are also very stable and have no additional problems [1], such as sedimentation, erosion, additional pressure drop and non-Newtonian behavior. Numerous methods have been taken to develop advanced heat transfer fluids with substantially higher conductive properties by suspending nano(usually less than 100 nm)/micro or larger-sized particle materials in liquids [2]. The comprehensive references and the broad range of current and future applications on nanofluids can be found in the recent book [3] and in the review papers by Buongiorno [4], Wang and Mujumdar [5], Kakac and Pramanjaroenkij [6], Wong and Leon [7].

Nanofluid has been used to improve heat transfer for some convection problems due to its wide applications in electronics cooling, heat exchangers, and double pane windows. Furthermore, mixed convection is preferred and then many numerical studies about heat and mass transfer of nanofluids in enclosures have been studied [8–11]. Some similarity solutions or analytical solutions also have been done. For example, Nield and Kuznetsov [12,13] studied the free convection of viscous and incompressible nano-fluid past a vertical plate embedded in porous medium or not. Loganathan et al. [14] discussed unsteady natural convection flow of nanofluids past an infinite vertical plate with considering the radiation. Ahmad et al. [15] extended Blasius problem and Sakiadis problem to the case of nanofluids. Norfifah Bachok et al. [16] made another extension and extended the Blasius and Sakiadis problems in nanofluids by considering a uniform free stream parallel to a fixed or moving flat plate. Ahmad and Pop [17] investigated steady mixed convection boundary layer flow past a vertical flat plate embedded in a porous medium filled with nanofluids. One of interesting problems, the existence of dual solutions, also have been proposed, which bring more insight on engineering applications. For example, Subhashini et al. [18–20] pointed out that the upper branch solutions are most physically relevant solution whereas the lower branch solutions seem to deprive physical significance or may have realistic meaning in different situations. Furthermore, the stability of the numerical solutions for the mixed convection also has been analyzed. Since Mahmood and Merkin [21] done the classical work and investigated the mixed convection on a vertical circular cylinder, Merkin [22], Merkin and Pop [23] also discussed the dual solutions occurring in mixed convection in a porous medium by perturbation method and analyzed the stability of the upper and lower branch of the solutions. Similarly, some interesting works also have been done [24,25].
In contrary, the number of studies on natural and mixed convection of nanofluids, where Non-Newtonian power law fluids is considered as the base fluid, is very small. Lin and Zheng et al. [26–28] studied the Marangoni convection of power law nanofluids, where they considered the thermal conductivity to be dependent on velocity or temperature gradient. Some other mass and heat transfer models about power law nanofluids [29–32] also have been proposed and solved numerically or analytically.

The aim of this paper is to investigate the mixed convection mass and heat transfer of power law nanofluids past a stretching vertical plate. Here the CMC-Water (0.0–0.4%) is considered as a pseudo-plastic power law fluid [26–28] and regarded as the base fluid. Three types of nanoparticles are considered: copper (Cu), aluminum oxide (Al2O3) and titanium oxide (TiO2). According to the experimental studies, the thermophysical properties of CMC-Water (<0.4%) are similar to water [33,34]. Their related properties are given in Table 1 and Table 2. The generalized Fourier law proposed by Zheng [26–28,35,36] for varying thermal conductivity of nanofluids is taken into account. The similar equations are solved numerically with Matlab. Effects of different parameters on velocity and temperature are discussed in detail.

2. Governing equations

Consider the steady two-dimensional mixed convection flow and heat transfer of power law nanofluid over a vertical porous stretching plate. The base fluid and the nanoparticles are in thermal equilibrium and no slip occurs between them. Here we assume that x-axis is along the vertical surface and y-axis is normal to the plate. u and v are the velocity components in the x and y directions, respectively. As shown in Fig. 1, the free stream velocity and the stretching velocity are assumed to be $u_\infty = a$ and $u_w = b$. The temperature $T_\infty$ at the wall is a constant, $T_\infty$ is the temperature of the fluid far away from the plate. The suction/injection velocity along the plate is $v_w = -f_w \frac{1}{\mu_2} (\nu y U_a x - x^n)^{\mu_2}$, where $U = a + b$ is the composite velocity.

Under these assumptions, the governing equations about the mixed convective flow and heat transfer of power-law nanofluid can be written as follow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho_\text{nf}} \frac{\partial}{\partial y} \left( \mu_\text{nf} \frac{\partial u}{\partial y} \right) + \frac{g (\rho_\text{nf})}{\rho_\text{nf}} (T - T_\infty),$$

(2.3)

$$\rho c_p \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left( \frac{k}{\eta} \frac{\partial T}{\partial y} \right),$$

where $\rho_\text{nf}$, $k_\text{nf}$, $\eta_\text{nf}$, $(\rho c_p)_\text{nf}$, $\beta_\text{nf}$ are the density of the nanofluid, the thermal conductivity of the nanfluid, the modified viscosity of the nanofluid, the heat capacitance of the nanofluid and the coefficient of the thermal expansion, respectively. $\mu_\text{nf} = \mu_\text{nf}(\frac{\partial T}{\partial y})^{-\mu}$ is the effective viscosity of the nanofluid. $K = z_\text{nf}|\frac{\partial T}{\partial y}|^{-\mu}$ is the thermal diffusivity proposed by Zheng et al. [26–28,35,36], where $z_\text{nf} = \frac{k_\text{nf}}{(\rho c_p)_\text{nf}}$ is the modified thermal diffusivity of the nanofluid. The expression of above nanofluid parameters are given as follow:

$$\rho_\text{nf} = (1 - \phi) \rho_f + \phi \rho_s; \ (\rho c_p)_\text{nf} = (1 - \phi) (\rho c_p)_f + \phi (\rho c_p)_s; \ (\rho_\text{nf}) = (1 - \phi) \rho_f + \phi \rho_s; \ \mu_\text{nf} = \frac{\mu_f}{(1 - \phi)^{\mu}};$$

$$K_\text{nf} = \frac{(k_f + 2k_s) - 2\phi(k_f - k_s)}{(k_f + 2k_f + \phi(k_f - k_s)}.$$  \hspace{1cm} (2.4)

where $\rho_f$, $\rho_s$ are the reference density of the fluid fraction and the solid fraction, and $k_f$, $k_s$ are the thermal conductivity of the fluid fraction and solid fraction, respectively. $\phi$ is the solid volume fraction parameter of the nanofluid, $\mu_f$ is the viscosity of the fluid fraction. Here the modified viscosity $\mu_\text{nf}$ of the nanofluid can be approximated as viscosity of the base fluid $\mu_f$ containing dilute

Table 1

<table>
<thead>
<tr>
<th>Property</th>
<th>CMC-water (0.0–0.4%)</th>
<th>Copper (Cu)</th>
<th>Aluminum Oxide (Al2O3)</th>
<th>Titanium Oxide (TiO2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_p$ (J/kg K)</td>
<td>4179</td>
<td>385</td>
<td>765</td>
<td>686.2</td>
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<tr>
<td>$\rho$ (kg/m$^3$)</td>
<td>997.1</td>
<td>8933</td>
<td>3970</td>
<td>4250</td>
</tr>
<tr>
<td>$\alpha$ (W/mK)</td>
<td>0.613</td>
<td>400</td>
<td>40</td>
<td>8.9538</td>
</tr>
<tr>
<td>$\beta \times 10^{-5}$ (1/K)</td>
<td>21</td>
<td>1.67</td>
<td>0.85</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>Power law index</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$ (N/m$^2$)</td>
<td>$8.55 \times 10^{-4}$</td>
<td>$6.319 \times 10^{-3}$</td>
<td>$1.754 \times 10^{-2}$</td>
<td>$3.136 \times 10^{-2}$</td>
<td>$7.853 \times 10^{-2}$</td>
</tr>
</tbody>
</table>
suspension of fine spherical particles and is given by Brinkman [37]. The effective thermal conductivity of the nanofluid is approximated by the Maxwell–Garnett model, which restricts to spherical nanoparticles and does not account for other shapes of nanoparticles [38–46].

The corresponding boundary conditions are
\[ u = u_w = b, \quad v = v_w(x), \quad T = T_w; y = 0, \]
\[ u = u_0 = a, \quad T = T_w; y = \infty. \tag{2.5} \]

Introduce the stream function \( \psi \) such that \( u = \partial \psi / \partial y \) and \( v = -\partial \psi / \partial x \), which are listed as follow:
\[ \psi = (\gamma U^2 - x) \psi (\eta), \quad \eta = U^2 - x \psi (\eta), \quad \theta = T - T_w. \tag{2.6} \]
where \( \gamma = \frac{8}{v} \).

Substituting above transformations into the governing Eqs. (2.1)–(2.3), the partial differential equations can be transformed into the following ordinary equations
\[ \frac{1}{1 - \phi} \left( f''(\eta)^{n-1} f'(\eta) \right) + \theta(x) f'(\eta) + \frac{1}{n + 1} \theta f(\eta) f''(\eta) = 0, \tag{2.7} \]
\[ \frac{1}{\Pr} \frac{k_f}{k} \left( f'(\eta)^{n-1} f''(\eta) \right) + \alpha + \frac{1}{n + 1} \theta f(\eta) = 0, \tag{2.8} \]
where the coefficients are \( \gamma = 1 - \phi + \phi \frac{\partial \psi}{\partial \eta}, \beta = 1 - \phi + \phi \frac{\partial \psi}{\partial \eta}, \alpha = 1 - \phi + \phi \frac{\partial \psi}{\partial \eta}, \) respectively. The Prandtl number \( \Pr \), the local Grashof number \( Gr_n \), the local Reynolds number \( Re_n \) and the mixed convection parameter \( \lambda(x) \) are given by
\[ Gr_n = \frac{Pr pf_j (T_w - T_w) x}{\mu_f^2}, \quad Re_n = \frac{pf_j dx}{\mu_f^2}, \quad Pr = \frac{\varepsilon \mu_f}{k_f}, \tag{2.9} \]
where \( \lambda(x) \) is a function of \( x \). Then the numerical solutions represent local similarity solutions since the parameters still depend on an independent variable \( x \). It should be noted that the sign of \( \lambda \) depends on the nature of the flow arising out from this situation. \( \lambda > 0 \) corresponds to buoyancy assisting flow. In contrast, when \( \lambda < 0 \), it corresponds to buoyancy opposing flow. Three special cases as \( \lambda = 0, n = 1 \) are discussed by Norfiah Bachok et al. [16], Ahmad et al. [15] and Subhashini and Sumathi [19].

The corresponding boundary conditions become
\[ f(0) = f_w, \quad f'(0) = \varepsilon, \quad \theta(0) = 1, \quad f'(\infty) = 1 - \varepsilon, \quad \theta(\infty) = 0. \tag{2.10} \]
where velocity ratio is \( \varepsilon = \frac{\beta}{\alpha} \).

The shear stress and local Nusselt number on the surface are derived as follow:
\[ \tau_w = \mu_f \frac{\partial f}{\partial \eta} = \mu_f U^2 f''(0) \frac{f'(0)}{f''(0)}, \tag{2.11} \]
\[ Nu_x = \frac{\alpha v_x}{k_f \frac{\partial f}{\partial \eta}} \frac{f'(0)}{f''(0)} \theta(0) = \frac{1}{\gamma U^2} \theta(0) \sim -\theta(0). \]

3. Numerical methods

3.1. Transformation of equations

In order to solve the nonlinear differential Eqs. (2.7) and (2.8), we transfer this problem to a system of first-order equations. Here we denote \( y_1 = f, \quad y_2 = f', \quad y_3 = f'', \quad y_4 = \theta, \quad y_5 = \varepsilon \), then
\[ y_1' = y_2, \]
\[ y_2' = y_3, \]
\[ y_3' = \frac{1}{n} (-\gamma \lambda(x) y_4 - \frac{1}{n + 1} \beta \gamma y_2 y_3) (1 - \phi)^{2n} [y_3]^{1-n}, \tag{3.12} \]
\[ y_4' = y_5, \]
\[ y_5' = -\frac{2 \Pr \frac{k_f}{k}}{n + 1} \frac{n}{\gamma - 1} y_3 y_1 - (n - 1) y_3^{n-2} y_3 y_5. \tag{3.13} \]
The boundary conditions can be written to be
\[ y_1(0) = f_w, \quad y_2(0) = \varepsilon, \quad y_3(\infty) = 1 - \varepsilon, \quad y_4(0) = 1, \quad y_5(\infty) = 0. \]

The system of nonlinear differential equations can be solved by Bvp4c with Matlab. Here we assume that the Maximum residual is \( 10^{-5} \). In order to obtain the dual solutions, the number of the mesh, the infinity replaced by finite point and the initial guess values need to be adjusted according to the different physical parameters. Some solutions for special cases also are compared with previous ones to validate the accuracy and correctness, which is shown in Table 3.

4. Numerical solutions and discussion

In this following section, we assume that the base fluid is 0.4% CMC-Water and will consider the velocity and temperature distribution influenced by different nano-particles and different physical parameters. In every figure, we will give illustration of velocity and temperature distribution corresponding to dual solutions for some fixed values.

Table 4 gives some values of the dual solutions influenced by different physical parameters. And according to the Fig. 2, we can find that dual solutions exist for power law nanofluids and there is no solution as \( \varepsilon < \varepsilon_w \). Both types of nanofluids have the similar trends as the base fluids are water or power law fluid for \( f'(0) \). However, an interesting phenomenon also can be found for the power law nanofluids. Local Nusselt number decreases first and then increases again near the point \( \varepsilon = 0.5 \), which is different dramatically from the case as the base fluid is water. Furthermore, the first solutions of \( f'(0) \) and \( \theta'(0) \) are stable and physically, while the second solutions are not [48,19]. The stability of the dual solutions will be discussed in the next section.

The influence of different nanoparticles on the velocity and temperature distribution can be found in Fig. 3 as other parameters are \( \varepsilon = -0.5, \lambda = -0.01, f_w = 0.1, Pr = 6.2, \phi = 0.1 \) and \( n = 0.85 \). For the first solution, it is observed that the thickness of thermal boundary layer for Cu nanofluid is the more thinner than other two cases. The reason is that Cu has the highest thermal conductivity compared to TiO_2 and Al_2O_3. The higher temperature gradients is caused from the reduced value of thermal diffusivity and, therefore, higher improvement in heat transfers. However, the difference between TiO_2 and Al_2O_3 can be negligible. For the second solution, the velocity and temperature distribution shows a more obvious difference influenced by different nanoparticles.

Fig. 4 gives the illustration of dual solutions of Cu-CMC nanofluid influenced by power law index as other parameters are \( \varepsilon = -0.01, Pr = 6.2, \phi = 0.001, \lambda = -0.5, f_w = 0.1 \). For the first solution, it is observed that the velocity decreases and the thickness of velocity boundary layer increases with increasing power law index. The reason may be that the effective viscosity of the power law fluids decreases with the increasing power law number. The similar trends also happen on the second solution. Furthermore, the influence of the power law index on the second solution is more obvious.
Table 3
The comparison of Cu-water nanofluid with previous works as \(Pr = 6.2\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\phi)</th>
<th>(\varepsilon)</th>
<th>Present results</th>
<th>Ishak et al. [47]</th>
<th>Bachok et al. [16]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1st solution</td>
<td>2nd solution</td>
<td>1st solution</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-0.5</td>
<td>0.397851</td>
<td>0.171021</td>
<td>0.3990</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.2</td>
<td>0.412369</td>
<td>0.011421</td>
<td>0.4124</td>
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<td></td>
<td>0</td>
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<td>0.332059</td>
<td>0.3321</td>
<td>0.3321</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td>-0.443762</td>
<td>-0.4438</td>
<td>0.3321</td>
</tr>
<tr>
<td>0.1</td>
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<td></td>
<td>0.467380</td>
<td>0.209017</td>
<td>0.4674</td>
</tr>
<tr>
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<td></td>
<td>0.484476</td>
<td>0.013433</td>
<td>0.4844</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td></td>
<td>0.390124</td>
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</tr>
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<td>-0.521295</td>
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<tr>
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<tr>
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<td>1</td>
<td></td>
<td>-0.540537</td>
<td>-0.5405</td>
<td></td>
</tr>
</tbody>
</table>

Table 4
Dual solutions of \(f'(0)\) and \(\theta'(0)\) for Cu-CMC power law nanofluid as \(Pr = 3, \lambda = -0.02, f_w = 0.1\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\varepsilon)</th>
<th>(f'(0))</th>
<th>(\theta'(0))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st solution</td>
<td>2nd solution</td>
</tr>
<tr>
<td>0.91</td>
<td>-0.5</td>
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<td></td>
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<td>0</td>
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<td>-0.2</td>
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<tr>
<td></td>
<td>1</td>
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<td>-0.55465</td>
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</tbody>
</table>

Fig. 5 illustrates the effects of solid volume fraction on the velocity and temperature as other parameters are \(\lambda = -0.001, f_w = 0.1, Pr = 6.2, \varepsilon = -0.5, \phi = 0.1, n = 0.85\). For the first solution, the temperature decreases and the temperature gradient increases near the plate with the increasing solid volume fraction. This means that the increasing solid volume fraction results in the increase of heat transfer, which reduces to the thinning of the thermal boundary layers near the vertical walls. Meanwhile, the velocity boundary layer thickness also decreases with the solid volume fraction. Furthermore, the second solution for velocity and temperature shows the similar trends compared with the first solution.

Fig. 6 gives the profiles of suction parameter on the velocity and temperature as other parameter are \(\lambda = -0.001, Pr = 6.2, \varepsilon = -0.5, \phi = 0.1, n = 0.85\). For the first solution, with the increasing suction parameter, much more fluid is carried away from the surface.
Fig. 3. Dual solutions for different power law nanofluids.

Fig. 4. Dual solutions for different $\eta$ in Cu power law nanofluid.

Fig. 5. Dual solutions for different Cu volume fraction in power law nanofluid.
which causes reduction in velocity gradient as it tries to maintain the same velocity over a small region near the surface. The velocity increases with the increasing value of permeability parameter. Moreover, the thermal boundary layer thickness decreases sharply with the increasing parameter \( f_w \). This also shows that the increasing suction velocity makes the temperature boundary layer thinner and the boundary layer becomes more stable. However, for the second solution, the opposite trends can be observed influenced by suction velocity.

Fig. 7 illustrates the effects the velocity ratio on the velocity and temperature distribution as other parameters are \( \lambda = -0.01, \ Pr = 6.2, \ e = -0.5, \ n = 0.91, \ f_w = 0.1 \). For the first solution, we can find that the temperature gradient and the velocity at the wall will increase with increasing \( e \). However, as the distance is far away from the plate, the velocity will decrease with increasing \( e \). For the second solution, the profile corresponding to temperature is put toward the right side with increasing mixed convection parameter \( \lambda \). The velocity has contrary trend corresponding to the case of first solution.

The stability of the numerical solution

In order to analyze the stability of the numerical solution, the unsteady mixed convection problem is introduced. The governing equations are written as

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho_{nf}} \left( \mu_{nf} \left( \frac{\partial u}{\partial y} \right)^{n-1} \frac{\partial u}{\partial y} \right) + g \left( \frac{\rho_\beta}{\rho_{nf}} \right) (T - T_w),
\]

(5.14)

\[
\left( \rho c_p \right)_{nf} \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left( \kappa_{nf} \left( \frac{\partial u}{\partial y} \right)^{n-1} \frac{\partial T}{\partial y} \right).
\]

(5.15)

Fig. 7. Dual solutions for different \( e \) in Cu power law nanofluid.
growth (or decay) rate of disturbances [22, 23, 25].

Table 5
The smallest eigenvalue \( \delta_0 \) for Cu-CMC power law nanofluid as Pr = 6.2, \( \lambda = -0.01 f_\alpha = -0.1 \).

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \nu )</th>
<th>1st solution</th>
<th>2nd solution</th>
<th>1st solution</th>
<th>2nd solution</th>
<th>1st solution</th>
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</thead>
<tbody>
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<td>0.1</td>
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<td>-0.10595</td>
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<td>0.190353</td>
<td>0.10005</td>
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</tr>
<tr>
<td></td>
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<td>0.521968</td>
<td>0.521968</td>
<td>0.527553</td>
<td>0.527553</td>
</tr>
</tbody>
</table>

where \( t \) represents time. The new dimensionless variables are introduced:

\[
\psi = (\eta U^{2n-1} x)^{1/n} f(\eta, \tau), \quad \theta(\eta, \tau) = \frac{T - T_\infty}{T_s - T_\infty},
\]

\[
\eta = \left( \frac{U^{2n-1} x}{Y} \right)^{1/n}, \quad \tau = U t x^{-1}, \tag{5.16}
\]

then the partial differential equations can be obtained after one substitutes dimensionless transformation (5.16) into Eqs. (5.14) and (5.15):

\[
\frac{1}{(1 - \phi)^2} \left( (f_{yyy})^{n-1} f_{yy} \right) + \frac{1}{n + 1} \beta g f_{yy} - \beta f f'_{yy} - f f'_{yy} \tau + f f_{yy} \tau = 0, \tag{5.17}
\]

\[
\frac{1}{Pr} \frac{K_{sl}}{K_{j}} \left( (f_{yy})^{n-1} \theta_{y} \right) + \alpha \frac{1}{n + 1} f'_{y} - \beta f_{y} \beta_{y} + f f_{y} \beta_{y} = 0. \tag{5.18}
\]

To investigate the stability of the steady flow solution, we express

\[
f(\eta, \tau) = F(\eta) = \exp(-\delta \tau) g(\eta, \tau), \quad \theta(\eta, \tau) = \Theta(\eta) = \exp(-\delta \tau) \omega(\eta, \tau), \tag{5.19}
\]

where \( F(\eta), \Theta(\eta) \) correspond to steady state solutions and \( \delta \) is the growth (or decay) rate of disturbances [22, 23, 25]. Here, functions \( g, \omega \) and their derivatives are assumed to be small compared to the steady solutions, respectively. Following Refs. [22, 23, 25], substituting Eq. (5.19) into differential equations system (5.17) and (5.18) and linearizing, we can obtain:

\[
\frac{1}{(1 - \phi)^2} \left( n(n-1)|F'|^{n-2}F''g'' + n|F'|^{n-1}g'''' \right) + \gamma \lambda \omega + \delta \rho g' + \frac{1}{n+1} \beta (F''g' + gF') = 0, \tag{5.20}
\]

\[
\frac{k_{sl}}{K_{j} Pr} \left( (n-1)|F'|^{n-2}F''\omega' + (n-1)(n-2)|F'|^{n-3}g''F''\omega'' \right) + (n-1)\rho (F''|F'|\omega' + |F'|^{n-1}\omega'' + (n-1)|F'|^{n-2}g''\omega'') + \sigma \lambda \omega + \frac{1}{n+1} \alpha (F\omega' + \Theta \omega) = 0. \tag{5.21}
\]

The corresponding boundary conditions are

\[
g(0) = 0, \quad g'(0) = 0, \quad g'(\infty) = 0, \quad \omega(0) = 0, \quad \omega(\infty) = 0. \tag{5.22}
\]

The homogeneous linear Eqs. (5.20) and (5.21) and the homogeneous boundary conditions (5.22) constitute an eigenvalue system problem with \( \delta \) as the eigenvalue. As a result, we need to get the smallest eigenvalue \( \delta_0 \). If \( \delta_0 \) is negative, then there is an initial growth of disturbance and the flow is unstable. While if the value of \( \delta_0 \) is positive, there is an initial decay and the solution is stable. In Table 5, the smallest eigenvalues for some values of physical parameters also have been given, which show the stability of the dual solutions.
6. Conclusion

A mixed convection flow and heat transfer of pseudo-plastic power law nanofluid past a stretching vertical plate is investigated. The generalized thermal conductivity proposed by Zheng is applied. The similar equations are solved by Matlab and the stability of the dual solutions is analyzed. Some conclusions can be drawn:

- Dual solutions are obtained for some values of the physical parameters.
- Compared both solutions, it is observed that the boundary layer thickness of the velocity for the first solution is thinner than the one for the second solution. This phenomenon also happens on the temperature boundary layer.
- Local Nusselt number shows the linearity near the point \( e = 0.5 \) no longer for power law nanofluids.
- \( Cu \) has the highest thermal conductivity compared to TiO2 and Al2O3. The reduced value of thermal diffusivity leads to higher temperature gradients and, therefore, higher enhancement in heat transfers.
- Increasing power law index leads to the decreasing velocity and increasing temperature gradient.
- The second solution is unstable, and the first solution is stable.

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References